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ON THE RING STRUCTURE OF $U_*(BU(1))$

Dedicated to Professor Keizo Asano on his 60th birthday

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The complex bordism group $U_k(BU(1))$ consists of the bordism classes of the pair (M^k, ξ), [1], where M^k is a k-dimensional U-manifold and ξ is a complex line bundle over M^k. We define the multiplication in $U_*(BU(1))$ as follows,

$$[M^k, \xi][N^l, \eta] = [M^k \times N^l, \xi \hat{\otimes} \eta],$$

where $\xi \hat{\otimes} \eta$ is the external tensor product of ξ and η. In this paper, we study the ring structure of $U_*(BU(1))$ with this multiplication.

1. The relation formula in $U_*(BU(1))$

At first we recall the Mischenko series [3], which is essential in the determination of the relation formula in $U_*(BU(1))$.

Theorem 1.1 (Mischenko). *For a complex line bundle ξ over a CW complex X, define a series $g(c_1(\xi))$ by*

$$g(c_1(\xi)) = \sum_{k=0}^{\infty} \frac{x_k}{k+1} c_1(\xi)^{k+1} \in U^*(X) \otimes Q,$$

where x_k is the class of $2k$-dimensional complex projective space CP^k, and $c_1(\xi)$ is a cobordism 1-st Chern class of ξ. This satisfies, for line bundles ξ and η, the relation

$$g(c_1(\xi \hat{\otimes} \eta)) = g(c_1(\xi)) \times 1 + 1 \times g(c_1(\eta)).$$

Denote by η_n the canonical line bundle over the $2n$-dimensional complex projective space CP^n. It is well known that $\tilde{U}_*(BU(1))$ is a free U_* module with a basis $\{[CP^n, \eta_n], n=1, 2, \cdots\}$. We put

$$\{n\} = [CP^n, \eta_n].$$

Consider the duality isomorphism

$$D: U^*(CP^n \times M^m) \to U_*(CP^n \times M^m),$$

where M^m is a $2m$-dimensional U-manifold. The classifying map f of $\eta_n \hat{\otimes} 1_M$, where 1_M is the trivial complex line bundle over M^m, induces the homomorphism

$$f_*: U_*(CP^n \times M^m) \to U_*(BU(1)) .$$

Then, we have the following

Lemma 1.2. $f_* D(c_1(\eta_n)^k \times 1) = \{n-k\}[M^m]$.

Proof. It is obtained immediately that

$$D(c_1(\eta_n)^k \times 1) = [N^{n-k} \times M^m, j \times id], \quad [N^{n-k}, j] = D(c_1(\eta_n)^k) .$$

And it is obtained in parallel with the case of the lens space, [2], that $D(c_1(\eta_n)^k) = [CP^{n-k}, i]$, where $i: CP^{n-k} \to CP^n$ is the inclusion map. Therefore, $f_*([CP^{n-k} \times M^m, i \times id]) = \{n-k\}[M^m]$. q. e. d.

Consider the duality isomorphism

$$D: U^*(CP^m \times CP^n) \to U_*(CP^m \times CP^n) ,$$

and the homomorphism

$$f_*^{m,n}: U_*(CP^m \times CP^n) \to U_*(BU(1)) ,$$

where $f^{m,n}$ is the classifying map of $\eta_m \hat{\otimes} \eta_n$. Noting that

$$f_*^{m,n}[CP^j \times CP^n, i \times id] = \{j\}\{n\} ,$$

we have the following

Lemma 1.3. $f_*^{m,n} D(c_1(\eta_m)^k \times 1) = \{m-k\}\{n\}$.

For $[M^k, \xi] \in U_k(BU(1))$, consider the following homomorphisms

$$\bar{D} = D \otimes id: U^*(M^k) \otimes Q \to U_*(M^k) \otimes Q ,$$

where D is the duality isomorphism, and

$$\bar{f}_*^\xi = f_*^\xi \otimes id: U_*(M^k) \otimes Q \to U_*(BU(1)) \otimes Q ,$$

where f_*^ξ is the homomorphism induced by the classifying map of ξ. Then, we define the homomorphism

$$\Theta: U_k(BU(1)) \to U_{k-2}(BU(1)) \otimes Q$$

by

$$\Theta[M^k, \xi] = \bar{f}_*^\xi \bar{D} g(c_1(\xi)) ,$$

where $g(c_1(\xi))$ is the Mischenko series. Using the standard technique, we can prove that Θ is well defined and it is the U_* homomorphism.

