Hotta, R. and Murakami, S.
Osaka J. Math.
12 (1975), 555-564

ON A VANISHING THEOREM FOR CERTAIN COHOMOLOGY GROUPS

RYOSHI HOTTA AND SHINGO MURAKAMI

(Received September 17, 1974)

Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup of G. We assume that the quotient manifold $X=G/K$ carries a G-invariant complex structure, so that X is holomorphically isomorphic to a symmetric bounded domain in $\boldsymbol{C}^N$. Let Γ be a discrete subgroup of G acting on X freely and such that the quotient $M=\Gamma\backslash X$ is compact. The quotient $\Gamma\backslash G$ is then also compact. Suppose now an irreducible representation τ of K be given in a finite-dimensional complex vector space V. We know that τ defines an automorphic factor J_τ on X, called the canonical automorphic factor of type τ, and this defines in turn a holomorphic vector bundle $E(J_\tau)$ over the complex manifold $M=\Gamma\backslash X$. The vector bundle $E(J_\tau)$ is in fact differentiably equivalent to the vector bundle over M which is associated to the principal bundle $\Gamma\backslash G$ over M with group K by the representation τ of K in V. We shall denote by $\boldsymbol{E}(J_\tau)$ the sheaf of germs of holomorphic sections of the vector bundle $E(J_\tau)$, and by $H^q(M, \boldsymbol{E}(J_\tau))$ $(q=0, 1, \cdots)$ the q-th cohomology group of M with coefficients in the sheaf $\boldsymbol{E}(J_\tau)$.

In a series of papers [6], [7], [8] (cf. also [9]), Y. Matsushima and one of the present authors have discussed the cohomology groups $H^q(M, \boldsymbol{E}(J_\tau))$ and in particular the vanishing of these cohomology groups. The aim of this note is to prove anew a vanishing theorem for these cohomology groups which generalizes one of the main results in [7]. In [7] (and also in [8]), the result has been obtained after proving the following two kind of assertions. (1) Vanishing theorems for the cohomology groups of M with coefficients in certain locally constant sheaves, and (2) Isomorphisms between cohomology groups of this type and the groups $H^q(M, \boldsymbol{E}(J_\tau))$. In this note we will apply a formula proved in [8] which expresses the dimension of the space of automorphic forms in terms of the unitary representation of G in $L^2(\Gamma\backslash G)$. As this formula has nothing to do with the earlier results as (1), (2), we get in this way a direct proof to a theorem in [7]. We note that N. Wallach and one of the present authors [3] have recently applied a similar kind of formula proved by Matsushima [5], thus giving a completely new proof to a theorem of Matsushima concerning the first Betti number of the space $\Gamma\backslash X$. The method used in this note generalizes that of [3] and depends on an argument used by R. Parthasarathy [11] who treated "L^2-cohomologies" of X. We remark also that a different kind of vanishing theorem for the cohomology groups is found in [2].

We need also a formula on a laplacian operator, which is essentially the same as the one given by K. Okamoto and H. Ozeki [10]. The proof of this formula given here may be considered as a simplification of the method developped by them (cf. [1] for another proof).

1. Preliminaries on Lie algebras. We retain the notation in the introduction. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ the subalgebra corresponding to the subgroup K. Since G/K carries a G-invariant complex structure, $\mathfrak{k}$ contains a Cartan subalgera $\mathfrak{h}$ of $\mathfrak{g}$. We denote by $\mathfrak{g}^C$ the complexification of $\mathfrak{g}$ and by $\mathfrak{h}^C$ and $\mathfrak{k}^C$ the subspaces of $\mathfrak{g}^C$ spanned by $\mathfrak{h}$ and $\mathfrak{k}$ respectively.

Let Δ be the root system of $\mathfrak{g}^C$ relative to $\mathfrak{h}^C$ and

$$\mathfrak{g}^C = \mathfrak{h}^C + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

be the root space decomposition. Then

$$\mathfrak{k}^C = \mathfrak{h}^C + \sum_{\alpha\in\Delta_k} \mathfrak{g}_\alpha$$

for a subset $\Delta_k \subset \Delta$. Moreover, by our assumption on G/K, there exist abelian subalgebras $\mathfrak{n}^+$ and $\mathfrak{n}^-$ of $\mathfrak{g}^C$ such that

$$\begin{aligned} &\mathfrak{g}^C = \mathfrak{n}^+ \oplus \mathfrak{n}^- \oplus \mathfrak{k}^C\,, \\ &[\mathfrak{k}^C, \mathfrak{n}^\pm] \subset \mathfrak{n}^\pm, \ [\mathfrak{n}^+, \mathfrak{n}^-] \subset \mathfrak{k}^C\,, \\ &\overline{\mathfrak{n}^+} = \mathfrak{n}^-\,, \end{aligned}$$

