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A NON-STABLE SECONDARY OPERATION AND HOMOTOPY CLASSIFICATION OF MAPS

Dedicated to Professor A. Komatu on his 60th birthday

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1. Introduction

Let $p: E \to A$ be the principal fibration with classifying map $\theta: A \to B$ and let $q: T \to E$ be the principal fibration induced by a map $\rho: E \to C$. We assume that A, B and C are H-spaces. Given a map $v: X \to E$, I. M. James and E. Thomas [5, 7] have defined the homomorphisms

$$\Delta(\theta, u): [X, \Omega A] \to [X, \Omega B]$$
$$\Delta_p(\rho, v): [X, \Omega^2 B] \to [X, \Omega C],$$

where u is the composite $p \circ v$, Ω is the loop functor and $[Y, Z]$ denotes the set of based homotopy classes of based maps $Y \to Z$.

The action $\Omega C \times T \to T$ of the principal fibration q induces the function

$$[X, \Omega C] \times [X, T] \to [X, T],$$

the image of $(\tau, w) \in [X, \Omega C] \times [X, T]$ under which is denoted by $\tau \cdot w$. The subgroup

$$\mathrm{I}(w) = \{\tau \in [X, \Omega C];\ \tau \cdot w = w\}$$

of $[X, \Omega C]$ is called the *isotropy group* of w under the action of $[X, \Omega C]$ on $[X, T]$. Our first main result is the following:

Theorem A. *Suppose $w: X \to T$ is a lifting of v. If $\Delta(\theta, u)$ is injective, then $\mathrm{I}(w)$ coincides with the image of $\Delta_p(\rho, v)$.*

This is obtained as a direct consequence of a property (Theorem 4.2) of a non-stable secondary operation $\Phi_\theta(\rho, v)$ which is inspired by an operation due to N. Shimada [14, p. 141].

The prime concern in this paper is to examine a few situations to which Theorem A is applicable.

Consider first the real projective space $P_n(R)$, where the dimension n is odd>1. Let X be a *path-connected* $(n+1)$-dimensional complex and let δ^*

denote the Bockstein homomorphism associated with $0 \to Z \to Z \to Z_2 \to 0$. We define

$$\varphi\colon [X, P_n(R)] \to H^1(X; Z_2)$$

by $\varphi(g)=g^*\iota$, where $g\colon X \to P_n(R)$ and ι denotes the generator of $H^1(P_n(R); Z_2)$. The following extends a result due to P. Olum [10].

Theorem B. *Let u be an element of $H^1(X; Z_2)$ such that $(\delta^* u)^{(n+1)/2}=0$. Then $\varphi^{-1}(u)$ is equivalent to*:

$$H^n(X; Z)\times H^{n+1}(X; Z_2)/(Sq^2+u^2\cup)H^{n-1}(X; Z) \qquad \textit{for} \quad n\equiv 1 \pmod 4$$
$$H^n(X; Z)\times H^{n+1}(X; Z_2)/Sq^2H^{n-1}(X; Z) \qquad \textit{for} \quad n\equiv 3 \pmod 4$$

Next, let η be the reduced stable class of the canonical line bundle over the real projective space and let $N_n(\xi; X)$ denote the number of classes of n-plane bundles over X which are stably equivalent to a stable reduced bundle ξ over X. The following is a partial extension of a result due to I. M. James and E. Thomas [5].

Theorem C. *Let k be an integer and let $\binom{k}{n+1}$ be even. Then*

(1) *for $n\equiv 1 \pmod 4$ and $\binom{k}{2}$ odd, $N_n(k\eta; P_{n+1}(R))=1$ or 2 according as $\binom{k-1}{n-1}$ is odd or even*;

(2) *$N_n(k\eta; P_{n+1}(R))=2$ if $n\equiv 1 \pmod 4$, $\binom{k}{2}$ even and $\binom{k-1}{n-1}$ odd*;

(3) *$N_n(k\eta; P_{n+2}(R))=2$ if $n\equiv 3 \pmod 4$ and $\binom{k-1}{n-1}$ odd.*

Finally, let $\mathcal{E}(X)$ denote the group of homotopy classes of homotopy equivalences of X whose group structure is induced by map-composition. Consider the tower

$$\begin{array}{ccccc}
\Omega C & \longrightarrow & T & & \\
 & & \downarrow{\scriptstyle q} & & \\
\Omega B & \longrightarrow & E & \xrightarrow{\rho} & C \\
 & & \downarrow{\scriptstyle p} & & \\
 & & K(\pi, n) & \xrightarrow{\theta} & B\,,
\end{array}$$

where B and C are H-spaces such that $\pi_r(B)\neq 0$ only for $n+2\leqq r\leqq m$ $(n>1)$ and $\pi_s(C)\neq 0$ only for $m+1\leqq s\leqq m+n-1$. The following theorem can be obtained by applying the results of [9] to the fibration q and observing the fact $H^{n-1}(T; \pi)=0$.

Theorem D. *The following sequence of groups and homomorphisms is exact*:

$$[T, \Omega^2 B] \xrightarrow{\Delta_p(\rho, q)} q^*[E, \Omega C] \to \mathcal{E}(T) \to \mathcal{E}(E) \times \mathcal{E}(\Omega C)$$

in which the image of the last homomorphism consists of $(\bar{g}, \Omega h) \in \mathcal{E}(E) \times \mathcal{E}(\Omega C)$ *such that* $\rho\bar{g} \simeq h\rho$.

The results stated in sections 2, 3 and 4 can be dualized in a cofibre space and will be considered elsewhere.

2. Preliminaries

We shall here fix the notations and recall some definitions given in [5, 6, 7]. We work in the category of spaces with basepoints (usually denoted by $*$) and basepoint preserving maps. The spaces considered are assumed to have the homotopy type of a CW complex. We blur the distinction between maps and their homotopy classes. We use the additive notation for path-composition and path-inversion. The suspension functor is denoted by S.

For a space X, let $\Omega^* X$ denote the space of free loops in X. One has a fibration

$$\Omega X \xrightarrow{i_X} \Omega^* X \xrightarrow{r_X} X$$

with section s_X: $X \to \Omega^* X$ given by $s_X(x)$=the constant loop at $x \in X$. A map f: $X \to Y$ induces the map $\Omega^* f$: $\Omega^* X \to \Omega^* Y$ in an obvious way.

Lemma 2.1. $(i_X)_* \xi = (i_X)_* \xi'$ *for* ξ, $\xi' \in [V, \Omega X]$ *if and only if* ξ *and* ξ' *are conjugate to each other.*

This can be proved directly or by replacing i_X by a principal fibration.

