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ON RELATIVELY SEPARABLE SUBALGEBRAS

Dedicated to the memory of the Late Professor T. Honda

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We shall treat, in this paper, with a relatively separable subalgebra in a certain algebra which is introduced by Azumaya [2]. Let A be an algebra over a commutative ring R, and B an R-subalgebra of A. A can be regarded as a $B\otimes A^0$-module by the natural way, where A^0 denotes the opposite copy of A. According to Azumaya, B is called *a relatively separable subalgebra in A* if A is a left $B\otimes A^0$-projective module. It seems, however, that such a sublagebra should be called *left* relatively separable because by the symmetric manner $A\otimes B^0$-projectivity of A naturally gives another relative separability which we may call *right* relative separability. In his paper [2], Azumaya has shown that every (left) relatively separable subalgebra B in A has the property of, say, (left) relative semisimplicity, that is, every left A-module is (B, R)-projective in the sense of Hochschild's relative homological algebra. We shall study some relations between relative separability and relative semisimplicity, and also study some properties of two sided (i.e. left and right) relatively separable subalgebras. We refer Auslander-Goldman [1] and DeMeyer-Ingraham [4] for separable algebras, Hattori [8] for semisimple algebras, and Hochschild [9] for relative homological algebra.

In this paper, every ring is assumed to have the unit and every module to be unitary.

1. Let R be a commutative ring and A an R-algebra. An R-subalgebra B of A is called *left* (*right*) *relatively separable in A* if A is left $B\otimes A^0$- ($A\otimes B^0$-) projective. μ denotes the canonical mapping $B\otimes A^0\rightarrow A$; $\mu(b\otimes a^0)=ba$. μ is a $B\otimes A^0$-epimorphism. So B is left relatively separable in A if and only if the mapping μ has a $B\otimes A^0$-right inverse, and this is also equivalent to that A is $(B\otimes A^0, R)$-projective, for μ always has an R-right inverse; $a\mapsto 1\otimes a^0$. A itself is a left (or equivalently right) relatively separable subalgebra in A if and only if A is a separable R-algebra.

Proposition 1. *Let A and B be as above. Then the following conditions are equivalent.*

1) *B is a left relatively separable subalgebra in A*

2) *The mapping μ has a $B\otimes B^0$-right inverse*

3) *There exists a system of elements $b_i\in B$, $a_i\in A$ $(i=1, 2, \cdots, n)$ which satisfies the equations*; $\sum_{i=i}^{n} b_i a_i=1$ *and* $\sum_{i=1}^{n} bb_i\otimes a_i{}^0=\sum_{i=1}^{n} b_i\otimes(a_i b)^0$ *in* $B\otimes A^0$ *for any* $b\in B$.

Proof. 1)$\Rightarrow$2). Trivial.

2)$\Rightarrow$3). Let λ be a $B\otimes B^0$-right inverse of μ, and put $\lambda(1)=\sum_{i=1}^{n} b_i\otimes a_i{}^0$. Then the elements b_i, a_i $(i=1, 2, \cdots, n)$ satisfy the required equations.

3)$\Rightarrow$1). We define a mapping λ: $A\rightarrow B\otimes A^0$ by $\lambda(x)=\sum b_i\otimes(a_i x)^0$. λ is a $B\otimes A^0$-homomorphism since $\lambda(bxa)=\sum b_i\otimes(a_i bxa)^0=\sum bb_i\otimes(a_i xa)^0=(b\otimes a^0)\lambda(x)$. The equality $\mu\lambda(x)=x$ is easily seen.

The similar proposition holds for right relative separability. From this proposition we immediately get

Proposition 2. *If B itself is a separable algebra, B is a left and right relatively separable subalgebra in any algebra which contains B as a sublagebra.*

Proposition 3. *If B is left relatively separable in some A, B is also left relatively separable in any algebra which is bigger than A.*

In general let B, B' be subalgebras of A, A' respectively. We shall denote the canonical image of $B\otimes B'$ in $A\otimes A'$ simply by $B\otimes B'$ unless there is a cofusion.

Proposition 4. *If B is left relatively separable in A, $S\otimes B$ is a left relatively separable S-subalgebra in $S\otimes A$ for any commutative R-algebra S.*

Proof. Let b_i, a_i $(i=1, \cdots, n)$ be as in Proposition 1, then a system of elements $1\otimes b_i$, $1\otimes a_i$ gives left relative separability of $S\otimes B$ in $S\otimes A$.