where — denotes the conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$. It follows in particular that

$$\mathfrak{n}^+ = \sum_{\alpha\in\Psi} \mathfrak{g}_\alpha\,, \quad \mathfrak{n}^- = \sum_{\alpha\in\Psi} \mathfrak{g}_{-\alpha}$$

for a subset $\Psi \subset \Delta$. For each root $\alpha \in \Psi$ we can choose a vector $X_\alpha \in \mathfrak{g}_\alpha$ in such a way that $\bar{X}_\alpha = X_{-\alpha}$ and $\varphi(X_\alpha, X_{-\alpha}) = 1$, φ being the Killing form of $\mathfrak{g}^C$.

Let $\mathfrak{h}_0$ be the real part of $\mathfrak{h}^C$. All roots of $\mathfrak{g}^C$ and more generally any weight of a finite-dimensional irreducible representation of the reductive Lie algebra $\mathfrak{k}^C$ relative to the Cartan subalgebra $\mathfrak{h}^C$ are real-valued on $\mathfrak{h}_0$ and so are considered as elements of the dual space $\mathfrak{h}_0^*$ of $\mathfrak{h}_0$. We know that there exists a linear ordering in $\mathfrak{h}_0^*$ such that the roots in Ψ are all positive. Choosing such an ordering once and for all, let Δ_+ be the set of all positive roots. Put $\Theta = \Delta_+ \cap \Delta_k$ and

$$\delta = \frac{1}{2}\langle\Delta_+\rangle, \quad \delta_k = \frac{1}{2}\langle\Theta\rangle, \quad \delta_n = \frac{1}{2}\langle\Psi\rangle$$

where $\langle Q \rangle$ denotes the sum of roots belonging to Q for any subset Q of Δ. Then $\Delta_+ = \Theta \cup \Psi$ and $\delta = \delta_k + \delta_n$.

The Killing form φ defines a positive definite inner product on $\mathfrak{h}_0$ and this induces in turn a linear isomorphism $\mathfrak{h}_0^* \cong \mathfrak{h}_0$ which assigns to $\lambda \in \mathfrak{h}_0^*$ the element $H_\lambda \in \mathfrak{h}_0$ such that $\lambda(H) = \varphi(H_\lambda, H)$ for all $H \in \mathfrak{h}_0$. Then

$$[X_\alpha, X_{-\alpha}] = H_\alpha$$

for any root $\alpha \in \Psi$. We define an inner product in $\mathfrak{h}_0^*$ by putting

$$\langle \lambda, \mu \rangle = \varphi(H_\lambda, H_\mu)$$

for $\lambda, \mu \in \mathfrak{h}_0^*$.

2. The cohomology groups $H^{0,q}(\Gamma, X, J_\tau)$. We recall some results obtained in [6], [7]. Let τ be a representation of the group K on a finite-dimensional complex vector space V, and J_τ the canonical automorphic factor of type τ on the space $X = G/K$ (Cf. [6], [9]). We denote by $A^{0,q}(\Gamma, X, J_\tau)$ the vector space of V-valued C^∞-differential forms η of type $(0, q)$ on X such that

$$(\eta \circ L_\gamma)_x = J_\tau(\gamma, x)\eta_x$$

for all $\gamma \in \Gamma$ and $x \in X$, where L_γ denotes the transformation of X defined by γ. Then we get a complex $\sum_{q \geq 0} A^{0,q}(\Gamma, X, J_\tau)$ with coboundary operator d''. The cohomology groups of this complex, which were denoted by $H^{0,q}_{d''}(\Gamma, X, J_\tau)$ in [6] [7], will be here denoted by $H^{0,q}(\Gamma, X, J_\tau)$ $(q = 0, 1, \cdots)$. The group $H^{0,q}(\Gamma, X, J_\tau)$ is isomorphic, via the Dolbeault's isomorphism, to the cohomology group $H^q(M, \boldsymbol{E}(J_\tau))$ defined in the introduction. Now in the space $A^{0,q}(\Gamma, X, J_\tau)$ we can introduce a "laplacian" operator $\square$ in a canonical way and we know that each cohomology class of $H^{0,q}(\Gamma, X, J_\tau)$ is represented by a unique harmonic form, i.e. a form η such that $\square\eta = 0$. The group $H^{0,q}(\Gamma, X, J_\tau)$ is thus isomorphic to the group $\mathcal{H}^{0,q}(\Gamma, X, J_\tau)$ formed by the harmonic forms in $A^{0,q}(\Gamma, X, J_\tau)$.