Corollary 2.2. (Theorem 2.6 of [5]) *The group* $[V, \Omega X]$ *is abelian if and only if* $(i_X)_*$: $[V, \Omega X] \to [V, \Omega^* X]$ *is injective.*

Given a fibration p: $E \to A$ and a map u: $X \to A$, we denote by $[X, E; u]$ the set of u-homotopy classes of u-maps $X \to E$ (see [6])

Let p: $E \to A$ be a fibration with fibre inclusion j: $F \to E$ and let q: $T \to E$ be the principal fibration with classifying map ρ: $E \to C$. Given a map w: $X \to T$, let $v = q \circ w$ and $u = p \circ v$. We define the *u-isotropy group* $\mathrm{I}_u(w)$ of w by setting

$$\mathrm{I}_u(w) = \{\tau \in [X, \Omega C]; \qquad \tau \cdot w = w \text{ in } [X, T; u]\}.$$

q induces the function

$$q_*: [X, T; u] \to [X, E; u]$$

and there is a bijection between $q_*^{-1}(v)$ and the totality of left cosets $[X, \Omega C]/\mathrm{I}_u(w)$. The following is obvious:

Proposition 2.3. (1) *For the trivial map* $*$: $X \to T$, $\mathrm{I}_*(*)$ *is the image of*

$(\Omega\rho)_*(\Omega j)_*$: $[X, \Omega F]\to[X, \Omega C]$;

(2) $\mathrm{I}_u(\tau\cdot w)=\tau+\mathrm{I}_u(w)-\tau$ *for* $\tau\in[X, \Omega C]$;

(3) $g^*\mathrm{I}_u(w)\subset\mathrm{I}_{u\circ g}(g^*w)$ *for* $g\colon Y\to X$;

(4) *Let* $i\colon \Omega C\to T$ *denote the inclusion. Then* $\mathrm{I}_*(i)=1+(\Omega\rho)_*(\Omega j)_*[\Omega C, \Omega F]-1$, *where* 1 *is the identity map of* ΩC.

Let Ω_p^*E denote the subspace of Ω^*E consisting of free loops λ such that $(\Omega^*p)\lambda\in s_A(A)$. Then one obtains a fibration $r\colon \Omega_p^*E\to E$ with fibre ΩF.

Theorem 2.4. $\tau\in[X, \Omega C]$ *is* ρ*-correlated* (*see* §3 *of* [7]) *to* $v\in[X, E; u]$, *if and only if there exists an* $\eta\in[X, T; u]$ *such that* $\tau\in\mathrm{I}_u(\eta)$ *and* $q_*\eta=v$.

Proof. The "if" part is proved in Lemma (3.3) of [7]. We shall prove the "only if" part. Assume that there exists a $\psi\in[X, \Omega_p^*E; u]$ such that $r_*\psi=v$ and $\rho'_*\psi=(i_C)_*\tau$ in the following commutative diagram

$$\begin{array}{ccccc} & & [X, \Omega_p^*E; u] & \xrightarrow{r_*} & [X, E; u] \\ & & \big\downarrow \rho'_* & & \big\downarrow \rho_* \\ [X, \Omega C] & \xrightarrow{(i_C)_*} & [X, \Omega^*C] & \xrightarrow{(r_C)_*} & [X, C]\,, \end{array}$$

where $\rho'\colon \Omega_p^*E\to\Omega^*C$ denotes the restriction of $\Omega^*\rho$ to Ω_p^*E.

Since r is a fibration, we may assume $\psi\colon X\to\Omega_p^*E$ is a lift of v. Let $F_t\colon X\to\Omega^*C$ be a homotopy with $F_0=i_C\tau$, $F_1=\rho'\psi$. Consider $\gamma\colon X\to C^I$ defined by $\gamma(x)(t)=r_C\circ F_t(x)$, $x\in X$, $0\leqq t\leqq 1$. It is easy to see that $\tau=\gamma+\rho'\psi-\gamma$ in $[X, \Omega C]$. Take $\eta\colon X\to T$ given by $\eta(x)=(v, \gamma)$; then it follows that $(\gamma+\rho'\psi-\gamma)\cdot\eta=\eta$ in $[X, T; u]$. This proves the only if part.

Corollary 2.5. *Suppose* $p\colon E\to A$ *is a principal fibration with fibre* F *in the sense of* [12]. *Then* $[X, \Omega F]=0$ *implies* $\mathrm{I}_u(w)=0$.

Proof. Let $\mu\colon F\times E\to E$ denote the action map and let $\mu'\colon \Omega F\times E\to\Omega_p^*E$ be the induced map defined as in §4 of [7]. Then, by Theorem (4.1) of [7],

$$\mu'_*\colon [X, \Omega F]\times[X, E; u]\to[X, \Omega_p^*E; u]$$

is bijective. Note that $\mu'\{*, 1_E\}\colon E\to\Omega_p^*E$ is p-homotopic to the canonical section $s\colon E\to\Omega_p^*E$ of r, so that both s and r induce the bijections between $[X, E; u]$ and $[X, \Omega_p^*E; u]$ because of $[X, \Omega F]=0$.

Now let $\tau\in[X, \Omega C]$ be ρ-correlated to v, i.e., there is an element $\psi\in[X, \Omega_p^*E; u]$ such that $r_*\psi=v$, $\rho'_*\psi=(i_C)_*\tau$. Then, since $q_*w=v$,

$$(i_C)_*\tau=\rho'_*s_*v=(s_C)_*\rho_*v=0\,,$$

which implies $\tau=0$ by virtue of $\ker(i_C)_*=0$.

Taking $A=*$ in the above situation, we obtain

Corollary 2.6. *$\tau \in [X, \Omega C]$ is ρ-correlated to $v \in [X, E]$ (see* §2 *of* [5]), *if and only if there is an element $\eta \in [X, T]$ such that $\tau \in \mathrm{I}(\eta)$ and $q_* \eta = v$. If E is an H-space with $[X, \Omega E]=0$, then $\mathrm{I}(\eta)=0$ for any lifting η of v: $X \to E$.*

3. The homomorphisms $\Delta(\theta, u)$ and $\Delta_p(\rho, v)$

Consider the situation

$$\begin{array}{ccccc} & & \Omega C & \xrightarrow{i} & T \\ & & & & \downarrow{\scriptstyle q} \\ F = \Omega B & \xrightarrow{j} & & & E & \xrightarrow{\rho} & C \\ & & & & \downarrow{\scriptstyle p} \\ & & & & A & \xrightarrow{\theta} & B, \end{array} \tag{3.1}$$

in which p and q are the principal fibrations with classifying maps θ and ρ respectively, and B and C are H-spaces with multiplications t: $B \times B \to B$ and n: $C \times C \to C$. Let μ: $F \times E \to E$ denote the action of F on E.

In case A is an H-space with multiplication m: $A \times A \to A$ and there is given a map v: $X \to E$ with $u = p \circ v$, I. M. James and E. Thomas have defined in [5, 7] the homomorphisms

$$\Delta(\theta, u): [X, \Omega A] \to [X, \Omega B],$$
$$\Delta_p(\rho, v): [X, \Omega F] \to [X, \Omega C]$$

as follows. Let

$$\mu': (\Omega F \times E, \Omega F \times F) \to (\Omega_p^* E, \Omega^* F)$$
$$m': \Omega A \times A \to \Omega^* A, \quad t': \Omega B \times B \to \Omega^* B, \quad n': \Omega C \times C \to \Omega^* C$$

denote the "right translations" determined by μ, m, t and n; then the equations

$$(\Omega^* \theta)_* m'_* \{\alpha, u\} = t'_* \{\Delta(\theta, u)\alpha, \theta_* u\} \qquad \text{for} \quad \alpha \in [X, \Omega A]$$
$$\rho'_* \mu'_* \{\beta, v\} = n'_* \{\Delta_p(\rho, v)\beta, \rho_* v\} \qquad \text{for} \quad \beta \in [X, \Omega F]$$

determine $\Delta(\theta, u)\alpha$ and $\Delta_p(\rho, v)\beta$ uniquely by virtue of Theorem 2.7 of [5]. The following is proven in [7, Theorem (4.2)]:

Theorem 3.2. *$\tau \in [X, \Omega C]$ is ρ-correlated to $v \in [X, E; u]$ if and only if, first, $\rho_* v = 0$ and, secondly, τ lies in the image of $\Delta_p(\rho, v)$. Thus, $\mathrm{I}_u(w) = \Delta_p(\rho, v)$ $[X, \Omega F]$ for any lift w of v.*

The homomorphism $\Delta(\theta, u)$ has also been introduced by J. W. Rutter [13], who has examined various properties of it. In the similar way we can obtain analogous theorems for $\Delta_p(\rho, v)$, so that the proofs are mostly omitted.