Suppose that

$$\{m\}\{n\} = \sum_{i=0}^{m+n} \alpha_i(m, n)\{i\} \qquad \cdots\cdots\cdots\cdots(1),$$

where $\alpha_i(m, n) \in U_{2(m+n-i)}$ and $\{0\} = 1$. We can compute the coefficient $\alpha_i(m, n)$ from the following

Theorem 1.4.

(i) $\displaystyle\sum_{k=r+1}^{m+n} \frac{x_{k-r-1}}{k-r}\alpha_k(m, n) = \sum_{k=0}^{m+n-1-r} \frac{x_k}{k+1}\{\alpha_r(m-k-1, n)+\alpha_r(m, n-k-1)\}$.

(ii) $\alpha_{m+n}(m, n) = \binom{m+n}{m}$.

(iii) $\displaystyle\alpha_0(m, n) = [CP^m][CP^n] - \sum_{k=1}^{m+n} \alpha_k(m, n)[CP^k]$.

Proof. We apply the homomorphism Θ to the equation (1).

$$\begin{aligned}
\Theta\{m\}\{n\} &= f_*^{m,n}\bar{D}g(c_1(\eta_m\hat{\otimes}\eta_n)) \\
&= f_*^{m,n}\bar{D}\{g(c_1(\eta_m))\times 1 + 1\times g(c_1(\eta_n))\}, \quad \text{by Theorem 1.1,} \\
&= \sum_{k=0}^{m-1} \frac{x_k}{k+1} f_*^{m,n}D(c_1(\eta_m)^{k+1}\times 1) + \sum_{k=0}^{n-1} \frac{x_k}{k+1} f_*^{m,n}D(1\times c_1(\eta_n)^{k+1}) \\
&= \sum_{k=0}^{m-1} \frac{x_k}{k+1}\{m-k-1\}\{n\} + \sum_{k=0}^{n-1} \frac{x_k}{k+1}\{m\}\{n-k-1\}, \quad \text{by Lemma 1.3,} \\
&= \sum_{k=0}^{m-1} \frac{x_k}{k+1}\Big(\sum_{i=0}^{m+n-k-1} \alpha_i(m-k-1, n)\{i\}\Big) \\
&\qquad + \sum_{k=0}^{n-1} \frac{x_k}{k+1}\Big(\sum_{i=0}^{m+n-k-1} \alpha_i(m, n-k-1)\{i\}\Big).
\end{aligned}$$

Suppose that $\alpha_i(m, n)$ is the bordism class of M_i. Denote by f^i the classifying map of $1_{M_i}\hat{\otimes}\eta_i$, where 1_{M_i} is the trivial line bundle over M_i.

$$\begin{aligned}
\Theta\Big(\sum_{i=0}^{m+n} \alpha_i(m, n)\{i\}\Big) &= \sum_{i=0}^{m+n} \Theta(\alpha_i(m, n)\{i\}) \\
&= \sum_{i=0}^{m+n} f_*^i\bar{D}(1\times g(c_1(\eta_i))) \\
&= \sum_{i=0}^{m+n}\sum_{k=0}^{i-1} \frac{x_k}{k+1}(f_*^iD(1\times c_1(\eta_i)^{k+1}) \\
&= \sum_{i=0}^{m+n}\sum_{k=0}^{i-1} \frac{x_k}{k+1}\alpha_i(m, n)\{i-k-1\}, \quad \text{by Lemma 1.2.}
\end{aligned}$$

Since $\{\{k\},\ k=1, 2, \cdots\}$ is the basis of U_* free module $\tilde{U}_*(BU(1))$, comparing the coefficient of $\{r\}$ of $\Theta\{m\}\{n\}$ with that of $\Theta(\sum_{i=0}^{m+n}\alpha_i(m, n)\{i\})$, (i) follows. Putting $r=m+n-1$ on the equation (i),

$$\alpha_{m+n}(m, n) = \alpha_{m+n-1}(m-1, n)+\alpha_{m+n+1}(m, n-1)\,.$$

Hence, by induction (ii) follows. Applying the homomorphism

$$c_*: U_*(BU(1)) \to U_*\,,$$

given by the collapsing map $c: BU(1) \to$ a point, to the equation (1), (iii) follows.

