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# ON GENERALIZED DECOMPOSITION NUMBERS AND FONG'S REDUCTIONS

Dedicated to Professor Hirosi Nagao on his 60th birthday

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## Introduction

In this paper we investigate how generalized decomposition numbers behave under Fong's reductions.

Let $G$ be a finite group and $p$ be a fixed prime number. If $\pi$ is a $p$-element of $G$ and $B$ is a $p$-block of $G$, then for an ordinary irreducible character $\chi$ in $B$ and for each $p$-regular element $\rho$ of the centralizer $C_G(\pi)$ of $\pi$, we have

$$\chi(\pi\rho)=\sum_{\phi} d(\chi, \pi, \phi)\phi(\rho)\,.$$

Here $\phi$ ranges over the irreducible Brauer characters in the $p$-blocks of $C_G(\pi)$ associated with $B$. We have the following theorem related to the Fong's first reduction.

**Theorem 1.** *Let $H$ be a subgroup of $G$, and let $B$ and $\tilde{B}$ be $p$-blocks of $G$ and $H$, respectively. We assume that $\tilde{\chi}\rightarrow\tilde{\chi}^G$ is a 1-1 correspondence between the ordinary irreducible characters in $\tilde{B}$ and those in $B$, where $\tilde{\chi}^G$ is the character of $G$ induced from $\tilde{\chi}$. Then the following holds.*

(i) *$B$ and $\tilde{B}$ have a common defect group $D$.*

(ii) *Let $\tilde{b}$ be a root of $\tilde{B}$ in $C_H(D)D$. Then $\tilde{b}^{C_G(D)D}$ is defined in the sense of Brauer [2]. We put $b=\tilde{b}^{C_G(D)D}$. Then $b$ is a root of $B$ in $C_G(D)D$ and $T(b)=T(\tilde{b})C_G(D)$ where $T(b)$ is the inertial group of $b$ in $N_G(D)$ and $T(\tilde{b})$ is the inertial group of $\tilde{b}$ in $N_H(D)$. In particular $T(b)/C_G(D)D\simeq T(\tilde{b})/C_H(D)D$.*

(iii) *Let $\{(\pi_i, \tilde{b}_i), i=1, 2, \cdots, n\}$ be a set of representatives for the conjugacy classes of subsections associated with $\tilde{B}$. Then $\tilde{b}_i^{C_G(\pi_i)}$ is defined and $\tilde{\phi}\rightarrow\tilde{\phi}^{C_G(\pi_i)}$ is a 1-1 correspondence between the irreducible Brauer characters in $\tilde{b}_i$ and those in $\tilde{b}_i^{C_G(\pi_i)}$. Furthermore $\{(\pi_i, \tilde{b}_i^{C_G(\pi_i)}), i=1, 2, \cdots, n\}$ is a set of representatives for the conjugacy classes of subsections associated with $B$.*

(iv) *Let $\tilde{\chi}$ be an ordinary irreducible character in $\tilde{B}$ and $\tilde{\phi}$ be an irreducible Brauer character in $\tilde{b}_i$. Then*

$$d(\tilde{\chi}^G,\ \pi_i,\ \tilde{\phi}^{C_G(\pi_i)}) = d(\tilde{\chi},\ \pi_i,\ \tilde{\phi})\,.$$

Let $\zeta$ be an irreducible character of a normal $p'$-subgroup $N$ of $G$ and suppose that $\zeta$ is extendible to a character $\hat{\zeta}$ of $G$. Let $\bar{B}$ be a $p$-block of the factor group $\bar{G}$ of $G$ by $N$ and $\bar{\chi}_0$ be an ordinary irreducible character in $\bar{B}$. $\bar{\chi}_0$ can be viewed as a character of $G$. We denote by $\hat{\zeta}\bar{B}$ the $p$-blosk of $G$ which contains $\hat{\zeta}\bar{\chi}_0$. The ordinary irreducible characters in $\hat{\zeta}\bar{B}$ are $\hat{\zeta}\bar{\chi}$'s, where $\bar{\chi}$ runs over the ordinary irreducible characters in $\bar{B}$ and the irreducible Brauer characters in $\hat{\zeta}\bar{B}$ are $\hat{\zeta}\bar{\phi}$'s, where $\bar{\phi}$ runs over the irreducible Brauer characters in $\bar{B}$. If $\bar{B}_1$ and $\bar{B}_2$ are different $p$-blocks of $\bar{G}$, then $\hat{\zeta}\bar{B}_1 \neq \hat{\zeta}\bar{B}_2$. For an element $x$ of $G$, we put $\bar{x}=xN\ (\in\bar{G})$ and for a subgroup $Q$ of $G$, we put $\bar{Q}=QN/N$. If $Q$ is a $p$-subgroup, then $C_{\bar{G}}(\bar{Q})=\overline{C_G(Q)}$ and $N_{\bar{G}}(\bar{Q})=\overline{N_G(Q)}$. We have the following theorem related to the Fong's second reduction.