Triviality Theorem 3.3. $\Delta_p(\rho, *)=(\Omega\rho)_*(\Omega j)_*$. *More generally, if* $j\circ\hat{v}=v$ *for a map* $\hat{v}\colon X\to F$, *then* $\Delta_p(\rho, v)=\Delta(\rho j, \hat{v})$.

In order to state the next theorem we define the *dual Hopf invariant* $\nu(\rho)\in[F\times E, C]$ of ρ to be

$$\nu(\rho)=-(j\circ p_1)^*\rho+\mu^*\rho-p_2{}^*\rho\,.$$

where $p_1\colon F\times E\to F$ and $p_2\colon F\times E\to E$ are the projections. If $\nu(\rho)=0$, we say that ρ is *primitive with respect to* μ.

Primitivity Theorem 3.4. *If* ρ *is primitive with respect to* μ, *then* $\Delta_p(\rho, v)=(\Omega\rho)_*(\Omega j)_*$.

Theorem 3.5. *If* X *is an* H *cogroup (see* [15]), *then* $\Delta_p(\rho, v)=(\Omega\rho)_*(\Omega j)_*$.

Composition Theorem 3.6. *Let* $\alpha\colon C\to D$ *be a map to an* H *space. Then* $\Delta_p(\alpha\rho, v)=\Delta(\alpha, \rho v)\Delta_p(\rho, v)$.

Cartesian Product Theorem 3.7. *Let* $\rho_1\colon E\to C_1$, $\rho_2\colon E\to C_2$ *be maps to* H*-spaces and let* $\{\rho_1, \rho_2\}=(\rho_1\times\rho_2)\circ d\colon E\to C_1\times C_2$ *be the composite with diagonal map* d. *Then*

$$\Delta_p(\{\rho_1, \rho_2\}, v)\beta=(\Delta_p(\rho_1, v)\beta, \Delta_p(\rho_2, v)\beta)\,.$$

Additivity Theorem 3.8. *Let* $\rho_1, \rho_2\colon E\to C$ *be maps. Then*

$$\Delta_p(\rho_1+\rho_2, v)\beta=\Delta_p(\rho_1, v)\beta+(\rho_1 v)_\square\Delta_p(\rho_2, v)\beta\,,$$
$$\Delta_p(-\rho, v)\beta=-(-\rho v)_\square\Delta_p(\rho, v)\beta\,,$$

where $(\rho_1 v)_\square$ *and* $(-\rho v)_\square$ *are the endomorphisms as defined in* [13, p. 383].

Cup Product Theorem 3.9. *Given* $\rho\in H^a(E;\pi)$ *and* $\rho'\in H^b(E;\pi')$, *let* $\rho\cup\rho'$ *denote the cup product with respect to a pairing* $\pi\otimes\pi'\to G$. *Then*

$$\Delta_p(\rho\cup\rho', v)\beta=\Delta_p(\rho, v)\beta\cup v^*\rho'+(-1)^a v^*\rho\cup\Delta_p(\rho', v)\beta\,.$$

The following theorem will be useful in computing $\Delta_p(\rho, v)$ in terms of deviation of ρ from primitivity and corresponds to Corollary 1.4 of [5] or Theorem 2.4.1 of [13].

Theorem 3.10. *Suppose* C *is homotopy abelian or* $\rho_* v=0$. *Then*

$$\Delta_p(\rho, v)\beta=(\Omega\rho)_*(\Omega j)_*\beta+\Delta_{pp_2}(\nu(\rho), \{*, v\})\{\beta, *\}\,.$$

Assume further that C *is an Eilenberg-MacLane space and that*

$$\nu(\rho)=\sum u_i\times v_i+\sum\delta^\sharp(u'_j\times v'_j)\,,$$

where $\delta^\sharp$ *is the Bockstein. Then*

$$\Delta_p(\rho, v)\beta = (\Omega\rho)_*(\Omega j)_*\beta + \sum (\Omega u_i)\beta \cup v^* v_i - \sum \delta^\sharp(\Omega u'_j)\beta \cup v^* v'_j .$$

Proof. We imitate the proof of Theorem 2.4.1 of Rutter [13]. Consider the diagram

$$\begin{tikzcd}
 & F\times E \arrow[r, shift left, "\rho\mu"] \arrow[r, shift right, "\rho p_2"'] \arrow[d, "pp_2"] & C \\
X \arrow[ur, "\{*, v\}"] \arrow[r, "u"] & A \arrow[r, "\{*, \theta\}"] & B\times B
\end{tikzcd}$$

Note that the action $\bar{\mu}$: $(F\times F)\times(F\times E)\to F\times E$ of the principal fibration $p\circ p_2$ is given by $\bar{\mu}(x, x'; x'', y)=(x+x'', \mu(x', y))$. Thus, the right translation $\bar{\mu}'$: $\Omega(F\times F)\times(F\times E)\to\Omega^*_{pp_2}(F\times E)$ satisfies the following:

$$\begin{aligned}
(\rho\mu)'_*\bar{\mu}'_*(\{\beta, *\}, \{*, v\}) &= \rho'_*\mu'_*\{\beta, v\} , \\
(\rho p_2)'_*\bar{\mu}'_*(\{\beta, *\}, \{*, v\}) &= (s_C)_*\rho_* v , \\
p'_{1*}\mu'_*(\{\beta, *\}, \{*, v\}) &= (i_F)_*\beta .
\end{aligned}$$

Using these and by 3.8 and 3.6 we have that

$$\begin{aligned}
\Delta_p(\rho, v)\beta &= \Delta_{pp_2}(\rho\mu, \{*, v\})\{\beta, *\} \\
&= \Delta_{pp_2}(\rho j p_1, \{*, v\})\{\beta, *)+\Delta_{pp_2}(\nu(\rho), \{*, v\})\{\beta, *) \\
&\quad +\Delta_{pp_2}(\rho p_2, \{*, v\})\{\beta, *\} \\
&= \Delta(\rho j, *)\Delta_{pp_2}(p_1, \{*, v\})\{\beta, *\}+\Delta_{pp_2}(\nu(\rho), \{*, v\})\{\beta, *\} \\
&= (\Omega\rho)_*(\Omega j)_*\beta+\Delta_{pp_2}(\nu(\rho), \{*, v\})\{\beta, *\} .
\end{aligned}$$

Now it follows from 3.9, 3.6 and 1.4.1 of [13] that

$$\begin{aligned}
\Delta_{pp_2}(u_i\times v_i, \{*, v\})\{\beta, *\} &= \Delta_{pp_2}(p_1^* u_i, \{*, v\})\{\beta, *\} \cup v_i p_2\{*, v\} \\
&\quad +(-1)^{\dim u_i} u_i p_1\{*, v\} \cup \Delta_{pp_2}(p_2^* v_i, \{*, v\})\{\beta, *\} \\
&= \Delta(u_i, *)\Delta_{pp_2}(p_1, \{*, v\})\{\beta, *\} \cup v^* v_i \\
&= (\Omega u_i)\beta \cup v^* v_i .
\end{aligned}$$

Similarly,

$$\Delta_{pp_2}(\delta^\sharp(u'_j\times v'_j), \{*, v\})\{\beta, *\} = (\Omega\delta^\sharp)_*(\Omega u'_j)_*\beta \cup v^* v'_j .$$

This completes the proof of 3.10.