We remark that there is no good relation between our relative separability and vanishing of Hochschild's relative cohomology. In fact, if B is left relatively separable in A and relative cohomological dimension of (A, B) is zero, A is $B\otimes A^0$-projective and $(A\otimes A^0, B\otimes A^0)$-projective. Then A must be $A\otimes A^0$-projective; A is a separable algebra.

We shall slightly modify the notion of *mean* of a mapping introduced by Hattori [8] which plays an important role also in this paper. Let B be a left relatively separable R-subalgebra in an R-algebra A, and let b_i, a_i $(i=1, 2, \cdots, n)$ be as in Proposition 1. For an R-homomorphism g from a left A-module M to a left B-module N, we define the *mean* $t(g)$ of g as $\sum_{i=1}^{n} b_i\circ g\circ a_i$, i.e. $[t(g)](m)=\sum b_i g(a_i m)$. $t(g)$ is a B-homomorphism since $[t(g)](bm)=\sum b_i g(a_i bm)=b\sum b_i g(a_i m)=b[t(g)](m)$. Similarly for an R-homomorphism g from a right

B-module M to a right A-module N, we get a right B-homomorphism $t'(g)\colon M\rightarrow N$ by defining $t'(g)=\sum a_i\circ g\circ b_i$, i.e. $[t'(g)](m)=\sum g(mb_i)a_i$.

Lemma 5. *Let B be a left relatively separable R-subalgebra in A. Then,*

1) *a B-epimorphism f from a left B-module M to a left A-module N is B-split if it is R-split.*

2) *a B-monomorphism f from a right A-module M to a right B-module N is B-split if it is R-split.*

Proof. Let g be an R-splitting homomorphism of f. The mean $t(g)$ of g can be defined, and $[f\circ t(g)](n)=f(\sum b_i g(a_i n))\sum b_i a_i n=n$. The second half is similar.

For a left A-module M, the canonical B-epimorphism $B\otimes M\rightarrow M$ is always R-split and for a right A-module M, the canonical B-monomorphism $M\rightarrow \mathrm{Hom}_R(B, M)$ is also R-split. So we get

Theorem 6. (Azumaya) *For a left relatively separable R-subalgebra B in A, any left A-module is (B, R)-projective and any right A-module is (B, R)-injective.*

From the theorem above we can naturally get the notion of relative semisimplicity of a sublagebra. We call a subalgebra B of A *a left relatively semisimple subalgebra* if and only if any left A-module is (B, R)-projective. A left relatively separable subalgebra is left relatively semisimple by Theorem 6. A left semisimple algebra B is left relatively semisimple in any algebra which contains B as a subalgebra. A left relatively semisimple subalgebra B in A is also left relatively semisimple in any algebra bigger than A. A itself is left relatively semisimple in A if and only if A is a left semisimple algebra.

Lemma 7. *Let B be an R-subalgebra of an R-algebra A.*

1) *If A is left (B, R)-projective, any left (A, R)-projective module is left (B, R)-projective*

2) *If A is right (B, R)-injective and B is finitely generaetd projective as an R-module, any left (A, R)-injective module is (B, R)-projective.*

Proof. 1) Let M be an (A, R)-projective module. The canonical A-epimorphism $A\underset{R}{\otimes}M\rightarrow M$ is A-split, and the canonical B-epimorphism $B\underset{R}{\otimes}A\rightarrow B$ is B-split. So M is a B-direct summand of a left B-module $B\underset{R}{\otimes}A\underset{R}{\otimes}M$ which is (B, R)-projective.

2) Let M be a left (A, R)-injective module. The canonical A-monomorphism $M\rightarrow \mathrm{Hom}_R(A, M)$ is A-split. While A is a right B-direct summand of a right B-module $\mathrm{Hom}_R(B, A)$, and the left B-module structure of $\mathrm{Hom}_R(A, M)$ is defined by the right multiplication of B on A, so M is a left B-

direct summand of $\mathrm{Hom}_R(\mathrm{Hom}_R(B, A), M)$. By our assumption on B, a natural mapping

$$\sigma\colon B\underset{R}{\otimes}\mathrm{Hom}_R(A, M)\to\mathrm{Hom}_R(\mathrm{Hom}_R(B, A), M)$$

defined by $[\sigma(b\otimes f)](g)=fg(b)$ for $b\in B, f\in\mathrm{Hom}_R(A, M)$ and $g\in\mathrm{Hom}_R(B, A)$ is an R-isomorphism. σ is actually a B-isomorphism since $[\sigma(b'b\otimes f)](g)=fg(b'b)=(f\circ gb')(b)=[\sigma(b\underset{R}{\otimes}f)](gb')=[b'\sigma(b\otimes f)](g)$ for any $b'\in B$. $B\otimes\mathrm{Hom}_R(A, M)$ is a left (B, R)-projective, so we have proved our lemma.