**Theorem 2.** *Let $\zeta$ be an irreducible character of a normal $p'$-subgroup $N$ of $G$ and $\hat{\zeta}$ be an extension of $\zeta$ to $G$ such that $(o\,(\det \hat{\zeta}), p)=1$. If $B$ is a $p$-block of $G$ and $B=\hat{\zeta}\bar{B}$ for some $p$-block $\bar{B}$ of the factor group $\bar{G}$, then the following holds.*

(i) *If $D$ is a defect group of $B$, then $\bar{D}$ is a defect group of $\bar{B}$.*

(ii) *Let $\bar{b}$ be a root of $\bar{B}$ in $C_{\bar{G}}(\bar{D})\bar{D}$ and let $b$ e be a $p$-block of $C_G(D)D$ such that $b^{NC_G(D)D}=\hat{\zeta}\bar{b}$. Then $b$ is a root of $B$ in $C_G(D)D$ and $T(\bar{b})=\overline{T(b)}$. In particular $T(\bar{b})/C_{\bar{G}}(\bar{D})\bar{D}\cong T(b)/C_G(D)D$.*

(iii) *Let $\pi$ be a $p$-element of $G$ and $\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_s$ be the $p$-blocks of $C_{\bar{G}}(\bar{\pi})$ associated with $\bar{B}$. If $b_i$ is a $p$-block of $C_G(\pi)$ such that $b_i^{NC_G(\pi)}=\hat{\zeta}\bar{b}_i$, then $b_1, b_2, \cdots, b_s$ are the $p$-blocks of $C_G(\pi)$ associated with $B$. Furthermore $b_i=\theta_\pi\bar{b}_i$ $(i=1, 2, \cdots, s)$ when $\bar{b}_i$ is viewed as a $p$-block of $C_G(\pi)/C_N(\pi)$, where $\theta_\pi$ is an ordinary irreducible character of $C_G(\pi)$ such that $\theta_{\pi|C_N(\pi)}$ is irreducible.*

(iv) *For each ordinary irreducible character $\bar{\chi}$ in $\bar{B}$, for the above $p$-element $\pi$ and for each irreducible Brauer character $\bar{\phi}$ in $\bar{b}_i$, there exists a sign $\varepsilon_\pi=\pm 1$ such that*

$$d(\hat{\zeta}\bar{\chi},\ \pi,\ \theta_\pi\bar{\phi}) = \varepsilon_\pi d(\bar{\chi},\ \bar{\pi},\ \bar{\phi})\,.$$

We remark that (ii) and (iii) in the above theorems are stated by Puig [8, Theorems 1 and 2] without proofs.

Let $K$ be the algebraic closure of the $p$-adic number field $Q_p$ and $R$ be the ring of local integers in $K$. Let $P$ denote the maximal ideal of $R$ and $F$ denote the residue class field $R/P$. For a $p$-block $B$ of $G$, we denote the block idempotent of $FG$ corresponding to $B$ by $E_B$ and for an ordinary irreducible character $\chi$ of $G$, we denote the centrally primitive idempotent of $KG$ corresponding to $\chi$ by $e_\chi$. The number of ordinary irreducible characters in $B$ and the number of irreducible Brauer characters in $B$ are denoted by $k(B)$ and $l(B)$, respectively.