In [7], James and Thomas have called $\pi=p\circ q$ of (3.1) a *stable decomposition* of π if there exists a map c: $F\times A\to C$ such that the composite

$$A \xrightarrow{i_2} F\times A \xrightarrow{c} C$$

is null-homotopic and $\rho\mu \simeq c(1\times p)+\rho p_2$, where i_2 denotes the injection. Note that, if ρ is primitive with respect to μ, then $\pi=p\circ q$ is stable with $c=\rho j p_1$, where p_1 denotes the projection $F\times A\to F$. As stated in p. 104 of [7], for $v\colon X\to E$ liftable to T, $\tau\cdot v$ is liftable to T if and only if $c_*\{\tau, p_*v\}=0$. The following theorem can also be proved in a way similar to 3.10.

Theorem 3.11. (James and Thomas [7]) *If* (3.1) *is a stable decomposition of* $\pi=p\circ q$, *then*

$$\Delta_p(\rho, v)\beta = c'_*\{\beta, p_*v\}\,,$$

where $c'\colon \Omega F\times A\to\Omega C$ *is determined by the composite*

$$\Omega F\times A \xrightarrow{\text{injection}} \Omega(F\times A)\times F\times A \xrightarrow{\approx} \Omega^*(F\times A) \xrightarrow{\Omega^* c} \Omega^* C$$

4. Secondary operation $\Phi_\theta(\rho, v)$ and proof of Theorem A

Consider the situation (3.1) in which A, B and C are H-spaces with multiplications m, t and n respectively. Given a map $v\colon X\to E$, we set $u=p\circ v$. We shall now define a sort of secondary operation

$$\Phi_\theta(\rho, v)\colon \ker\Delta(\theta, u)\to \operatorname{coker}\Delta_p(\rho, v)$$

as follows.

Take an element $\alpha\in[X, \Omega A]$ such that $(\Omega^*\theta)_* m'_*\{\alpha, u\}=0$; then it follows from the next Sublemma (i) that there exists an element $\psi\in[X, \Omega^*E; u]$ such that

$$(4.1)\qquad (\Omega^*p)_*\psi = hm'_*\{\alpha, u\} \quad\text{and}\quad (r_E)_*\psi = v\in[X, E; u]\,,$$

where $h\colon [X, \Omega^*A; m\{*, u\}]\to[X, \Omega^*A; u]$ is the canonical bijection. The coset of $\gamma\in[X, \Omega C]$ determined by $(\Omega^*\rho)_*\psi=n'_*\{\gamma, \rho_*v\}$ is, by definition, $\Phi_\theta(\rho, v)\alpha$.

Observe that, if there exists another ψ' such that $(\Omega^*p)\psi'=(\Omega^*p)_*\psi$ in $[X, \Omega^*A; u]$ and $(r_E)_*\psi'=v$ in $[X, E; u]$, then we may assume $(r_E)\psi'=v=(r_E)\psi$ as maps and, applying Sublemma (ii) to $\varphi=\psi+(-\psi')$, we conclude that $(\Omega^*\rho)_*\psi-(\Omega^*\rho)_*\psi'$ lies in the image of $\Delta_p(\rho, v)$. This ensures that $\Phi_\theta(\rho, v)$ is well defined.

Sublemma. (i) *Given an element* $\beta\in[X, \Omega^*A; u]$ *lying in the image of* $(\Omega^*p)_*$; $[X, \Omega^*E]\to[X, \Omega^*A]$, *there exists a* $\psi\in[X, \Omega^*E; u]$ *such that* $(\Omega^*p)_*\psi=\beta$ *and* $(r_E)_*\psi=v$.

(ii) *If* $\varphi\in[X, \Omega^*E; u]$ *satisfies* $(\Omega^*p)_*\varphi=(s_A)_*u$, *then* φ *is contained in the image of the natural map* $[X, \Omega^*_pE; u]\to[X, \Omega^*E; u]$.

Proof. (i) Let $\hat{\mu}\colon F\times\Omega^*E\to\Omega^*E$ be the map induced by the action $\mu\colon F\times E\to E$. It is easily verified that

$$(\Omega^*p)\hat{\mu}(x, \gamma) = (\Omega^*p)\gamma, \ \mu(1\times r_E) = r_E\hat{\mu} \ (x\in F, \ \gamma\in\Omega^*E)$$

By assumption we can take $\tilde{\psi}$: $X\to\Omega^*E$ with $(\Omega^*p)\tilde{\psi}\simeq\beta$, so that there is a u-map ψ_0: $X\to\Omega^*E$ with $(\Omega^*p)\psi_0=\beta$, since Ω^*p is a fibration. Choose ω: $X\to F$ such that $\mu\{\omega, r_E\psi_0\}$ is u-homotopic to v. Then $\psi=\hat{\mu}\{\omega, \psi_0\}$ has the desired property.

(ii) This is a simple application of homotopy lifting property.

Theorem A in the introduction follows immediately from 3.2 and the following theorem which states a main property of $\Phi_\theta(\rho, v)$.

Theorem 4.2. *If w: $X\to T$ is a lifting of v, then the image of $\Phi_\theta(\rho, v)$ coincides with the factor group* $\mathrm{I}(w)/\mathrm{I}_u(w)$.

Proof. Let $\gamma\in[X, \Omega C]$ lie in the coset $\Phi_\theta(\rho, v)\alpha$. Since $\rho_*v=0$, we have that

$$(\Omega^*\rho)_*\psi = (i_C)_*\gamma \quad \text{and} \quad (r_E)_*\psi = v\,,$$

which shows that γ is ρ-correlated to v. Hence it follows from Theorem 3.3 of [5] that γ lies in $\mathrm{I}(w)$.

Conversely, suppose $\gamma\in\mathrm{I}(w)$. Then, by Theorem 3.2 of [5], there exists $\psi'\in[X, \Omega^*E]$ such that $(\Omega^*\rho)_*\psi'=(i_C)_*\gamma$ and $(r_E)_*\psi'=v$ in $[X, E]$. By the homotopy lifting property of r_E we see that $\psi\simeq\psi'$ for a map ψ: $X\to\Omega^*E$ with $r_E\psi=v$. Then there is an $\alpha\in[X, \Omega A]$ with $h^{-1}(\Omega^*p)_*\psi=m'_*\{\alpha, u\}$ and, moreover, $(\Omega^*\theta)_*m'_*\{\alpha, u\}=0$. This means that γ lies in in the coset $\Phi_\theta(\rho, v)\alpha$.

As a special case of Theorem A we obtain

Corollary 4.3. *If $[X, \Omega A]=0$, then $\mathrm{I}(w)=\mathrm{I}_u(w)$ for any lifting w of v.*

REMARK. The conclusion of 4.3 remains valid without assuming that B is an H-space, as shown in what follows. Since $\mathrm{I}_u(w)\subset\mathrm{I}(w)$, it suffices to prove $\mathrm{I}(w)\subset\mathrm{I}_u(w)$. Let $\tau\in\mathrm{I}(w)$, then, by Theorem 3.2 of [5], there is a $\psi\in[X, \Omega^*E]$ such that $(\Omega^*\rho)_*\psi=(i_C)_*\tau$ and $(r_E)_*\psi=v$. As above we may represent ψ by a v-map $\tilde{\psi}$: $X\to\Omega^*E$. The assumption implies that $(r_A)_*$: $[X, \Omega^*A]\to[X, A]$ is a bijection with inverse $(s_A)_*$ and hence there is a homotopy deforming $(\Omega^*p)\tilde{\psi}$ into a map $X\to s_A(A)$. Since Ω^*p is a fibration, it follows that there is a map ψ_0: $X\to\Omega^*_pE$ homotopic to $\tilde{\psi}$ in Ω^*E. Now the composite $\Omega^*_pE\xrightarrow{r}E\xrightarrow{p}A$ is a fibration, so that we can find a map ψ_1: $X\to\Omega^*_pE$ such that $pr\psi_1=u$ and $\psi_0\simeq\psi_1$ in Ω^*_pE. Consequently, if we can show that $r\psi_1$: $X\to E$ is u-homotopic to v, then we infer from 2.4 that $\tau\in\mathrm{I}_u(w)$. Now, since $pr\psi_1=pv$, there is an ω: $X\to F$ such that $\omega\cdot v$ and $r\psi_1$ are u-homotopic, whence $r\psi_1\simeq r_E\tilde{\psi}=v$ implies $\omega\in\mathrm{I}(v)$. Thus, by 2.6, $\omega=0$, which shows that $r\psi_1$ is u-homotopic to v.