The space $A^{0,q}(\Gamma, X, J_\tau)$ is canonically isomorphic to the space of V-valued q-forms η on the manifold $\Gamma\backslash G$ satisfying the following conditions. An element $X \in \mathfrak{g}^C$ being a left-invariant complex vector field on G, X projects to a vector field on $\Gamma\backslash G$ which we write also by X. Let $i(X)$ be the operator of taking interior product by X for differential forms on $\Gamma\backslash G$. Then the conditions to be satisfied by the forms η are the followings.

$$\text{(2.1)} \qquad \begin{cases} \eta \circ R_k = \tau^{-1}(k)\eta & \text{for} \quad k \in K\,, \\ i(X)\eta = 0 & \text{for} \quad X \in \mathfrak{n}^+\,, \\ i(Y)\eta = 0 & \text{for} \quad Y \in \mathfrak{k}\,, \end{cases}$$

where R_k is the transformation of $\Gamma\backslash G$ defined by an element $k\in K$. Now, there exists a bijection between V-valued q-forms satisfying (2.1) and $V\otimes\Lambda^q\mathfrak{n}^+$-valued C^∞-functions f on $\Gamma\backslash G$ which verify

$$f(xk)=(\tau\otimes\mathrm{ad}_+^q)(k^{-1})f(x) \tag{2.2}$$

for $x\in X$ and $k\in K$, where ad_+^q is the representation of K on $\Lambda^q\mathfrak{n}^+$ induced from the adjoint action of K on $\mathfrak{n}^+$. To be more precise, put $\Psi=\{\alpha_1,\cdots,\alpha_N\}$ and write X_i, $X_{\bar{i}}$ for X_{α_i}, $\bar{X}_{\alpha_i}(1\leqq i\leqq N)$ respectively. Then the function corresponding to a form η is given by

$$f(x)=\sum\eta_{\bar{j}_1\cdots\bar{j}_q}(x)X_{j_1}\wedge\cdots\wedge X_{j_q},$$

where

$$\eta_{\bar{j}_1\cdots\bar{j}_q}=\eta(X_{\bar{j}_1},\cdots,X_{\bar{j}_q})$$

and $j_1,\cdots,j_q$ run over integers such that $1\leqq j_1<\cdots<j_q\leqq N$. We shall denote this function also by η and identify the space $A^{0,q}(\Gamma,X,J_\tau)$ with the space of $V\otimes\Lambda^q\mathfrak{n}^+$-valued C^∞-functions satisfying the condition (2.2). If we denote by $C^\infty(\Gamma\backslash G)$ the space of all complex-valued C^∞-functions on $\Gamma\backslash G$, the space of all $V\otimes\Lambda^q\mathfrak{n}^+$-valued C^∞-functions on $\Gamma\backslash G$ may be identified with the tensor product space $C^\infty(\Gamma\backslash G)\otimes V\otimes\Lambda^q\mathfrak{n}^+$. The group K acts on this space by $R_k\otimes\tau(k)\otimes\mathrm{ad}_+^q(k)$ $(k\in K)$, and then $A^{0,q}(\Gamma,X,J_\tau)$ coincides with the subspace of $C^\infty(\Gamma\backslash G)\otimes V\otimes\Lambda^q\mathfrak{n}^+$ consisting of all K-invariant elements.

Each vector field on $\Gamma\backslash G$ acting on $C^\infty(\Gamma\backslash G)$ in a natural way, we get a natural representation l of the Lie algebra $\mathfrak{g}^C$ in $C^\infty(\Gamma\backslash G)$. The restriction of l to $\mathfrak{k}$ is denoted by l_k, and the representations of $\mathfrak{k}$ induced from the representations τ, ad_+^q of the group K will be denoted by the same letters. The action of the group K on $C^\infty(\Gamma\backslash G)\otimes V\otimes\Lambda^q\mathfrak{n}^+$ defines as its differential the tensor product representation $l_k\otimes\tau\otimes\mathrm{ad}_+^q$ of the representations l_k, τ, ad_+^q of the Lie algebra $\mathfrak{k}$. It follows that an element $\eta\in C^\infty(\Gamma\backslash G)\otimes V\otimes\Lambda^q\mathfrak{n}^+$ belongs to the subspace $A^{0,q}(\Gamma,X,J_\tau)$, if and only if

$$(l_k\otimes\tau\otimes\mathrm{ad}_+^q)(Y)\eta=0 \tag{2.3}$$

holds for all $Y\in\mathfrak{k}$.