## 1. Proof of Theorem 1

**Lemma 1.** *Let $H$ be a subgroup of $G$ and $x_1, x_2, \cdots, x_h$ be a set of representatives for the right cosets of $H$ in $G$. For a $p$-block $\tilde{B}$ of $H$, we assume that*

$$E_{\tilde{B}}x^{-1}E_{\tilde{B}}x = 0 \qquad \textit{for all} \quad x \in G-H .$$

*Then $\sum_{i=1}^{h} x_i^{-1}E_{\tilde{B}}x_i$ is a block idempotent of $FG$. If we put $\sum_{i=1}^{h} x_i^{-1}E_{\tilde{B}}x_i = E_B$, where $B$ is a $p$-block of $G$, then $\tilde{\phi} \to \tilde{\phi}^G$ is a 1–1 correspondence between the irreducible Brauer characters in $\tilde{B}$ and those in $B$.*

REMARK. By Iizuka, Ohmori and Watanabe [6, Theorem 2], the following (i) and (ii) are equivalent.

(i) $\tilde{\phi} \to \tilde{\phi}^G$ is a 1–1 correspondence between the irreducible Brauer characters in $\tilde{B}$ and those in $B$.

(ii) $\tilde{\chi} \to \tilde{\chi}^G$ is a 1-1 correspondence between the ordinary irreducible characters in $\tilde{B}$ and those in $B$.

Proof. We put $E = \sum_{i=1}^{h} E_{\tilde{B}}^{x_i}$, where $E_{\tilde{B}}^{x_i} = x_i^{-1}E_{\tilde{B}}x_i$. Then $E$ is a central idempotent of $FG$. By the assumption we can show $\mathfrak{L} \to \mathfrak{L}^G$ defines a 1–1 correspondence between the isomorphism classes of (right) $FH$-modules $\mathfrak{L}$ with $\mathfrak{L}E_{\tilde{B}} = \mathfrak{L}$ and the isomorphism classes of $FG$-modules $\mathfrak{M}$ with $\mathfrak{M}E = \mathfrak{M}$, where $\mathfrak{L}^G$ is the induced $FG$-module. Furthermore if $\mathfrak{L}$ is an irreducible or a principal indecomposable $FH$-module, then $\mathfrak{L}^G$ is an irreducible or a principal indecomposable $FG$-module. Hence by the indecomposability of Cartan matrices, $E$ is a block idempotent. This completes the proof.

Proof of Theorem 1. (i) is well known. It is also well known that if $E_{\tilde{B}}'$ is the block idempotent of $RH$ which corresponds to $\tilde{B}$, then $E_{\tilde{B}}' = \sum_{\tilde{\chi}} e_{\tilde{\chi}}$, $\tilde{\chi}$ ranges over the ordinary irreducible characters in $\tilde{B}$. Let $x_1, x_2, \cdots, x_h$ be a set of representatives for the cosets of $H$ in $G$, where $x_1 = 1$. We can show that $e_{\tilde{\chi}^G} = \sum_{i=1}^{h} e_{\tilde{\chi}}^{x_i}$, so we have $E_B = \sum_{i=1}^{h} E_{\tilde{B}}^{x_i}$. By the assumption, $E_{\tilde{B}}FGE_B = E_{\tilde{B}}FG$ and hence $E_{\tilde{B}}E_B = E_{\tilde{B}}$ and

$$E_{\tilde{B}} \sum_{i=2}^{h} E_{\tilde{B}}^{x_i} = 0 . \tag{1}$$

By the proof of Watanabe [10, Theorem 2] and the fact $\dim_F (E_BFG) = |G{:}\,H|^2 \dim_F (E_{\tilde{B}}FH)$, we have

$$E_BFG = \sum_{i,j=1}^{h} \oplus x_i^{-1}E_{\tilde{B}}FHx_j . \tag{2}$$

From (2), we obtain $E_{\tilde{B}}E_{\tilde{B}}^{x} = 0$ for all $x \in G-H$.