Now let

$$p_*\colon [X, E]\to[X, A]\,, \quad q_*\colon [X, T]\to[X, E]$$

be the induced functions and let $v: X\to E$ be liftable to T. We set $u=p\circ v$. We see that $(p\circ q)_*^{-1}(u)$ coincides with the union

$$\bigcup q_*^{-1}(\omega\cdot v)\,,$$

where ω runs over the cosets in $[X, F]/\Delta(\theta, u)[X, \Omega A]$ such that $\nu(\rho)_*\{\omega, v\}+(\rho j)_*\omega=0$. We conclude from Theorem A that

A Classification Theorem 4.4. *Let $\pi=p\circ q$ in* (3.1) *be a stable decomposition with $c: F\times A\to C$ such that $\Delta(\theta, u)$ is injective. Then $(p\circ q)_*^{-1}(u)$ is equivalent to the product*

$$\{\omega\in[X, F];\ c_*\{\omega, u\}=0\}/\Delta(\theta, u)[X, \Omega A]$$
$$\times[X, \Omega C]/\{c'_*\{\beta, u\};\ \beta\in[X, \Omega F]\}\,.$$

Finally we list some properties of $\Phi_\theta(\rho, v)$.

Triviality Theorem 4.5. *$\Phi_\theta(\rho, v)$ is the usual (stable) secondary operation Φ determined by $\Omega\rho: \Omega E\to\Omega C$ and the image of $\Phi_\theta(\rho, v)$ is $(\Omega\rho)_*[X, \Omega E]/\Delta_p(\rho, v)[X, \Omega F]$ in each of the following cases:*

(i) *v is the constant map;*

(ii) *X is an H cogroup;*

(iii) *$p_*v=0$, ρ is primitive with respect to μ and C is homotopy abelian;*

(iv) *θ is primitive and ρ is primitive with respect to H structure of E.*

Proof. (i) follows from the fact that $[X, \Omega^*A; *]$ can be identified with $[X, \Omega A]$.

In order to prove (ii), note that the following diagram is homotopy-commutative:

$$\begin{tikzcd}
X \arrow[r, "d"] \arrow[dr, "g"'] & X\times X \arrow[r, "(\Omega\rho)\omega\times\rho v"] & \Omega C\times C \arrow[r, "n'"] & \Omega^*C \\
 & X\vee X \arrow[u] \arrow[r, "(\Omega\rho)\omega\vee\rho v"] & \Omega C\vee C \arrow[u] \arrow[r, "i_C\vee s_C"] & \Omega^*C\vee\Omega^*C \arrow[u, "d'"']
\end{tikzcd}$$

where g is H' structure map, d the diagonal map, d' the folding map and $\omega: X\to\Omega E$. Let ψ be an element of $[X, \Omega^*E; u]$ which corresponds to $(i_E)_*\omega+(s_E)_*v\in[X, \Omega^*E; *+u]$ under the canonical isomorphism. Then,

$$(\Omega^*\rho)_*\psi=(i_C)_*(\Omega\rho)_*\omega+(s_C)_*\rho_*v=n'_*\{(\Omega\rho)_*\omega,\ \rho_*v\}$$

and we see that $(\Omega^*p)_*\psi$ corresponds to $(i_A)_*(\Omega p)_*\omega+(s_A)_*u$, i.e., $(\Omega^*p)_*\psi=hm'_*\{\alpha, u\}$ with $\alpha=(\Omega p)_*\omega$. These show that $(\Omega\rho)_*\omega$ represents $\Phi_\theta(\rho, v)\alpha$.

We prove (iii). Given $\alpha\in[X, \Omega A]$ with $(\Omega\theta)_*\alpha=0$ (which is equivalent to $(\Omega^*\theta)_* m'_*\{\alpha, *\}=0$), take $\gamma\in[X, \Omega E]$ such that $(\Omega p)_*\gamma=\alpha$. Since $p_*v=0$, there is a $\tau\in[X, F]$ with $j_*\tau=v$. Using the map $\hat{\mu}$: $F\times\Omega^*E\to\Omega^*E$ in the proof of Sublemma, we set $\psi=\hat{\mu}_*\{\tau, (i_E)_*\gamma\}\in[X, \Omega^*E; *]$; then, $(r_E)_*\psi=v$. Thus,

$$\begin{aligned}(\Omega^*\rho)_*\psi &= (\Omega^*\rho)_*\hat{\mu}_*\{\tau, (i_E)_*\gamma\} = \bar{n}_*\{\rho_* j_*\tau, (\Omega\rho)_*\gamma\} \\ &= n'_*\{(\Omega\rho)_*\gamma, \rho_* v\}\,,\end{aligned}$$

where $\bar{n}$: $C\times\Omega C\to\Omega^*C$ is the "left translation". This proves (iii). The proof of (iv) is left to the reader.

Composition Theorem 4.6. *Let* $\sigma: C\to D$ *be a map of* C *to an* H *space* D. *Let*

$$\hat{\Delta}(\sigma, \rho v)\colon \operatorname{coker} \Delta_p(\rho, v) \to \operatorname{coker} \Delta_p(\sigma\rho, v)$$

denote the homomorphism induced by $\Delta(\sigma, \rho v)$. *Then*

$$\Phi_\theta(\sigma\rho, v) = \hat{\Delta}(\sigma, \rho v)\Phi_\theta(\rho, v)\,.$$

Cartesian Product Theorem 4.7. *Suppose* ρ_1 *and* ρ_2 *are as in* 3.7. *Then*

$$\Phi_\theta(\{\rho_1, \rho_2\}, v)\alpha = (\Phi_\theta(\rho_1, v)\alpha, \Phi_\theta(\rho_2, v)\alpha)\,.$$

Cup Product Theorem 4.8. *Let* ρ, ρ' *and* $\rho\cup\rho'$ *be as in* 3.9. *Then*

$$\Phi_\theta(\rho\cup\rho', v)\alpha = \Phi_\theta(\rho, v)\alpha\cup\rho' v+(-1)^a\rho v\cup\Phi_\theta(\rho', v)\alpha\,.$$

Naturality Theorem 4.9. *Let* α *lie in ker* $\Delta(\theta, u)$ *and let* f: $Y\to X$ *be a map. Then*

$$\Phi_\theta(\rho, f^*v)(f^*\alpha) = f^*\Phi_\theta(\rho, v)\alpha \bmod \Delta_p(\rho, f^*v)[Y, \Omega F]\,.$$

5. Proof of Theorem B

Consider a Postnikov tower for $P_n(R)$, n odd>1. In (3.1) we take

$$A=K(Z_2, 1),\quad B=K(Z, n+1),\quad C=K(Z_2, n+2),\quad \theta=(\delta^*\iota_1)^{(n+1)/2}\,,$$

where $\iota_1\in H^1(Z_2, 1; Z_2)$ is the fundamental class; then we have the first two stages and $H^{n+2}(E; Z_2)=Z_2$ whose generator is the second invariant ρ with $j^*\rho=Sq^2\iota_n$, ι_n being the non-zero element of $H^n(Z, n; Z_2)$. We claim that

$$\mu^*(\rho) = \begin{cases} Sq^2\iota_n\times1+1\times\rho+\iota_n\times p^*\iota_1^2 & \text{for}\quad n\equiv1\ (4)\\ Sq^2\iota_n\times1+1\times\rho & \text{for}\quad n\equiv3\ (4)\end{cases}$$

and hence (3.1) in this case is a stable decomposition with $c=Sq^2\iota_n\times1+\iota_n\times\iota_1^2$ or $Sq^2\iota_n\times1$.