Let $\{Y_1,\cdots,Y_m\}$ be a basis of $\mathfrak{k}$ such that $\varphi(Y_a,Y_b)=-\delta_{ab}$ $(1\leqq a,b\leqq m)$. Then the laplacian operator $\square$ in $A^{0,q}(\Gamma,X,J_\tau)$ is induced from the operator, denoted also by $\square$, in $C^\infty(\Gamma\backslash G)\otimes V\otimes\Lambda^q\mathfrak{n}^+$ defined as follows.

$$\square=-\sum_{i=1}^{N}l(X_i)l(X_{\bar{i}})\otimes\mathbf{1}\otimes\mathbf{1}-\sum_{a=1}^{m}l(Y_a)\otimes\mathbf{1}\otimes\mathrm{ad}_+^q(Y_a), \tag{2.4}$$

where $\mathbf{1}$ denotes the identity operator in each space (See [6], [7], [9]).

3. An expression of the laplacian operator. Let C be the Casimir operator of the Lie algebra $\mathfrak{g}^C$. This is an element in the enveloping algebra $U(\mathfrak{g}^C)$ of $\mathfrak{g}^C$ and, according to our choice of the basis $\{X_1, \cdots, X_N, X_{\bar{1}}, \cdots, X_{\bar{N}}, Y_1, \cdots, Y_m\}$ of $\mathfrak{g}^C$, it is written as

$$C = -\sum_{a=1}^{m} Y_a^2 + \sum_{i=1}^{N} (X_i X_{\bar{i}} + X_{\bar{i}} X_i) .$$

The operator C acts on $C^\infty(\Gamma\backslash G) \otimes V \otimes \Lambda^q \mathfrak{n}^+$ via the canonical action of $U(\mathfrak{g}^C)$ on the first factor $C^\infty(\Gamma\backslash G)$. Analogously we define an element $C_k \in U(\mathfrak{g}^C)$ as follows.

$$C_k = -\sum_{a=1}^{m} Y_a^2 ,$$

and put

$$\tau(C_k) = -\sum_{a=1}^{m} \tau(Y_a)^2 .$$

From now on we assume that the representation τ of K is irreducible. Then τ induces an irreducible representation, denoted also by τ, of the reductive Lie algebra $\mathfrak{k}^C$. Let Λ (resp. Λ') be the highest (resp. lowest) weight of τ with respect to the ordering in $\mathfrak{h}_0^*$ chosen in §1. Then we see easily

$$\tau(C_k) = \langle \Lambda, \Lambda + 2\delta_k \rangle \mathbf{1} \ ; \quad \tau(H_\lambda) = \langle \Lambda, \lambda \rangle \mathbf{1} \tag{3.1}$$

for H_λ belonging to the center of $\mathfrak{k}^C$.

The formula given in the following lemma is essentially the same as the one of Okamoto-Ozeki [10] established for "L^2-cohomologies".

Lemma 1. *In the subspace $A^{0,q}(\Gamma, X, J_\tau) \subset C^\infty(\Gamma\backslash G) \otimes V \otimes \Lambda^q \mathfrak{n}^+$, we have*

$$\square = \frac{1}{2} \{-C + \langle \Lambda, \Lambda + 2\delta \rangle \mathbf{1}\} . \tag{3.2}$$

Proof. For simplicity, we write X for $l(X)$ $(X \in \mathfrak{g}^C)$. In the following summations a runs over $1, \cdots, m$ and i over $1, \cdots, N$. We shall use the following formula proved in [7. Lemma 4.1].

$$\sum_a \mathrm{ad}_+^q(Y_a)^2 = -\sum_i \mathrm{ad}_+^q([X_i, X_{\bar{i}}]) . \tag{3.3}$$

Now, in the subspace $A^{0,q}(\Gamma, X, J_\tau)$, we have

$$\begin{aligned} &2\sum_a (Y_a \otimes \mathbf{1} \otimes \mathrm{ad}_+^q(Y_a)) \\ &= \sum_a (Y_a \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \mathrm{ad}_+^q(Y_a))^2 - \sum_a Y_a^2 \otimes \mathbf{1} \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{1} \otimes \sum_a \mathrm{ad}_+^q(Y_a)^2 \\ &= \sum_a (-\mathbf{1} \otimes \tau(Y_a) \otimes \mathbf{1})^2 - \sum_a Y_a^2 \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \sum_i \mathrm{ad}_+^q([X_i, X_{\bar{i}}]) \\ &\qquad\qquad\qquad\qquad [\text{by (2.3) and (3.3)}] \end{aligned}$$

$$= \mathbf{1}\otimes\sum_a \tau(Y_a)^2\otimes\mathbf{1}-\sum_a Y_a^2\otimes\mathbf{1}\otimes\mathbf{1}$$
$$-\sum_i [X_i, X_{\bar{i}}]\otimes\mathbf{1}\otimes\mathbf{1}-\mathbf{1}\otimes\sum_i \tau([X_i, X_{\bar{i}}])\otimes\mathbf{1} \qquad \text{[by (2.3)]}$$