Let $Q$ be a $p$-subgroup of $H$, $\tilde{b}$ be a $p$-block of $C_H(Q)Q$ with $\tilde{b}^H=\tilde{B}$ and $\mathrm{Br}_Q$ be the Brauer morphism from $(FG)^Q$ onto $FC_G(Q)$, where $(FG)^Q=\{a\in FG \mid ya=ay \text{ for all } y\in Q\}$ (see Alperin and Broué [1]). Then we have

$$\mathrm{Br}_Q(E_{\tilde{B}})\mathrm{Br}_Q(E_{\tilde{B}})^x=0 \qquad (x\in C_G(Q)Q-C_H(Q)Q),$$
$$\mathrm{Br}_Q(E_{\tilde{B}})E_{\tilde{b}}=E_{\tilde{b}}.$$

So $E_{\tilde{b}}E_{\tilde{b}}{}^x=0$ for all $x\in C_G(Q)Q-C_H(Q)Q$. By Reynolds [9, Theorem 2] and Lemma 1, $\tilde{b}^{C_G(Q)Q}$ is defined and $\tilde{\phi}\to\tilde{\phi}^{C_G(Q)Q}$ is a 1–1 correspondence between the irreducible Brauer characters in $\tilde{b}$ and those in $\tilde{b}^{C_G(Q)Q}$.

Let $z_1, z_2, \cdots, z_v$ be a set of respresentatives for the cosets of $T(\tilde{b})$ in $N_H(D)$, where $z_1=1$. Then

$$\mathrm{Br}_D(E_{\tilde{B}})=\sum_{j=1}^{v}E_{\tilde{b}}{}^{z_j}, \quad E_{\tilde{b}}E_{\tilde{b}}{}^{z_j}=0 \tag{3}$$

for $j\geqq 2$. We assume that $C_G(D)D=\bigcup_{i=1}^{t}C_H(D)Dx_i$ and $N_G(D)=\bigcup_{i=1}^{u}N_H(D)x_i$. So $N_G(D)=\bigcup_{i=1}^{u}\bigcup_{j=1}^{v}T(\tilde{b})z_jx_i$. For $(i,j)\neq(1,1)$ we have

$$E_{\tilde{b}}E_{\tilde{b}}{}^{z_jx_i}=E_{\tilde{b}}\mathrm{Br}_D(E_{\tilde{B}})\mathrm{Br}_D(E_{\tilde{B}})^{x_i}E_{\tilde{b}}{}^{z_jx_i}=0 \tag{4}$$

from (3). By the above argument $\tilde{b}^{C_G(D)D}$ is defined. We put $b=\tilde{b}^{C_G(D)D}$. Then $b^G=\tilde{b}^G=\tilde{B}^G=B$. Hence $b$ is a root of $B$ in $C_G(D)D$ and $E_b=\sum_{i=1}^{t}E_{\tilde{b}}{}^{x_i}$. If $y\in T(b)$, then

$$E_b=E_bE_b^y=\sum_{i,j=1}^{t}E_{\tilde{b}}{}^{x_i}E_{\tilde{b}}{}^{x_jy}.$$

From (4), there exist $i$ and $j$, $1\leqq i,\ j\leqq t$, such that $x_jyx_i^{-1}\in T(\tilde{b})$, hence $y\in T(\tilde{b})C_G(D)$. Conversely if $w\in T(\tilde{b})$, then

$$E_b^w=\sum_{i=1}^{t}E_{\tilde{b}}{}^{x_iw}=\sum_{i=1}^{t}E_{\tilde{b}}{}^{w^{-1}x_iw}=E_b.$$

Therefore $T(b)=T(\tilde{b})C_G(D)$. This completes the proof of (ii).

Next we prove (iii) and (iv). $\tilde{b}_i^{C_G(\pi_i)}$ is defined and $\tilde{\phi}\to\tilde{\phi}^{C_G(\pi_i)}$ is a 1–1 correspondence between the irreducible Brauer characters in $\tilde{b}_i$ and those in $\tilde{b}_i^{C_G(\pi_i)}$. Let $\pi$ be a $p$-element of $G$. We assume that exactly $m$ elements $\pi_1, \pi_2, \cdots, \pi_m$ are conjugate to $\pi$ in $G$. We put $\pi_i^{a_i}=\pi$ ($a_i\in G$, $i=1, 2, \cdots, m$). Since $\tilde{\chi}=\sum_{i=1}^{n}\tilde{\chi}^{(\pi_i,\tilde{b}_i)}$, $\tilde{\chi}^G=\sum_{i=1}^{n}(\tilde{\chi}^{(\pi_i,\tilde{b}_i)})^G$. Here $\tilde{\chi}^{(\pi_i,\tilde{b}_i)}(\pi_i\rho)=\sum_{\tilde{\phi}}d(\tilde{\chi}, \pi_i, \tilde{\phi})\tilde{\phi}(\rho)$ for all $p$-regular elements $\rho$ of $C_H(\pi_i)$ with $\tilde{\phi}$ ranging over the irreducible Brauer characters in $\tilde{b}_i$ (see Brauer [2, §1]). So we can show