Now it follows from Cartan's formula that

$$Sq^2(\delta^*\iota_1)^{(n+1)/2} = Sq^2(\iota_1{}^{n+1}) = \frac{(n+1)n}{2}\iota_1{}^{n+3} = \begin{cases} \iota_1{}^{n+3} & \text{if } n\equiv 1\ (4) \\ 0 & \text{if } n\equiv 3\ (4) \end{cases}$$

We shall use an exact sequence due to E. Thomas [16, p. 187]. We see from the above equality that, for the morphism τ: $H^{n+2}(F\times E,\ E)\to H^{n+3}(A)$,

$$\tau(Sq^2\iota_n\times 1+\iota_n\times p^*\iota_1{}^2) = 0 \quad \text{or} \quad \tau(Sq^2\iota_n\times 1) = 0$$

according as $n\equiv 1$ or 3 (4), and hence there is a $\tilde{\rho}\in H^{n+2}(E;\ Z_2)$ such that

$$\bar{\mu}(\tilde{\rho}) = Sq^2\iota_n\times 1+\iota_n\times p^*\iota_1{}^2 \quad \text{or} \quad Sq^2\iota_n\times 1$$

for the operator $\bar{\mu}$: $H^{n+2}(E)\to H^{n+2}(F\times E,\ E)$ (which is essentially equal to $\mu^*-p_2{}^*$). Since $\ker p^*=\ker l^*$ in $\dim n+2$ for a map l: $P_n(R)\to A$ representing ι, we infer that $\rho=\tilde{\rho}$, which proves our assertion. Theorem B now follows from 4.4 and the fact $[X,\ P_n(R)]\approx[X,\ T]$.

6. Proof of Theorem C

Consider the first two stages of a Moore-Postnikov tower for the inclusion $BO(n)\subset BO$ between the classifying spaces for n-plane bundles and stable ones:

$$\begin{tikzcd} & T \arrow[d, "q"] & \\ K(Z_2,\ n) \arrow[r, "j"] & E \arrow[r] \arrow[d, "p"] & K(Z_2,\ m) \\ & BO \arrow[r, "w_{n+1}"] & K(Z_2,\ n+1)\ , \end{tikzcd}$$

where $m=n+2$ or $n+3$ according as $n\equiv 1$ (4) or $n\equiv 3$ (4), $n>2$, and w_i denotes the universal Stiefel-Whitney class of dimension i. As shown in [7, p. 110], $p\circ q$ forms a stable decomposition with

$$c = \begin{cases} Sq^2\iota_n\times 1+\iota_n\times w_2 & \text{if } n\equiv 1\ (4) \\ Sq^2Sq^1\iota_n\times 1+Sq^1\iota_n\times(w_1{}^2+w_2) & \text{if } n\equiv 3\ (4)\,, \end{cases}$$

where $\iota_n\in H^n(Z_2,\ n;\ Z_2)$ is the fundamental class (This can also be shown using Thomas' exact sequence). Thus, we conclude from 4.4 that, for a stable bundle $\xi\in\widetilde{KO}(X)$ with $w_{n+1}(\xi)=0$ such that $\Delta(w_{n+1},\ \xi)$: $\widetilde{KO}^{-1}(X)\to H^n(X;\ Z_2)$ is injective,

(a) if $n\equiv 1$ (4) and $\dim X\leqq n+1$, then $N_n(\xi;\ X)$ is equal to the cardinal of the direct product:

$$\operatorname{Coker}\Delta(w_{n+1},\ \xi)\times H^{n+1}(X;\ Z_2)/(Sq^2+w_2(\xi)\cup)H^{n-1}(X;\ Z_2)\,.$$

(b) if $n\equiv 3$ (4) and $\dim X\leqq n+2$, then $N_n(\xi;\ X)$ is equal to the cardinal of the direct product:

$$\operatorname{Coker} \Delta(w_{n+1}, \xi) \times H^{n+2}(X; Z_2) / (Sq^2 + (w_1(\xi)^2 + w_2(\xi)) \cup) Sq^1 H^{n-1}(X; Z_2).$$

Now take $X = P_{n+1}(R)$ or $P_{n+2}(R)$ according as $n \equiv 1$ or 3 (4). Then $\widetilde{KO}^{-1}(X) = Z_2$ by [3] and it follows from a formula of [5, p. 489] that $\Delta(w_{n+1}, k\eta)$ is injective for $\binom{k-1}{n-1}$ odd and $w_{n+1}(k\eta) = 0$ for $\binom{k}{n+1}$ even. Theorem C will be obtained by observing that

$$w_2(k\eta) = \binom{k}{2} x^2, \quad Sq^2 x^{n-1} = 0 \text{ for } n \equiv 1 \ (4), \quad Sq^1 x^{n-1} = 0 \text{ for odd } n,$$

where x denotes the non-zero element of $H^1(X; Z_2)$.

Note that, for $n \equiv 1$ (4) and $\binom{k}{2}$ odd, $\Delta_p(\rho, v)$: $H^{n-1}(P_{n+1}(R); Z_2) \to H^{n+1}(P_{n+1}(R); Z_2)$ is surjective for any lift v of $k\eta$ and hence q_*: $[P_{n+1}(R), T] \to [P_{n+1}(R), E]$ is bijective.

As another illustration, let $n \equiv 1$ (4) and let X be $P_{(n+1)/2}(C)$, the complex projective space of complex dimension $\frac{n+1}{2}$. Then $\widetilde{KO}^{-1}(X) = 0$ by [3]. Since $H^n(X; Z_2) = 0$ and $Sq^2(y^{(n-1)/2}) = 0$ for the non-zero element y of $H^2(X; Z_2)$, we see that $N_n(\xi; X) = 1$ or 2 according as $w_2(\xi) \neq 0$ or $w_2(\xi) = 0$.

7. Further examples

7.1. Suppose that, in (3.1), A is an $(n-1)$-connected space such that $\pi_k(A) = 0$ for $k \geqq n + n' - 2$ $(n' > n \geqq 2)$, $B = K(\pi', n'+1)$ and $C = K(G, n+n')$. Assume $\rho \in H^{n+n'}(E; G)$ represents $(\bar{\rho} + \psi \cup)_p(\theta)$ for $\bar{\rho} \in H^{n+n'+1}(\pi', n'+1; G)$, $\psi \in H^n(A; \pi)$, where the cup product is taken with respect to the Whitehead product pairing $\pi \otimes \pi' \to G$ in T, $\pi = \pi_n(T)$. Then it is proven by F. P. Peterson [11] that

$$\mu^*(\rho) = j^*\rho \times 1 + 1 \times \rho + x' \times p^*(\psi), \qquad x' \in H^{n'}(\pi', n; \pi'),$$

where $j^*\rho$ is the suspension of $\bar{\rho}$ and $j^*\rho(x') = j^*\rho(\iota')$, ι' being the fundamental class of $H^{n'}(\pi', n'; \pi')$. Thus, *the tower $p \circ q$ in this case is a stable decomposition with $c = j^*\rho \times 1 + x' \times \psi$* (cf. Theorem 3.1 of [4]). Hence, $\Delta_p(\rho, v)\beta = \Omega(\rho j)_*\beta + (\Omega x')\beta \cup u^*\psi$, $\beta \in H^{n'-1}(X; \pi')$.