2. The ring structure of $U_*(BU(1))\otimes Z_p$

In this section we study the ring structure of $U_*(BU(1))\otimes Z_p$, p a prime. We put $\overline{[M, \xi]}=[M, \xi]\otimes 1\in U_*(BU(1))\otimes Z_p$. We define the homomorphism

$$\mu_p: U_*(BU(1))\otimes Z_p \to H_*(BU(1))\otimes Z_p$$

by $\mu_p=\mu\otimes id$ with $\mu[M, \xi]=f_*^\xi\sigma(M)$, where $\sigma(M)$ is a fundamental class of M and $f_*^\xi: H_*(M)\to H_*(BU(1))$ is the homomorphism induced by the classifying map f^ξ of ξ. We have immediately the following

Lemma 2.1. *If* $n>0$, *then* $\mu([N^n][M, \xi]) = 0$.

Proposition 2.2. *For p a prime, $\overline{\{p^k\}}$ is indecomposable.*

Proof. Suppose that

$$\overline{\{p^k\}} = \sum_{\substack{t_1+\cdots+t_n=p^k \\ p^k>t_n>0}} \bar{\alpha}_{t_1,\cdots,t_n}\overline{\{t_1\}}\cdots\overline{\{t_n\}}+\sum_{\dim\beta_m>0}\bar{\beta}_m\overline{\{m\}}\,.$$

Using the relation (1) of § 1 and Theorem 1.4, (ii),

$$\overline{\{t_1\}}\cdots\overline{\{t_n\}}-\binom{t_1+t_2}{t_2}\binom{t_1+t_2+t_3}{t_3}\cdots\binom{p^k}{t_n}\overline{\{p^k\}}\in \bar{U}_*\cdot U_*(BU(1))\otimes Z_p\,,$$

where $\bar{U}_*=\sum_{i>0} U_i$. Since $\binom{p^k}{t_n}\equiv 0 \bmod p$,

$$\overline{\{p^k\}}\in \bar{U}_*\cdot U(BU(1))\otimes Z_p\,.$$

By Lemma 2.1, $\mu_p\overline{\{p^k\}}=0$. Denote by c the generator of $H^2(BU(1))$. Since $\langle c^{p^k}, \mu\{p^k\}\rangle=1$, $\mu_p\overline{\{p^k\}}$ is the generator of $H_*(BU(1))\otimes Z_p$. Therefore, the proposition follows. q.e.d.

Proposition 2.3. *For p a prime, $\overline{\{p^k\}}^p\in\bar{U}_*\cdot U_*(BU(1))\otimes Z_p$, where $\bar{U}_*=\sum_{i>0} U_i$.*

Proof. By Theorem 1.4, (ii), $\{p^k\}^p$ is represented as follows,

$$\{p^k\}^p = \binom{2p^k}{p^k}\cdots\binom{p^{k+1}}{p^k}\{p^{k+1}\} + \sum_{m<p^{k+1}} \beta_m\{m\}\,.$$

Since $\binom{2p^k}{p^k}\cdots\binom{p^{k+1}}{p^k}\equiv 0 \bmod p$,

$$\overline{\{p^k\}}^p = \sum_{m<p^{k+1}} \bar{\beta}_m \overline{\{m\}}\,,$$

where the dimension of β_m is positive. q. e. d.

Theorem 2.4. *Suppose that p is prime. Let Δ_* be U_* free module with a basis*

$$\{\{p^{k_1}\}^{i_1}\cdots\{p^{k_n}\}^{i_n}; 0\leqq k_1<\cdots<k_n,\ 0<i_j<p\}\,.$$

Then, $\Delta_*\otimes Z_p \approx \tilde{U}_*(BU(1))\otimes Z_p$.