Proposition 8. *Let A be a left semisimple algebra over a commutative ring R. For any subalgebra B of A, the following conditions* 1), 2) *are equivalent. Furthermore if A is a reflexive R-module and B is a finitely generated projective R-module, the conditions* 1), 2) *are equivalent to* 3).

1) *B is left relatively semisimple in A.*
2) *A is left (B, R)-projective.*
3) *A is right (B, R)-injective.*

Proof. 1)$\Rightarrow$2) is trivial and 2)$\Rightarrow$1), 3)$\Rightarrow$1) follows from the previous lemma, so we may only prove the implication 1)$\Rightarrow$3). $A^*=\mathrm{Hom}_R(A, R)$ has a natural left A-module structure and is (B, R)-projective. So, A^* is a left B-direct summand of $B\otimes A^*$, and the double dual $A^{**}=A$ is a right B-direct summand of $(B\otimes A^*)^*=\mathrm{Hom}_R(B\otimes A^*, R)$ which is B-isomorphic to $\mathrm{Hom}_R(B, A^{**})=\mathrm{Hom}_R(B, A)$. The last module is right (B, R)-injective.

In his paper [2], Azumaya has treated with a left relatively separable subalgebra contained in the center. Here we shall show that under a weak condition a left (or right) relatively separable subalgebra is (absolutely) separable if it is contained in the center.

Proposition 9. *Let A be an R-algebra which is finitely generated projective as an R-module, and C be the center of A. A subalgebra S contained in C is separable if and only if S is left* (*or right*) *relatively separable in A, and is semisimple if and only if S is left* (*or right*) *relatively semisimple in A.*

Proof. If S is left relatively separable, A is (S, R)-projective. Since A is R-projective, A is a finitely generated projective S-domule. So S is a S-direct summand of A. We denote by π an S-projection from A to S. Since S is left relatively separable in A, there exists a system of elements $s_i\in S$, $a_i\in A$ $(i=1, 2, \cdots, n)$ which satisfies $\sum s_ia_i=1$ and $\sum ss_i\otimes a_i{}^0=\sum s_i\otimes(a_is)^0$ for any $s\in S$. We put $\pi(a_i)=t_i\in S$. Then a system of elements s_i, $t_i\in S$ satisfies the separability conditions. For $\sum s_it_i=\sum s_i\pi(a_i)=\pi(\sum s_ia_i)=\pi(1)=1$, and $\sum ss_i\otimes t_i{}^0=\sum ss_i\otimes\pi(a_i)^0=\sum s_i\otimes\pi(a_is)^0=\sum s_i\otimes\pi(sa_i)^0=\sum s_i\otimes(s\pi(a_i))^0=$

$\sum s_i \otimes (\pi(a_i)s)^0 = \sum s_i \otimes (t_i s)^0$. Secondly we assume that S is left relatively semisimple in A. For any S-module M, $A \underset{S}{\otimes} M$ is (S, R)-projective. A can be decomposed to $A = S \oplus A'$ as an S-module, and $M = S \underset{S}{\otimes} M$ is an S-direct summand of $(S \oplus A') \underset{S}{\otimes} M = A \underset{R}{\otimes} M$, since S is contained in the center C. So M is (S, R)-projective as is required.

As an immediate corollary to Proposition 9, we get an assertion that *the center C of a finite generated projective left semisimple algebra A is a semisimple algebra if A is C-projective.* But there exists a semisimple algebra which is not projective over the center. ([6][7]).

According to Hattori [8], the (absolute) separability of an algebra A is characterized as the semisimplicity of the enveloping algebra $A \underset{R}{\otimes} A^0$ of A. Here we shall see a relation between relative separability and relative semisimplicity.

