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RING STRUCTURES OF K_U-COHOMOLOGIES OF DOLD MANIFOLDS

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Introduction

In [4] we determined the K_U-cohomologies of the Dold manifold $D(m, n)$ additively. But we could not determine the ring structures of them, because we could not find a generator of the 2-torsion part in $\tilde{K}_U^1(D(m, 2r+1))$. The purpose of this paper is to determine the ring structures of K_U-cohomologies of the Dold manifold $D(m, n)$. The stunted Dold manifold plays an important role in the present discussions.

Let S^k, $k \geqq 0$, denote the unit k-sphere in R^{k+1}, each point of which is represented by a sequence $(x_0, \cdots, x_k)$ of real numbers x_i with $\sum x_i^2=1$, and S^{2l+1}, $l \geqq 0$, denote the unit $(2l+1)$-sphere in C^{l+1}, each point of which is represented by a sequence $(z_0, \cdots, z_l)$ of complex numbers z_i with $\sum |z_i|^2=1$. Then the Dold manifold $D(k, l)$ is defined as the quotient space of the product space $S^k \times S^{2l+1}$ under the identification $(x, z)=(-x, \overline{\lambda z})$ for $x \in S^k$, $z \in S^{2l+1} \subset C^{l+1}$ and all $\lambda \in C$ with $|\lambda|=1$. Let $[x_0, \cdots, x_k, z_0, \cdots, z_l] \in D(k, l)$ denote the class of $(x_0, \cdots, x_k, z_0, \cdots, z_l) \in S^k \times S^{2l+1}$. The manifold $D(k', l')$, $k' \leqq k$ and $l' \leqq l$, is naturally imbedded in $D(k, l)$ by identifying $[x_0, \cdots, x_{k'}, z_0, \cdots, z_{l'}]$ with $[x_0, \cdots, x_{k'}\, 0, \cdots, 0, z_0, \cdots, z_{l'}, 0, \cdots, 0]$.

Denote by ξ the canonical real line bundle over the real projective k-space $RP(k)$, and $\xi_1=p^!\xi$ the induced bundle of ξ by the projection $p: D(k, l) \rightarrow RP(k)$; and denote by η_1 the canonical real 2-plane bundle over $D(k, l)$ (cf. [4], § 2).

Theorem 1. *The Thom space $T(m\xi_1 \oplus n\eta_1)$ and the stunted Dold manifold $D(k+m, l+n)/D(m-1, l+n) \cup D(k+m, n-1)$ are homeomorphic, where $m\xi_1$ and $n\eta_1$ are the m-fold and n-fold sum of ξ_1 and η_1 respectively.*

From this theorem we have the following

Proposition 2. *We have the following homeomorphisms:*

i) h: $D(k, n)/D(k, n-1) \approx S^n \wedge (RP(n+k)/RP(n-1))$,

ii) $D(m, l)/D(m-1, l) \approx S^m \wedge CP(l)^+$,

where $S^n \wedge (RP(n+k)/RP(n-1))$ *is the n-fold suspension of the stunted real projective space, and* $CP(l)^+$ *is the disjoint union of the complex projective l-space* $CP(l)$ *and a point.*

Let g is the generator of $\tilde{K}_U^0(S^2)$ given by the reduced Hopf bundle and $g^{[r]}$ is the generator of $\tilde{K}_U^0(S^{2r})$ given by the external product $g \wedge \cdots \wedge g$. Also, let $\nu^{(r+1)}$ is the generator of $\tilde{K}_U^0(RP(2r+s)/RP(2r))$ (cf. [1], Theorem 7.3), then $g^{[r]}\nu^{(r+1)}$ is the generator of $\tilde{K}_U^{-2r}(RP(2r+s)/RP(2r))$. Now, using Proposition 2, i), we can define a generator ω of the 2–torsion part in $\tilde{K}_U^1(D(m, 2r+1))$ as follows: $\omega = \pi^! h^! g^{[r]}\nu^{(r+1)}$, where π is the projection $D(m, 2r+1) \to D(m, 2r+1)/D(m, 2r)$, and determine the multiplicative structures of $\tilde{K}_U^*(D(m, n))$, namely