Note that $\sum_i [X_i, X_{\bar{i}}]=\sum_{\alpha\in\Psi} H_\alpha=H_{2\delta_n}$, which belongs to the center of $\mathfrak{k}^C$. From (2.4) we get therefore

$$\begin{aligned}\square &= -\sum_i X_i X_{\bar{i}}\otimes\mathbf{1}\otimes\mathbf{1}\\ &\quad -\frac{1}{2}\{\mathbf{1}\otimes\sum_a \tau(Y_a)^2\otimes\mathbf{1}-\sum_a Y_a^2\otimes\mathbf{1}\otimes\mathbf{1}\\ &\qquad\qquad -\sum_i [X_i, X_{\bar{i}}]\otimes\mathbf{1}\otimes\mathbf{1}-\mathbf{1}\otimes\tau(H_{2\delta_n})\otimes\mathbf{1}\}\\ &= \frac{1}{2}\{-(\sum_i (X_i X_{\bar{i}}+X_{\bar{i}}X_i)+\sum_a Y_a^2)\otimes\mathbf{1}\otimes\mathbf{1}\\ &\qquad\qquad +\mathbf{1}\otimes\tau(C_k)\otimes\mathbf{1}+\mathbf{1}\otimes\tau(H_{2\delta_n})\otimes\mathbf{1}\}\\ &= \frac{1}{2}(-C+\langle\Lambda, \Lambda+2\delta_k\rangle\mathbf{1}+\langle\Lambda, 2\delta_n\rangle\mathbf{1}) \qquad \text{[by (3.1)]}\\ &= \frac{1}{2}(-C+\langle\Lambda, \Lambda+2\delta\rangle\mathbf{1})\,.\end{aligned}$$

This proves the Lemma.

4. The theorem. We shall prove the following

Theorem. *The notation and hypotheses being as in the preceding sections, let τ be an irreducible representation of K whose highest weight Λ is a dominant integral form with respect to Δ_+. Then*

$$H^{0,q}(\Gamma, X, J_\tau) = (0)$$

for q satisfying one of the following conditions.

(I) *$q<q_\Lambda$, where q_Λ is the number of roots α such that $\langle\Lambda, \alpha\rangle>0$.*

(II) *$q<r$, if X is an irreducible symmetric space of rank r and unless $q=0$ nor $\Lambda=0$.*

As mentioned in the introduction, the case (I) has been proved in [7] (and also in [8]) in a different way, while the case (II) is a slight generalization of a result in [3].

To prove the theorem, we recall first a formula in [8, Part II] which we will apply. Let σ be a representation of K in a complex vector space V_σ. By an automorphic form of type $(\Gamma, \sigma, \lambda)$ we mean a V_σ-valued C^∞-function f on G satisfying the following conditions. (i) $f(gk)=\sigma(k^{-1})f(g)$ for $g\in G$, $k\in K$, (ii) $f(\gamma g)=f(g)$ for $\gamma\in\Gamma$, $g\in G$, and (iii) $Cf=\lambda f$, where λ is a complex number depending only on σ. By what we have observed in §§2 and 3, we can identify the space $\mathcal{H}^{0,q}(\Gamma, X, J_\tau)$ with the space of $V\otimes\Lambda^q\mathfrak{n}^+$-valued C^∞-functions f on $\Gamma\backslash G$ satisfying (2.2) and such that

$$Cf=\langle\Lambda, \Lambda+2\delta\rangle f.$$

Therefore, $\mathcal{H}^{0,q}(\Gamma, X, J_\tau)$ may be considered as the space of automorphic forms of type $(\Gamma, \tau\otimes\mathrm{ad}_+^q, \lambda)$ with $\lambda=\langle\Lambda, \Lambda+2\delta\rangle$.