Lemma 10. *Let B be a left relatively separable subalgebar in A, M be an $A \otimes A^0$-module and N be a $B \otimes A^0$-module. A $B \otimes A^0$-epimorphism $f : N \to M$ is $B \otimes A^0$-split whenever f is A-split.*

Proof. Let g be an A-right inverse of f, and let $b_i \in B$, $a_i \in A$ be as in Proposition 1. We define a mapping g' (the mean of g); $M \to N$ by $g'(m) = \sum b_i g(a_i m)$ for $m \in M$. For any $b \in B$ and $a \in A$, $g'(bma) = \sum b_i g(a_i bma) = \sum b_i g(a_i bm) a = bg(m) a$, so g' is a $B \otimes A^0$-homomorphism. g' is a $B \otimes A^0$-right inverse of f as is easily seen.

Proposition 11. *Let B be a subalgebra of A. B is left relatively separable in A if and only if every $A \otimes A^0$-module which is right (A, R)-projective is $(B \otimes A^0, R)$-projective.*

Proof. A is $(B \otimes A^0, R)$-projective since A itself is an $A \otimes A^0$-module which is right (A, R)-projective. Conversely, let M be an $A \otimes A^0$-module which is (A^0, R)-projective. The canonical $B \otimes A^0$-epimorphism $B \otimes A^0 \otimes M \to M$ is A^0-split since it is always R-split and M is (A^0, R)-projective. Our assertion follows directly from Lemma 10.

Corollary 12. *If A is a right semisimple algebra, a subalgebra B is left relatively separable in A if and only if $B \otimes A^0$ is left relatively semisimple in $A \otimes A^0$.*

Now we assume that A is a separable R-algebra. Then A is a left and right semisimple algebra.

Theorem 13. *A subalgebra B in a separable algebra A is left relatively separable if ands only if it is left relatively semisimple.*

Proof. Let B be a left relatively semisimple sublagebra in A, and M be an $A\otimes A^0$-module. The canonical mapping $f_1: B\otimes M\to M$ is B-split. And by right semisimplicity of a separable algebra A, the canonical mapping $f_2: B\otimes A^0\otimes M\to B\otimes M$, $f_2(b\otimes a^0\otimes m)=b\otimes ma$ is $B\otimes A^0$-split so that $B\otimes M$ is $(B\otimes A^0, R)$-projective. For a B-right inverse g of f_1, we set g' to be the mean of g, $g'(m)=\sum g(mu_i)v_i$ where $\sum u_i\otimes v_i{}^0$ is a separability idempotent of A. Then g' is a $B\otimes A^0$-homomorphism, for $g'(bma)=\sum g(bmau_i)v_i=b\sum g(mu_i)v_ia=bg'(m)a$, and $f_1g'=$the identity of M, that is, M is a $(B\otimes A^0, R)$-projective module. Therefore $B\otimes A^0$ is left relatively semisimple in $A\otimes A^0$ so that B is left relatively separable in A. The converse assertion has been already proved.

2. A left and right relatively separable subalgebra is called simply *a relatively separable subalgebra*. We now study such a subalgebra. For a relatively separable subalgebra B in A, every one sided A-module is (B, R)-projective and (B, R)-injective.

Theorem 14. *Let B be an R-subalgebra of A. Then the following conditions are equivalent.*

1) *B is a relatively separable subalgebra in A.*
2) *$B\otimes B^0$ is a left relatively separable subalgebra in $A\otimes A^0$.*
3) *$B\otimes B^0$ is a left relatively semisimple subalgebra in $A\otimes A^0$.*
4) *A is a $(B\otimes B^0, R)$-projective module.*

Proof. We will prove more generally that for two pairs of algebras and subalgebras $A\supset B$, $C\supset D$, $B\otimes D$ is left relatively separable in $A\otimes C$ if B and D are left relatively separable in A and C.