$$\tilde{\chi}^G(\pi\rho) = \sum_{i=1}^{m} \sum_{\tilde{\phi}} d(\tilde{\chi}, \pi_i, \tilde{\phi})\tilde{\phi}^{C_G(\pi_i)}(\rho^{a_i^{-1}})$$
$$= \sum_{i=1}^{m} \sum_{\tilde{\phi}} d(\tilde{\chi}, \pi_i, \tilde{\phi})(\tilde{\phi}^{C_G(\pi_i)})^{a_i}(\rho)$$

for all $p$-regular elements $\rho$ of $C_G(\pi)$. Hence a subsection associated with $B$ is conjugate to some subsection $(\pi_i, \tilde{b}_i^{C_G(\pi_i)})$ $(i=1, 2, \cdots, n)$. In partisular $k(B) \leqq \sum_{i=1}^{n} l(\tilde{b}_i)$. On the other hand, $k(B)=k(\tilde{B})=\sum_{i=1}^{n} l(\tilde{b}_i)$. Therefore, if $i \neq j$, then $(\pi_i, \tilde{b}_i^{C_G(\pi_i)})$ and $(\pi_j, \tilde{b}_j^{C_G(\pi_j)})$ are not conjugate and $d(\tilde{\chi}^G, \pi_i, \tilde{\phi}^{C_G(\pi_i)})=d(\tilde{\chi}, \pi_i, \tilde{\phi})$. This completes the proof of Theorem 1.

## 2. Proof of Theorem 2

We denote the set of $p$-regular elements of $G$ by $G_{p'}$.

If $\chi$ is a character of $G$ and $T$ is a matrix representation of $G$ affording $\chi$, then $x \rightarrow \det T(x)$ is a linear character of $G$. The linear character is denoted by $\det \chi$ and $o(\det \chi)$ means the order. The following lemma is a special case of Glauberman's theorem [5, Theorem 3].

**Lemma 2.** *Let $\pi$ be a $p$-element of $G$ and $N$ be a $p'$-subgroup of $G$ such that $N^\pi=N$. Suppose that $\zeta$ is an irreducible character of $N$ and $\hat{\zeta}$ is an extension of $\zeta$ to $N\langle\pi\rangle$ with $(o(\det \hat{\zeta}), p)=1$. Then there exist a unique sign $\varepsilon=\pm 1$ and a unique irreducible character $\beta$ of $C_N(\pi)$ with the property that*

$$\hat{\zeta}(\pi\rho) = \varepsilon\beta(\rho), \qquad \rho \in C_N(\pi).$$

**Lemma 3.** *Let $\zeta$ be an irreducible character of a normal $p'$-subgroup $N$ of $G$ and $\hat{\zeta}$ be an extension of $\zeta$ to $G$ such that $(o(\det \hat{\zeta}), p)=1$. For a $p$-element $\pi$ of $G$, there exist a sign $\varepsilon_\pi=\pm 1$ and an irreducible character $\theta_\pi$ of $C_G(\pi)$ with the property that $\theta_{\pi|C_N(\pi)}$ is irreducible and*

$$\hat{\zeta}(\pi\rho) = \varepsilon_\pi\theta_\pi(\rho) \qquad \rho \in (C_G(\pi))_{p'}.$$