Proof. Denote by ψ the natural homomorphism from $\Delta_*\otimes Z_p$ to $\tilde{U}_*(BU(1))\otimes Z_p$. Suppose that

$$\sum \bar{\alpha}(i_1, \cdots, i_m; k_1, \cdots, k_m)\overline{\{p^{k_1}\}}^{i_1}\cdots\overline{\{p^{k_m}\}}^{i_m} = 0 \qquad \cdots\cdots\cdots(2)$$

We define the order in the set consisting of $(i_1, \cdots, i_n; k_1, \cdots, k_n)$ as follows,

$$(i_1, \cdots, i_n; k_1, \cdots, k_n)<(i_1', \cdots, i_m'; k_1', \cdots, k_m') \quad \text{if} \sum_{j=1}^n i_j p^{k_j} < \sum_{j=1}^m i_j' p^{k_j'}\,.$$

Let $(\tilde{i}_1, \cdots, \tilde{i}_n; \tilde{k}_1, \cdots, \tilde{k}_n)$ be maximal in the set consisting of $(i_1, \cdots, i_m; k_1, \cdots, k_m)$ which is used in the equation (2). Put

$$q = \sum_{j=1}^n \tilde{i}_j p^{\tilde{k}_j}\,.$$

By Theorem 1.4,

$$\{p^{\tilde{k}_1}\}^{\tilde{i}_1}\{p^{\tilde{k}_2}\}^{\tilde{i}_2}\cdots\{p^{\tilde{k}_n}\}^{\tilde{i}_n} = c\{q\} + \sum_{s<q} \beta_s\{s\}\,,$$

where

$$\begin{aligned} c &= \binom{2p^{\tilde{k}_1}}{p^{\tilde{k}_1}}\cdots\binom{\tilde{i}_1 p^{\tilde{k}_1}}{p^{\tilde{k}_1}}\cdots\binom{q}{p^{\tilde{k}_n}} \\ &\equiv \tilde{i}_1!\cdots\tilde{i}_n! \bmod p \\ &\not\equiv 0 \bmod p. \end{aligned}$$

Then, the equation (2) becomes

$$\bar{\alpha}(\tilde{i}_1, \cdots, \tilde{i}_n; \tilde{k}_1, \cdots, \tilde{k}_n)c\overline{\{q\}} + \sum_{s<q} \bar{\gamma}_s\overline{\{s\}} = 0\,.$$

Since $\tilde{U}_*(BU(1))$ is the U_* free module with the basis $\{\{m\}, m=1, 2, \cdots\}$ and $c \not\equiv 0 \bmod p$, $\bar{\alpha}(\tilde{i}_1, \cdots, \tilde{i}_n; \tilde{k}_1, \cdots, \tilde{k}_n)=0$. By induction, it follows that $\bar{\alpha}(i_1, \cdots, i_m; k_1, \cdots, k_m)=0$ and ψ is monomorphism.

We show that each $\overline{\{n\}}$ belongs to the image of ψ. By the definition of Δ_*, $\{\bar{1}\} \in$ image ψ. Suppose that $\overline{\{m\}} \in$ image ψ for $m<n$. We represent n as follows,

$$n = j_1 p^{k_1} + \cdots + j_r p^{k_r} ,$$

where $0<j_s<p$, $0 \leqq k_1 < \cdots < k_r$. By Theorem 1.4,

$$\{p^{k_1}\}^{j_1} \cdots \{p^{k_r}\}^{j_r} = c\{n\} + \sum_{s<n} \beta_s \{s\} ,$$

where

$$\begin{aligned} c &= \binom{2p^{k_1}}{p^{k_1}} \cdots \binom{j_1 p^{k_1}}{p^{k_1}} \cdots \binom{n}{p^{k_r}} \\ &\equiv j_1! \cdots i_r! \bmod p. \end{aligned}$$

Since $c \not\equiv 0 \bmod p$ and $\overline{\{p^{k_1}\}}^{j_1} \cdots \overline{\{p^{k_r}\}}^{j_r} \in$ image ψ, the inductive hypothesis implies that $\overline{\{n\}} \in$ image ψ. q. e. d.

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