Let π be a unitary representation of G in a Hilbert space H_π. Then π gives rise to representations of the Lie algebra $\mathfrak{g}$ and of the universal enveloping algebra $U(\mathfrak{g})$ in H_π, which we shall denote also by π. The operator $\pi(C)$ is known to be a self-adjoint operator of H_π with a dense domain. Assume now that π is irreducible. There exists then a complex number λ_π such that $\pi(C)=\lambda_\pi\mathbf{1}$, i.e. that $\pi(C)u=\lambda_\pi u$ for all u in the domain of $\pi(C)$. On the other hand, the space H_π being considered as a K-module by the restriction of π to K, decomposes into a countable sum of irreducible K-submodules among which each irreducible K-module occurs with finite multiplicity. So we can define for a representation σ of K on a finite-dimensional complex vector space V_σ, the intertwining number $(\pi|K; \sigma)$ as the dimension of the space of all K-homomorphisms of H_π into V_σ. If σ is irreducible, $(\pi|K; \sigma)$ is equal to the multiplicity of σ in the restricition of π to K.

Let now ρ be the unitary representation of the group G in the Hilbert space $L^2(\Gamma\backslash G)$ induced from the action of G on $\Gamma\backslash G$. We know that ρ decomposes into sum of a countable number of irreducible representations, in which each irreducible representation π of G enters with a finite multiplicity that we denote by $m_\pi(\Gamma)$.

Now, for a representation σ of K, let $A(\Gamma, \sigma, \lambda)$ be the space of automorphic forms of type $(\Gamma, \sigma, \lambda)$. Then we have obtained the following formula [8, Theorem 3].

$$\dim A(\Gamma, \sigma, \lambda)=\sum_{\pi\in D_\lambda} m_\pi(\Gamma)(\pi|K; \sigma^*) \tag{4.1}$$

where σ^* denotes the representation of K contragredient to σ and D_λ is the set of irreducible unitary representations π of G such that $\pi(C)=\lambda\mathbf{1}$. Actually this formula is established in [8] for the case that σ is irreducible, but it follows that the same formula holds for any finite-dimensional representation σ of K, since σ decomposes into a finite sum of irreducible representations. Moreover, if π^* denotes the representation of G contragredient to an irreducible unitary representation π of G, we can easily see $(\pi|K; \sigma^*)=(\pi^*|K; \sigma)$ and that $\pi(C)$ and $\pi^*(C)$ are the same scalar multiple of the identity operators. The representation σ of G in $L^2(\Gamma\backslash G)$ is self-contragredient, from which it follows that $m_\pi(\Gamma)=m_{\pi^*}(\Gamma)$ for any irreducible representation π of G. Combining these results, the formula (4.1) can now be written as

$$\dim A(\Gamma, \sigma, \lambda) = \sum_{\pi \in D_\lambda} m_\pi(\Gamma)(\pi | K; \sigma).$$

Applying this to our case, we get the following formula.

$$\dim \mathcal{H}^{0,q}(\Gamma, X, J_\tau) = \sum_\pi m_\pi(\Gamma)(\pi | K; \tau \otimes \mathrm{ad}_+^q) \tag{4.2}$$

where π runs over the irreducible unitary representations of G for which $\pi(C) = \langle \Lambda, \Lambda + 2\delta \rangle \mathbf{1}$.

Using this interpretation, we have the following lemma whose proof depends on a computation similar to Parthasarathy's [11] (See also [3, Lemma 3.7]).

Lemma 2. *Assume $H^{0,q}(\Gamma, X, J_\tau) \neq (0)$. Then there exists a subset $Q \subset \Psi$ with cardinality q, satisfying the following conditions;*

(1) *there exists an irreducible unitary representation π_μ of G whose highest weight with respect to the (new) positive root system $\Delta_+' = \Theta \cup (-\Psi)$ is $\mu = \Lambda + \langle Q \rangle$. That is, there exists a non-zero vector v in the representation space of π_μ such that*

$$\pi_\mu(X_\alpha)v = 0 \qquad (\alpha \in \Delta_+'),$$
$$\pi_\mu(H)v = \mu(H)v \qquad (H \in \mathfrak{h}^C).$$

(2) *$\langle \Lambda, \alpha \rangle = 0$ for $\alpha \in \Psi - Q$ and $|\delta_k - \delta_n| = |\delta_k - \delta_n + \langle Q \rangle|$, where $|\lambda|^2 = \langle \lambda, \lambda \rangle$ for any $\lambda \in \mathfrak{h}_0^*$.*