Theorem 3. *As for the ring structures of* $\tilde{K}_U^*(D(m, n))$ *we have the following relations:*

a) $\gamma^2 = g'^2 = \beta^2 = g'\beta = 0, \quad g'\alpha = 2\beta$,

b) $\alpha^{r+1} = 0, \quad \gamma\alpha^r = 0$ (*for* $n = 2r$) or $\gamma\alpha^{r+1} = 0$ (*for* $n = 2r+1$),
$\beta\alpha^r = 0$ (*for* $n = 2r$) or $\beta\alpha^r = 2^t\omega$ (*for* $n = 2r+1$),

c) $\alpha\nu_1 = \gamma\nu_1 = g'\nu_1 = \beta\nu_1 = 0, \quad \alpha\omega = \gamma\omega = g'\omega = \beta\omega = 0$,

d) $\nu_1^2 = -2\nu_1, \quad \omega\nu_1 = -2\omega, \quad \omega^2 = 0$,

where α, γ, g', β *and* ν_1 *are the generators given in* [4], *Theorem* (3.14), *and* ω *is the generator of the* 2–*torsion part in* $\tilde{K}_U^1(D(m, 2r+1))$ *given by the above formula.*

1. Proof of Theorem 1

The total space $E(m\xi_1 \oplus n\eta_1)$ of $m\xi_1 \oplus n\eta_1$ is the quotient space of the product space $S^k \times S^{2l+1} \times R^m \times C^n$ under the identification $((x, z), (u, v)) = ((-x, \overline{\lambda z}), (-u, \overline{\lambda v}))$ for $x \in S^k$, $z \in S^{2l+1} \subset C^{l+1}$, $u \in R^m$, $v \in C^n$ and all $\lambda \in C$ with $|\lambda| = 1$. Moreover, the associated unit disk bundle $D(m\xi_1 \oplus n\eta_1)$ is homeomorphic to the quotient space of the product apace $S^k \times S^{2l+1} \times D^m \times D^{2n}$ under the identification $((x, z), (u, v)) = ((-x, \overline{\lambda z}), (-u, \overline{\lambda v}))$, where $x \in S^k$, $z \in S^{2l+1} \subset C^{l+1}$, $u \in D^m$, $v \in D^{2n} \subset C^n$ and λ is as above. Let $[(x, z), (u, v)]$ denote the class of $((x, z), (u, v))$ in $D(m\xi_1 \oplus n\eta_1)$. Then $[(x, z), (u, v)]$ is an element of the associated unit sphere bundle $S(m\xi_1 \oplus n\eta_1)$ if and only if $||u|| = 1$ or $||v|| = 1$.

We define a map

$$f: S^k \times S^{2l+1} \times D^m \times D^{2n} \to S^{k+m} \times S^{2l+2n+1}$$

by

$$f((x, z), (u, v)) = ((u, \sqrt{1-\|u\|^2}x), (v, \sqrt{1-\|v\|^2}z)) .$$

Since

$$f((-x, \overline{\lambda z}), (-u, \overline{\lambda v})) = (-(u, \sqrt{1-\|u\|^2}x), \overline{\lambda(v, \sqrt{1-\|v\|^2}z)}) ,$$

the map f defines a map

$$g\colon D(m\,\xi_1 \oplus n\eta_1) \to D(k+m, l+n)$$

such that $g(S(m\xi_1 \oplus n\eta_1)) \subset D(m-1, l+n) \cup D(k+m, n-1)$. The map

$$g\colon D(m\xi_1 \oplus n\eta_1) - S(m\xi_1 \oplus n\eta_1) \to D(k+m, l+n) - D(m-1, l+n) \\ \cup D(k+m, n-1)$$

is a homeomorphism. Therefore, the map g defines a quotient map

$$h\colon T(m\,\xi_1 \oplus n\eta_1) \to D(k+m, l+n)/D(m-1, l+n) \cup D(k+m, n-1)$$

which is a homeomorphism.