In case $A = K(\pi, n)$, we can take the basic classes ι' and $\iota \in H^n(\pi, n; \pi)$ for x' and ψ respectively.

7.2. Consider a Postnikov tower for the usual lens space $L = S^{2n+1}/Z_p$, where p is an *odd* prime. We take

$$A = K(Z_p, 1), \quad B = K(Z, 2n+2), \quad C = K(Z_2, 2n+3),$$

and let $\theta=(\delta^*\iota)^{n+1}$ where δ^* is the Bockstein associated with $0\to Z\to Z\to Z_p\to 0$ and ι denotes the basic class of $H^1(Z_p, 1; Z_p)$, and let ρ denote the generator of $H^{2n+3}(E; Z_2)=Z_2$.

Given a *path-connected* $(2n+2)$-dimensional complex X, we have $\Delta(\theta, u)=0$ by virtue of $H^1(SX; Z_p)=0$ and $j^*\rho=Sq^2\bar{\iota}$, $\bar{\iota}$ being the basic class of $H^{2n+1}(Z, 2n+1; Z)$. Since $H^{2n+3}(F\times E, F\vee E; Z_2)=0$, ρ is primitive with respect to the action μ, so that $\Delta_p(\rho, v)=Sq^2\colon H^{2n}(X; Z)\to H^{2n+2}(X; Z_2)$. Thus, we see from 4.4 that $[X, L]$ is equivalent to the product

$$\{u\in H^1(X; Z_p);\ (\delta^*u)^{n+1}=0\}\times H^{2n+1}(X; Z)\times H^{2n+2}(X; Z_2)/Sq^2H^{2n}(X; Z)\,.$$

This extends a result of P. Olum [10].

7.3. Consider a Postnikov tower for the n-sphere S^n, $n\geqq 4$. We take

$$A=K(Z, n),\quad B=K(Z_2, n+2),\quad C=K(Z_2, n+3)$$

and $Sq^2\iota$ and the unique non-zero element of $H^{n+3}(E; Z_2)=Z_2$ for θ and ρ, where ι is the basic class of $H^n(Z, n; Z)$. Then,

$$\Delta(\theta, u)=Sq^2\colon H^{n-1}(X; Z)\to H^{n+1}(X; Z_2)$$
$$\Delta_p(\rho, v)=Sq^2\colon H^n(X; Z_2)\to H^{n+2}(X; Z_2)\,.$$

Let X be a complex with dim $X\leqq n+2$; then $v\colon X\to E$ is always liftable to T. We conclude

(1) (Nakaoka [8, p. 94, Theorem 4]) Assume $Sq^2\colon H^n(X; Z_2)\to H^{n+2}(X; Z_2)$ is surjective; then $I(w)=I_u(w)=H^{n+2}(X; Z_2)$ and hence it follows that $[X, S^n]$ is equivalent to

$$\{u\in H^n(X; Z);\ Sq^2u=0\}\times H^{n+1}(X; Z_2)/Sq^2H^{n-1}(X; Z)\,.$$

(2) Assume $Sq^2\colon H^{n-1}(X; Z)\to H^{n+1}(X; Z_2)$ is injective; then it follows from 4.4 that $[X, S^n]$ is equivalent to the product

$$\{u\in H^n(X; Z);\ Sq^2u=0\}\times H^{n+1}(X; Z_2)/Sq^2H^{n-1}(X; Z)$$
$$\times H^{n+2}(X; Z_2)/Sq^2H^n(X; Z_2)\,.$$

7.4. Consider a Postnikov tower for the complex projective space $P_m(C)$. Let

$$A=K(Z, 2),\quad B=K(Z, 2m+2),\quad C=K(Z_2, 2m+3)\,,$$

and let $\theta=\iota^{m+1}$, where $\iota\in H^2(Z, 2; Z)$ is the basic class, and ρ be the unique non-zero element $\psi_m\in H^{2m+3}(E; Z_2)$ (cf. [12]). Then $j^*\rho=Sq^2\iota_{2m+1}$, where ι_{2m+1} is the generator of $H^{2m+1}(Z, 2m+1; Z_2)$. The dual Hopf invariant $\nu(\theta)$ with respect to H-structure of A is $(\iota\times 1+1\times\iota)^{m+1}$ and hence

$$\Delta(\theta, u)\alpha = (m+1)\alpha \cup u^m, \quad \alpha \in H^1(X; Z) \qquad \text{(cf. [15, p. 452])}$$

We see from 7.1 and [11] that

$$\mu^*(\rho) = \begin{cases} \rho j \times 1 + 1 \times \rho & \text{if } m \text{ is odd} \\ \rho j \times 1 + 1 \times \rho + \bar{\iota}_{2m+1} \times p^*\iota & \text{if } m \text{ is even}, \end{cases}$$

where the cross product is taken with respect to the nontrivial pairing $Z \otimes Z \to Z_2$ and $\bar{\iota}_{2m+1}$ denotes the basic class of $H^{2m+1}(Z, 2m+1; Z)$.

Given a $(2m+2)$-dimensional complex X, we assume that $(m+1)\alpha \cup u^m = 0$ implies $\alpha = 0$ for $\alpha \in H^1(X; Z)$ and a fixed $u \in H^2(X; Z)$. Then it follows from 4.4 that, for the function φ: $[X, P_m(C)] \to H^2(X; Z)$ assigning f^*z to f: $X \to P_m(C)$, z being a generator of $H^2(P_m(C); Z)$, $\varphi^{-1}(u)$ is equivalent to

$$H^{2m+1}(X; Z)/(m+1)u^m \cup H^1(X; Z) \times H^{2m+2}(X; Z_2)/Sq^2 H^{2m}(X; Z) \qquad \text{for } m \text{ odd}$$

$$H^{2m+1}(X; Z)/(m+1)u^m \cup H^1(X; Z) \times H^{2m+2}(X; Z_2)/(Sq^2 + u \cup) H^{2m}(X; Z) \qquad \text{for } m \text{ even}.$$

It seems likely that, for $m=1$ and dim $X=4$, our $\Phi_\theta(\rho, v)$ coincides with Φ_{C_2} introduced by N. Shimada [14, p. 141].

7.5. Let n be an even integer and let

$$\begin{array}{ccccc} K(Z_2, 2n+2) & \longrightarrow & T & & \\ & & \downarrow{\scriptstyle q} & & \\ K(Z, 2n+1) & \xrightarrow{j} & E & \xrightarrow{\rho} & K(Z_2, 2n+3) \\ & & \downarrow{\scriptstyle p} & & \\ & & BU & \xrightarrow{c_{n+1}} & K(Z, 2n+2) \end{array}$$

be part of a Moore-Postnikov tower for $BU(n) \subset BU$ between the classifying spaces for the unitary groups $U(n)$ and U, where c_{n+1} denotes the universal $(n+1)$ th Chern class. It is readily shown that $H^{2n+3}(E; Z_2) = Z_2$ is generated by ρ with $j^*\rho = Sq^2 \iota_{2n+1}$, where ι_{2n+1} is the generator of $H^{2n+1}(Z, 2n+1; Z_2)$.