Proof. By the assumption and (4.2), there exists an irreducible unitary representation π in a space H_π containing an irreducible K-module U intertwining with $\tau \otimes \mathrm{ad}_+^q$. Let μ be the highest weight of U and v be the non-zero eigenvector for μ. Note that there then exists $Q \subset \Psi$ such that $\mu = \Lambda + \langle Q \rangle$. We know that v is in the domain of all operators $\pi(X)$ ($X \in U(\mathfrak{g}^C)$). Since $\pi(C) = \langle \Lambda, \Lambda + 2\delta \rangle \mathbf{1}$, we have

$$\begin{aligned}
&2 \sum_i \pi(X_i)\pi(X_{\bar{i}})v \\
&= \sum_i \{\pi(X_i)\pi(X_{\bar{i}}) + \pi(X_{\bar{i}})\pi(X_i)\} v + \sum_i \pi([X_i, X_{\bar{i}}])v \\
&= \{\pi(C_k) - \pi(C) + \pi(H_{2\delta_n})\} v \\
&= \{\langle \Lambda, \Lambda + 2\delta \rangle - \langle \mu, \mu + 2\delta_k \rangle + \langle \mu, 2\delta_n \rangle\} v && \text{[by (3.1)]} \\
&= \{|\Lambda + \delta|^2 - |\mu + \delta_k - \delta_n|^2\} v && [\text{as } |\delta| = |\delta_k - \delta_n|] \\
&= \{2\langle \Lambda, 2\delta_n - \langle Q \rangle \rangle + |\delta_k - \delta_n|^2 - |\delta_k - \delta_n + \langle Q \rangle|^2\} v && [\text{as } \mu = \Lambda + \langle Q \rangle].
\end{aligned}$$

Since π is unitary, if follows

$$\begin{aligned}
&-2 \sum_i \|\pi(X_{\bar{i}})v\|^2 \\
&= \{2\langle \Lambda, 2\delta_n - \langle Q \rangle \rangle + (|\delta_k - \delta_n|^2 - |\delta_k - \delta_n + \langle Q \rangle|^2)\} \|v\|^2,
\end{aligned}$$

where $\|\cdot\|$ denotes the Hilbert norm on H_π. But by the assumption on Λ, $\langle \Lambda, 2\delta_n - \langle Q \rangle \rangle \geqq 0$ and by a result of Kostant [4],

$$|\delta_k-\delta_n|^2 \geqq |\delta_k-\delta_n+\langle Q\rangle|^2 .$$

Hence $\pi(X_{\bar{\gamma}})v=0$; thus π satisfies the requirement for π_μ in (1) and simultaneously Q satisfies (2). Q.E.D.

Proof of Theorem. We are now ready to prove the case (I). Assume $H^{0,q}(\Gamma, X, J_\tau)\neq(0)$. Let $Q\subset\Psi$ be as in Lemma 2. Then $\langle\Lambda, \alpha\rangle=0$ for $\alpha\in\Psi-Q$. Setting

$$Q_\Lambda = \{\alpha\in\psi;\ \langle\Lambda, \alpha\rangle>0\} ,$$

we thus have $Q_\Lambda\subset Q$. Hence $q_\Lambda\leqq q$.

We shall next prove the theorem for the case (II). Under the assumption of (II), the Lie algebra $\mathfrak{g}^C$ is simple and so there exists a unique root $\alpha_0\in\Psi$ which is a simple root with respect to the positive root system $\Delta_+=\Theta\cup\Psi$. If $\langle\Lambda, \alpha_0\rangle\neq 0$, then clearly $\langle\Lambda, \alpha\rangle>0$ for all $\alpha\in\Psi$, which means $q_\Lambda=N=\dim_C X$. Hence, in this case (I) implies the assertion in (II).

To treat the remaining case, i.e. the case $\langle\Lambda, \alpha_0\rangle=0$, we use a criterion of the unitarizability of representations with highest weights obtained in [3]. Assume again that $H^{0,q}(\Gamma, X, J_\tau)\neq(0)$. By Lemma 2, there exists an irreducible unitary representation π_μ with highest weight $\mu=\Lambda+\langle Q\rangle$ with respect to the positive root system $\Delta'_+=\Theta\cup(-\Psi)$. To simplify our notation, put $\delta'=\delta_k-\delta_n$ and $Q'=-Q\subset-\Psi$. Then by (2) of Lemma 2,

$$|\delta'|=|\delta'-\langle Q'\rangle| .$$

By Kostant [4], there then exists an element $s\in W$ such that $s(-\Delta'_+)\cap\Delta'_+=Q'$, where W is the Weyl group for $(\mathfrak{g}^C, \mathfrak{h}^C)$. Note that $\langle Q'\rangle=\delta'-s\delta'$ and $l(s)=q$, where $l(s)$ is the length of a minimal expression of s as product of Weyl reflections for simple roots in Δ'_+.