Since B is left relatively separable in A, there exists a system of elements $b_i\in B$, $a_i\in A$ $(i=1, \cdots, n)$ which satisfies $\sum b_ia_i=1$, $\sum bb_i\otimes a_i{}^0=\sum b_i\otimes(a_ib)^0$ for any $b\in B$. We can also find a system $d_j\in D$, $c_j\in C$ $(j=1, \cdots, m)$ with similar conditions, so we get a new system $b_i\otimes d_j$, $a_i\otimes c_j$ $(i=1, \cdots, n; j=1, \cdots, m)$. This system gives the left relative separability for $B\otimes D$ in $A\otimes C$. In fact, $\sum_{i,j}(b_i\otimes d_j)(a_i\otimes c_j)=\sum_{i,j}b_ia_i\otimes d_jc_j=\sum_i b_ia_i\otimes\sum_j d_jc_j=1\otimes 1$, and for any $b\in B$, $d\in D$, $\sum_{i,j}(b\otimes d)(b_i\otimes d_j)\otimes(a_i\otimes c_j)^0=\sum_{i,j}(bb_i\otimes dd_j)\otimes(a_i\otimes c_j)^0=\sum_{i,j}(b_i\otimes d_j)\otimes(a_ib\otimes c_jd)^0=\sum(b_i\otimes d_j)\otimes((a_i\otimes c_j)(b\otimes d))^0$. So we have proved 1)$\Rightarrow$2). The implications 2)$\Rightarrow$3) and 3)$\Rightarrow$4) are trivial. We shall prove 4)$\Rightarrow$1). The canonical mapping $\mu: B\otimes A^0\to A$ is a $B\otimes B^0$-epimorphism which is R-split. Hence μ is $B\otimes B^0$-split by $(B\otimes B^0, R)$-projectivity of A. The left relative separability now follows from Proposition 1. Similarly, using the canonical mapping $A\otimes B^0\to A$, we can show that B is right relatively separable in A.

By Proposition 8 and Theorem 13, we can add some more criteria for the relative separability in the case that A is a separable R-algebra.

By Proposition 8 and Theorem 13. we can add some more criteria for the reiative separability in the case that A is a separable R-algebra.

Proposition 15. *For a subalgebra B in a separable R-algebra A, the following conditions are equivalent*

1) *B is relatively separable in A,*

2) *A is left and right (B, R)-projective.*

Furthermore if A is reflexive as an R-module and B is finitely generated R-projective, the conditions 1), 2) *are equivalent to*

3) *A is left and right (B, R)-injective.*

Corollary 16. *Let A be an R-separable algebra which is a finitely generatde projective R-module and B a subalgebra of A which is also finitely generated R-projective. If A is left (B, R)-projective and B is a quasi-Frobenius algebra, B is a relatively separable subalgebra in A.*

Proof. We may only prove that A is right (B, R)-projective. Since A is R-projective and left (B, R)-projective, A is left B-projective so that A is a B-direct summand of a direct sum of a finite number of isomorphic copies of B which is left (B, R)-injective. A is right (A, R)-injective because A is a separable algebra, so from Lemma 7, A is right (B, R)-projective as is required.

Here, we shall consider a necessary condition for a subalgebra B of A to be relatively separable.

Proposition 17. *Let A be an R-projective R-algebra and B an R-subalgebra which is finitely generated R-projective. If A is finitely generated as a left B-module and B is relatively separable in A, B is necessarily quasi-Frobenius.*

Proof. We can localize this problem, for B is a quasi-Frobenius algebra over R if and only if $B_{\mathfrak{m}}$ is a quasi-Frobenius $R_{\mathfrak{m}}$-algebra for any maximal ideal $\mathfrak{m}$ of R (see Endo [5]). We assume R to be a local ring with the unique maximal ideal $\mathfrak{m}$. B is a semi-local ring so that it is noetherian modulo the radical. By virtue of Theorem 3.1 of H. Bass [3]*), a free left A-module F of countably infinite rank is B-free, since F is uniformly big (in the sence of Bass) projective B-module. By the relative separability of B in A, F is a left (B, R)-injective module. So B, a B-direct summand of F, is left (B, R)-injective; i.e. B is a quasi-Frobenius R-algebra.

Conversely, any quasi-Frobenius R-algebra B can be imbedded into an R-projective R-algebra as a relatively separable subalgebra if B is faithful as an R-module. For, if we put $A=\mathrm{Hom}_R(B, B)$, A contains B as the left homotheties

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*) If B is a commutative sualgebra, we do not need such a "big" heorem because B is a B-direct summand of A in this case and A is (B, R)-injective.

and A is left B-projective as is easily seen. So B is relatively separable by Corollary 16.

In the case that S is a maximal commutative subalgebra in an R-Azumaya alegbra A, if S is left relatively separable in A, $S\otimes A^0$ is isomorphic to an S-Azumaya algebra of left S-endomorphisms of A, $\mathrm{Hom}_S^l(A, A)$, by the canonical mapping so that S is a splitting ring of A. Hence the (two sided) relative separability of S in A means $S\otimes A^0\cong\mathrm{Hom}_S^l(A, A)$ and $A\otimes S^0\cong\mathrm{Hom}_S^r(A, A)$. Such a "good" splitting ring has been studied in Yokogawa [10].

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