*In particular $\theta_\pi$ is irreducible as a Brauer character.*

Proof. We fix a $p$-element $\pi$. By Lemma 2, there exist a unique sign $\varepsilon=\pm 1$ and a unique irreducible character $\beta$ of $C_N(\pi)$ with the property that $\hat{\zeta}(\pi\rho)=\varepsilon\beta(\rho)$ for all $\rho \in C_N(\pi)$. First of all we show that $\beta$ is etxendible to $C_G(\pi)$. Since $\hat{\zeta}(\pi\rho)=\hat{\zeta}(\pi\rho^c)$ for all $c \in C_G(\pi)$ and all $\rho \in C_N(\pi)$, $\beta$ is $C_G(\pi)$-invariant. Let $L$ be a subgroup of $C_G(\pi)$ such that $L/C_N(\pi)$ is a $p$-group. Then by Isaacs [7, (8.16)], $\beta$ is extendible to $L$. Let $M$ be a subgroup of $C_G(\pi)$ such that $M/C_N(\pi)$ is a $p'$-group. Then $(NM)^\pi=NM$. By Lemma 2, there exist a sign $\varepsilon_M=\pm 1$ and an irreducible character $\beta_M$ of $C_{NM}(\pi)$ with the property that

$$\hat{\zeta}(\pi\rho) = \varepsilon_M\beta_M(\rho), \qquad \rho \in C_{NM}(\pi).$$

Since $C_N(\pi)\subset C_{NM}(\pi)=M$, we have $\varepsilon_M=\varepsilon$ and $\beta_{M|C_N(\pi)}=\beta$ by the uhiqueness of $\varepsilon$ and $\beta$. Hence by [7, (11.31)], $\beta$ is extendible to $C_G(\pi)$.

Let $\theta_0$ be an extension of $\beta$. For a $p'$-subgroup $M$ of $C_G(\pi)$ with $M\supset C_N(\pi)$, there exists a unique linear character $\lambda_M$ of $M/C_N(\pi)$ which satisfies

$$\theta_{0|M}\lambda_M = \beta_M .$$

Furthermore for $p'$-subgroups $M$, $M'$ of $C_G(\pi)$ with $M$, $M'\supset C_N(\pi)$, if $M\supset M'$ then $\lambda_{M'}=(\lambda_M)_{|M'}$ and if $M'=M^x$ for some $x\in C_G(\pi)$ then $\lambda_{M'}=\lambda_M{}^x$. Here we define a class function $\lambda$ of $C_G(\pi)/C_N(\pi)$ as follows. For an element $c$ of $C_G(\pi)/C_N(\pi)$

$$\lambda(c) = \lambda_M(c_{p'}) ,$$

where $c_{p'}$ is the $p'$-part of $c$ and $M$ satisfies that $M/C_N(\pi)=\langle c_{p'}\rangle$. Then $\lambda$ is a generalized character of $C_G(\pi)/C_N(\pi)$ by Brauer's theorem on generalized characters. Since the inner product $(\lambda, \lambda)$ and $\lambda(1)$ are equal to 1, $\lambda$ is a linear character. If we put $\theta=\theta_0\lambda$, then $\hat{\xi}(\pi\rho)=\varepsilon\theta(\rho)$ for all $\rho\in(C_G(\pi))_{p'}$. This completes the proof.

Proof of Theorem 2. (i) is well known. Let $Q$ be a $p$-subgroup of $G$ and $\bar{b}$ be a $p$-block of $C_{\bar{G}}(\bar{Q})\bar{Q}$ associated with $\bar{B}$. We show $(\hat{\xi}\bar{b})^G=B$. Let $C$ be an arbitrary conjugacy class of $G$ and $\bar{\chi}$ and $\bar{\psi}$ be ordinary irreducible characters in $\bar{B}$ and $\bar{b}$, respectively. Since $\bar{b}^{\bar{G}}=\bar{B}$,

$$\bar{\chi}(\sum_{x\in C}\bar{x})/\bar{\chi}(1)\equiv\bar{\psi}(\sum_{x\in C\cap NC_G(Q)Q}\bar{x})/\bar{\psi}(1) \qquad (\mathrm{mod}\ P) .$$

If $x_0$ is an element of $C$, then

$$(\hat{\xi}\bar{\chi})(\sum_{x\in C}\bar{x})/\hat{\xi}(1)\bar{\chi}(1) = \hat{\xi}(x_0)\bar{\chi}(\sum_{x\in C}\bar{x})/\hat{\xi}(1)\bar{\chi}(1) ,$$
$$(\hat{\xi}\bar{\psi})(\sum_{x\in C\cap NC_G(Q)Q}\bar{x})/\hat{\xi}(1)\bar{\psi}(1) = \hat{\xi}(x_0)\bar{\psi}(\sum_{x\in C\cap NC_G(Q)Q}\bar{x})/\hat{\xi}(1)\bar{\psi}(1) .$$