Since π_μ is an irreducible unitary representation with highest weight μ with respect to the positive root system Δ'_+, we have by [3, Lemma 3.4],

$$\langle\mu, \beta_0\rangle\neq 0 ,$$

if $\mu\neq 0$, where β_0 is the highest root in Δ'_+. By what we have seen above,

$$\mu = \Lambda-(\delta'-s\delta')$$

with $l(s)=q$ and $s\Delta'_+\supset\Theta$.

Now, as we suppose $\langle\Lambda, \alpha_0\rangle=0$, $\langle\Lambda, \beta_0\rangle=0$. Hence applying [3. Lemma 3.6], we have

$$\begin{aligned}\langle\mu, \beta_0\rangle &= \langle\Lambda-(\delta'-s\delta'), \beta_0\rangle \\ &= \langle\Lambda, \beta_0\rangle+\langle s\delta'-\delta', \beta_0\rangle \\ &= \langle s\delta'-\delta', \beta_0\rangle = 0\end{aligned}$$

when $q=l(s)<r=\operatorname{rank} X$. Thus we should have $q\geqq r$ unless $\mu=0$.

We shall see that if $\langle\Lambda, \alpha_0\rangle=0$ and $\mu=0$, then $\Lambda=0$ and $q=0$. Since $s\Delta'_+\supset\Theta$,

$$\langle s\delta'-\delta', \alpha\rangle\geqq 0 \qquad (\alpha\in\Theta)$$

(see, for example, [3, Lemma 3.5]). But we assume that $\langle\Lambda, \alpha\rangle\geqq 0$ for $\alpha\in\Theta\cup\Psi$. Hence if $\mu=0$, i.e. if $\Lambda=\delta'-s\delta'$, then

$$\langle\Lambda, \alpha\rangle = 0 \qquad (\alpha\in\Theta)\,.$$

Since the center of $\mathfrak{k}$ is one-dimensional, it follows that there exists a scalar $c\in \boldsymbol{C}$ such that $\Lambda=c\delta_n$. By [9, p. 96, Corollary], we know

$$\langle\delta_n, \alpha_0\rangle>0$$

(actually, $\langle 2\delta_n, \alpha\rangle=\frac{1}{2}\,(\alpha\in\Psi)$). Hence $c=\langle\Lambda, \alpha_0\rangle/\langle\delta_n, \alpha_0\rangle$ and so $c=0$, because $\langle\Lambda, \alpha_0\rangle=0$. Thus we have $\Lambda=0$. We have also $q=0$, since $\mu=\Lambda+\langle Q\rangle=0$. We have thus completed the proof for the case (II).

HIROSHIMA UNIVERSITY
OSAKA UNIVERSITY

Bibliography

[1] R. Hotta: *A remark on the Laplace-Beltrami operators attached to Hermitian symmetric pairs,* Osaka J. Math. **8** (1971), 15–19.

[2] R. Hotta and R. Parthasarathy: *A geometric meaning of the multiplicity of integrable discrete classes in $L^2(\Gamma\backslash G)$,* Osaka J. Math. **10** (1973), 211–234.

[3] R. Hotta and N. Wallach: *On Matsushima's formula for the Betti numbers of a locally symmetric space,* Osaka J. Math. **12** (1975), 419–431.

[4] B. Kostant: *Lie algebra cohomology and the generalized Borel-Weil theorem,* Ann. of Math. **74** (1961), 329–387.

[5] Y. Matsushima: *A formula for the Betti numbers of compact locally symmetric Riemannian manifolds,* J. Differential Geometry **1** (1967), 99–109.

[6] Y. Matsushima and S. Murakami: *On vector bundle valued harmonic forms and automorphic forms on symmetric Riemannian manifolds,* Ann. of Math. **78** (1963), 365–416.

[7] ————: *On certain cohomology groups attached to hermitian symmetric spaces,* Osaka J. Math. **2** (1965), 1–35.

[8] ————: *On certain cohomology groups attached to hermitian symmetric spaces* (II), Osaka J. Math. **5** (1968), 223–241.

[9] S. Murakami: Cohomology groups of vector-valued forms on symmetric spaces, Lecture Notes, University of Chicago, 1966.

[10] K. Okamoto and H. Ozeki: *On square-integrable $\bar{\partial}$-cohomology spaces attached to hermitian symmetric spaces,* Osaka J. Math. **4** (1967), 95–110.

[11] R. Parthasarathy: *A note on the vanishing of certain "L^2-cohomologies",* J. Math. Soc. Japan **23** (1971), 676–691.