Since, for the realification $\hat{\gamma}$ of the canonical bundle γ over BU,

$$Sq^2 c_{n+1} = Sq^2 w_{2n+2}(\hat{\gamma}) = w_2(\hat{\gamma}) \cup c_{n+1}$$

where the cup product is with respect to the non-trivial pairing $Z_2 \otimes Z \to Z_2$, Thomas' exact sequence reveals that

$$\mu^*\rho = Sq^2 \iota_{2n+1} \times 1 + 1 \times \rho + \iota_{2n+1} \times p^* w_2(\hat{\gamma}).$$

Hence it follows from 4.4 that, for a complex X such that dim $X \leq 2n+2$ and $\Delta(c_{n+1}, u)$: $\tilde{K}^{-1}(X) \to H^{2n+1}(X; Z)$ is injective for $u \in \tilde{K}(X)$ with $c_{n+1}(u) = 0$, the number of n-dimensional complex vector bundles over X which are stably equivalent to u, is equal to the cardinal of the direct product

$$\operatorname{coker} \Delta(c_{n+1}, u) \times H^{2n+2}(X; Z_2)/(Sq^2 + w_2(\hat{u}) \cup) H^{2n}(X; Z),$$

where $\hat{u}$ is the realification of u.

For example, let $X = P_{2n+2}(R)$. Since $\tilde{K}^{-1}(X) = 0$ by Theorem 3.3 of [2] and since $K(X)$ consists of elements $k\nu$ $(k = 0, 1, \cdots, 2^{n-1} - 1)$, ν denoting the complexification of the canonical line bundle λ (see [1]), the number of classes of n-dimensional complex plane bundles which are stably equivalent to $k\nu$, is equal to 2 or 4 according as k is odd or even. This follows by observing that $Sq^2 x^{2n} = 0$ for the generator $x \in H^1(X; Z_2)$ and $w_2(k\hat{\nu}) = kw_2(2\lambda) = kx^2$.

8. Appendix: the group of fibre homotopy equivalences

Given a fibration $f: Y \to Z$, we denote by $\mathcal{E}(Y; f)$ the group of fibre homotopy classes of fibre homotopy equivalences of Y.

In the situation (3.1) we shall assume that $\pi_k(A) \neq 0$ only for $n \leqq k \leqq n' - 1$, $\pi_k(F) \neq 0$ only for $n \leqq k \leqq n' - 1$ and $\pi_r(\Omega C) \neq 0$ only for $n' \leqq r \leqq n' + n - 1$ $(n > 1)$. It is easily shown that there is an exact sequence

$$1 \to q^*[E, \Omega C] \to \mathcal{E}(T; q) \to \mathcal{E}(\Omega C).$$

We shall study $\mathcal{E}(T; p \circ q)$. First we need

Lemma 8.1. *The functions*

$$q^*: [E, E; p] \to [T, E; p \circ q], \quad i_*: [\Omega C, \Omega C] \to [\Omega C, T]$$

are bijective.

Proof. Introduce the commutative diagram

$$\begin{array}{ccc} [E, F] & \xrightarrow{q^*} & [T, F] \\ {\scriptstyle T_1}\downarrow & & \downarrow{\scriptstyle T_2} \\ [E, E; p] & \xrightarrow{q^*} & [T, E; p \circ q], \end{array}$$

where the vertical bijections T_1 and T_2 are given by

$$T_1(\tau) = \mu_*\{\tau, 1_E\}, \quad T_2(\omega) = \mu_*\{\omega, q\},$$

1_X being the identity map of X. Since the upper q^* is bijective, so is the bottom q^*. The second assertion can be proved by a classical obstruction argument or by using a Moore-Postnikov tower for i.

In the light of Lemma 8.1 we can now define homomorphisms

$$J: \mathcal{E}(T; p \circ q) \to \mathcal{E}(E; p), \quad J_0: \mathcal{E}(T; p \circ q) \to \mathcal{E}(\Omega C)$$

by requiring, for $g \in \mathcal{E}(T; p\circ q)$,

$$q_* g = q^* J(g) \quad \text{in} \quad [T, E; p\circ q], \qquad i_* J_0(g) = i^* g .$$

Let

$$\Delta\colon \ker i^* \to \mathcal{E}\,(T; p\circ q)$$

denote the homomorphism defined by $\Delta(\tau) = \tau \cdot 1_T$, where i^* is the homomorphism in the exact sequence

$$[\Omega C, \Omega C] \xleftarrow{i^*} [T, \Omega C] \xleftarrow{q^*} [E, \Omega C] .$$

Theorem 8.2. *The following sequence of groups and homomorphisms is exact:*

$$[T, \Omega F] \xrightarrow{\Delta_p(\rho, q)} q^*[E, \Omega C] \xrightarrow{\Delta} \mathcal{E}(T; p\circ q) \xrightarrow{\{J, J_0\}} \mathcal{E}(E; p) \times \mathcal{E}(\Omega C) ,$$

in which the image of $\{J, J_0\}$ *consists of* $(\bar{g}, \Omega h) \in \mathcal{E}(E; p) \times \mathcal{E}(\Omega C)$ *such that* $\rho \bar{g} \simeq h\rho$.

Proof. The exactness at the second term follows from the fact that the image of $\Delta_p(\rho, q)$ coincides with $\mathrm{I}_{p\circ q}(1_T)$ by 3.2. We shall prove the exactness at the third term.

Let $g\colon T \to T$ be a homotopy equivalence such that $gi \simeq i$, $pqg = pq$ and $qg \simeq q$ by a pq-homotopy $H_t\colon T \to E$, $0 \leqq t \leqq 1$, with $H_0 = qg$, $H_1 = q$. By the homotopy lifting property there exists a homotopy $\tilde{H}_t\colon T \to T$ with $\tilde{H}_0 = g$, $q\tilde{H}_t = H_t$. Since $pq\tilde{H}_t = pH_t = pq$, $\tilde{H}_t$ is a pq-homotopy. Put $g' = \tilde{H}_1$, then $qg' = q$ and so g' is q-homotopic to $\tau \cdot 1_T$ for some $\tau\colon T \to \Omega C$. Since

$$(\tau i) \cdot i \simeq g' i \simeq gi \simeq i$$

and $[\Omega C, \Omega E] = 0$, it follows from 2.3 that $\mathrm{I}(i) = 0$ and hence $i^* \tau = 0$.

The assertion about the image of $\{J, J_0\}$ can be proved by an argument similar to Theorem 2.9 of [9], noting that, if $qg \simeq \bar{g}q$ by a pq-homotopy, we can replace g by $\hat{g}$ which is pq-homotopic to g and which is such that $q\hat{g} = \bar{g}q$.

Consider the situation in which $A = K(\pi, n)$, $B = K(\pi', n'+1)$ and $C = K(G, n+n')$, $1 < n < n'$ in (3.1). Theorem 8.2, together with 7.1, gives rise to an exact sequence

$$1 \to R \to \mathcal{E}(T; p\circ q) \to \mathcal{E}(E; p) \times \mathrm{Aut}\, G ,$$

where R denotes the factor group

$$H^{n+n'-1}(E; G)/p^*(\Omega(\rho j) + \iota_n \cup) H^{n'-1}(\pi, n; \pi') ,$$

ι_n being the basic class in $H^n(\pi, n; \pi)$ and the cup product being taken with respect to the Whitehead product pairing of T.

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