Hence $(\hat{\xi}\bar{b})^G=\hat{\xi}\bar{B}=B$. Since a defect group of $\bar{b}$ is $\bar{D}$, $D$ is a defect group of $\hat{\xi}\bar{b}$. Let $b$ be a root of $\hat{\xi}\bar{b}$ in $C_G(D)D$. $b$ is a root of $B$ in $C_G(D)D$ and is determined uniquely, because $N_G(D)\cap NC_G(D)D=C_G(D)D$. If $x\in T(b)$, then

$$\hat{\xi}\bar{b} = (b^x)^{NC_G(D)D} = (b^{NC_G(D)D})^x = (\hat{\xi}\bar{b})^x = \hat{\xi}\bar{b}^{\bar{x}} .$$

Hence $\bar{b}=\bar{b}^{\bar{x}}$, so $\bar{x}\in T(\bar{b})$. If $y\in N_G(D)$ and $\bar{y}\in T(\bar{b})$, then

$$\hat{\xi}\bar{b} = (\hat{\xi}\bar{b})^y = (b^y)^{NC_G(D)D} .$$

By the uniqueness of a root $b$ of $\hat{\xi}\bar{b}$, $b=b^y$ and hence $y\in T(b)$. So we have $T(\bar{b})=\overline{T(b)}$.

Next we prove (iii) and (iv). By Lemma 3, there exist a sign $\varepsilon_\pi=\pm1$ and an ordinary irreducible character $\theta_\pi$ of $C_G(\pi)$ such that $\theta_\pi$ is irreducible as a

Bauer character and $\hat{\zeta}(\pi\rho)=\varepsilon_\pi\theta_\pi(\rho)$ for all $\rho\in(C_G(\pi))_{p'}$. $\bar{b}_i$ can be viewed as a $p$-block of $C_G(\pi)/C_N(\pi)$. We put $b_i=\theta_\pi\bar{b}_i$. Since

$$\bar{\chi}(\bar{\pi}\bar{\rho}) = \sum_{i=1}^{s}\sum_{\bar{\phi}} d(\bar{\chi}, \bar{\pi}, \bar{\phi})\bar{\phi}(\bar{\rho}) \qquad \rho\in(C_G(\pi))_{p'}.$$

we have

$$(5) \qquad (\hat{\zeta}\bar{\chi})(\pi\rho) = \sum_{i=1}^{s}\sum_{\bar{\phi}} \varepsilon_\pi d(\bar{\chi}, \bar{\pi}, \bar{\phi})(\theta_\pi\bar{\phi})(\rho) \qquad \rho\in(C_G(\pi))_{p'}.$$

Here $\bar{\phi}$ ranges over the irreducible Brauer characters in $\bar{b}_i$. By the second main theorem on $p$-blocks, $b_1, b_2, \cdots, b_s$ are the $p$-blocks of $C_G(\pi)$ associated with $B$. In particular we see that $b_i$ is a unique $p$-block of $C_G(\pi)$ such that $b_i^{NC_G(\pi)}=\hat{\zeta}\bar{b}_i$. From (5), $d(\hat{\zeta}\bar{\chi}, \pi, \theta_\pi\bar{\phi})=\varepsilon_\pi d(\bar{\chi}, \bar{\pi}, \bar{\phi})$. This completes the proof of Theorem 2.

We have the following as a corollary of Theorem 1 and 2.

**Corollary.** *Suppose that $G$ is a $p$-solvable group. Let $B$ be a $p$-block of $G$ with an abelian defect group $D$ and $b$ be a root of $B$ in $C_G(D)$. We assume that $T(b)/C_G(D)$ is cyclic and any element of $T(b)/C_G(D)-\{\bar{1}\}$ does not fix any element of $D-\{1\}$. Let $\pi_1, \pi_2, \cdots, \pi_t$ be a set of representatives for the $T(b)$-conjugacy classes of $D-\{1\}$ and $\Lambda$ be a set of representatives for the $T(b)$-conjugacy classes of non-trivial linear characters of $D$, where $t=(p^d-1)/e$, $e=|T(b):C_G(D)|$ and $p^d=|D|$. Then the following holds.*

(i) *$B$ contains exactly $e$ irreducible Brauer characters $\phi_1, \phi_2, \cdots, \phi_e$ and exactly $e+(p^d-1)/e$ ordinary irreducible characters $\chi_1, \chi_2, \cdots, \chi_e, \chi_\lambda$ ($\lambda\in\Lambda$).*

(ii) *For $i$, $1\leqq i\leqq e$, and $\lambda$, $\lambda\in\Lambda$,*

$$\chi_i = \phi_i \qquad \text{on } G_{p'},$$
$$\chi_\lambda = \phi_1+\cdots+\phi_e \qquad \text{on } G_{p'}.$$

(iii) *$(1, B)$, $(\pi_j, b^{C_G(\pi_j)})$ ($j=1, 2, \cdots, t$) form a set of representatives for the conjugacy classes of subsections associated with $B$. $b^{C_G(\pi_j)}$ contains a unique irreducible Brauer character $\phi^{(j)}$.*

(iv) *There exist $t$ signs $\varepsilon_j=\pm 1$ such that*

$$d(\chi_i, \pi_j, \phi^{(j)}) = \varepsilon_j,$$
$$d(\chi_\lambda, \pi_j, \phi^{(j)}) = (\varepsilon_j/|C_G(D)|)\sum_{x\in T(b)}\lambda^x(\pi_j).$$

*for $i$, $1\leqq i\leqq e$, $\lambda$, $\lambda\in\Lambda$ and $j$, $1\leqq j\leqq t$.*

Proof. If $\pi\in D-\{1\}$, then $C_G(\pi)\cap T(b)=C_G(D)$. Hence $b^{C_G(\pi)}$ contains a unique irreducible Brauer character by [2, (7A)] and Brauer [3, (6C)]. Hence (iii) follows from [3, (6C)]. By Fong's reductions (see Feit [4, Chapter X, Lemma 1.1]) and Theorems 1 and 2, we may assume that $D$ is a normal subgroup of $G$ and $T(b)=G$. Then $B$ is a unique $p$-block of $G$ which covers $b$. Let $\Lambda_0$ be the set of all linear characters of $D$. By [9, Theorem 3], $b$ contains a unique irreducible Brauer character $\phi$ and exactly $p^d$ ordinary irreducible characters $\tilde{\chi}_\mu$, $\mu \in \Lambda_0$, where if $\pi \in D$ and $\rho \in (C_G(D))_{p'}$ then $\tilde{\chi}_\mu(\pi\rho)=\mu(\pi)\phi(\rho)$. Since $\phi$ is $G$-invariant and $G/C_G(D)$ is cyclic, $B$ contains exactly $e$ irreducible Brauer characters $\phi_1, \phi_2, \cdots, \phi_e$. Since $\tilde{\chi}_1$ is also $G$-invariant, $B$ contains exactly $e$ ordinary irreducible characters $\chi_1, \chi_2, \cdots, \chi_e$ such that $\chi_{i|C_G(D)}=\tilde{\chi}_1$. We may assume $\chi_i=\phi_i$ on $G_{p'}$. For an element $\pi \in D-\{1\}$, $\chi_i(\pi\rho)=\phi(\rho)$ $(\rho \in (C_G(D))_{p'})$. Here we note $C_G(D)=C_G(\pi)$. By the assumption, if $\mu \neq 1$, then the stabilizer of $\tilde{\chi}_\mu$ in $G$ is equal to $C_G(D)$. Hence $\tilde{\chi}_\mu^G$ is irreducible and

$$\tilde{\chi}_\mu^G = \phi_1+\cdots+\phi_e \qquad \text{on } G_{p'},$$
$$\tilde{\chi}_\mu^G(\pi\rho) = (1/|C_G(D)|)\sum_{x\in G}\mu^x(\pi)\phi(\rho) \qquad \rho \in (C_G(D))_{p'}.$$

This completes the proof.

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