2. Proof of Proposition 2

i). By taking $m=l=0$ in Theorem 1, we have the homeomorphism

$$T(n\eta_1) \approx D(k, n)/D(k, n-1) .$$

Since η_1 over $D(k, 0)$ is the 2-plane bundle $1 \oplus \xi_1$ (cf. [4], Theorem (2.2)), we have

$$T(n\eta_1) = T(n \oplus n\xi_1) \approx S^n \wedge T(n\xi_1) .$$

If we identify $D(k, 0)$ with $RP(k)$, the line bundle ξ_1 is the canonical line bundle ξ over $RP(k)$. Therefore we have the homeomorphism

$$T(n\xi_1) \approx RP(n+k)/RP(n-1) .$$

Combining the above three homeomorphisms, we have the homeomorphism

$$h\colon D(k, n)/D(k, n-1) \approx S^n \wedge (RP(n+k)/RP(n-1)) .$$

ii). By taking $n=k=0$ in Theorem 1, we have the homeomorphism

$$T(m\xi_1) \approx D(m, l)/D(m-1, l) .$$

Since ξ_1 over $D(0, l)$ is the trivial line bundle, if we identify $D(0, l)$ with $CP(l)$, we have

$$T(m\xi_1) = T(m) \approx S^m \wedge CP(l)^+ .$$

Therefore we have the homeomorphism

$$D(m, l)/D(m-1, l) \approx S^m \wedge CP(l)^+ .$$

3. Proof of Theorem 3

Firstly we show that ω is a generator of the 2-torsion part in $\tilde{K}^1_U(D(m, 2r+1))$. Consider the exact sequence of the pair $(D(2t, 2r+1), D(2t, 2r))$

$$\tilde{K}^1_U(D(2t, 2r+1)/ D(2t, 2r)) \to \tilde{K}^1_U(D(2t, 2r+1)) \to \tilde{K}^1_U(D(2t, 2r)) .$$

According to [4], Theorem (3.14), we have $\tilde{K}^1_U(D(2t, 2r))=0$ and $\tilde{K}^1_U(D(2t, 2r+1))=Z_{2^t}$. Also, in virtue of Proposition 2, i), and [1], Theorem 7.3, we have

$$\tilde{K}^1_U(D(2t, 2r+1)/D(2t, 2r)) \cong \tilde{K}^{-2r}_U(RP(2t+2r+1)/RP(2r)) \cong Z_{2^t} ,$$

whose generator is $g^{[r]}\nu^{(r+1)}$. Therefore, ω is the generator of $\tilde{K}^1_U(D(2t, 2r+1))$.

Using the exact sequence of the pair $(D(2t+1, 2r+1), D(2t+1, 2r))$

$$\tilde{K}^1_U(D(2t+1, 2r+1)/D(2t+1, 2r)) \to \tilde{K}^1_U(D(2t+1, 2r+1)) \to \tilde{K}^1_U(D(2t+1, 2r)),$$

it is easy to see that ω is the generator of the 2-torsion part $Z_{2^{t+1}}$ of $\tilde{K}^1_U(D(2t+1, 2r+1))$ in the same way as the above case.

Next we show the relations. Since $(g^{[k]})^2=0$ in $\tilde{K}^0_U(S^{2k})$, the relations $\gamma^2=g'^2=\beta^2=g'\beta=0$ and $\omega^2=0$ follow from $g'=(sf)^! g^{[t+1]}$, $\beta=(sf)^! g^{[t+1]}\mu$, $\gamma=f^! g^{[t]}\mu$ and $\omega=\pi^! h^! g^{[r]}\nu^{(r+1)}$. The relation $\nu_1{}^2=-2\nu_1$ follows from the relation $\nu^2=-2\nu$ in $\tilde{K}^0_U(RP(m))$.

Since $\tilde{K}^1_U(D(2t+1, 2r))$ has no torsion, Chern character ch: $\tilde{K}^1_U(D(2t+1, 2r)) \to H^*(D(2t+1, 2r); Q)$ is monomorphic. Therefore the relations $g'\alpha=2\beta$ and $\beta\alpha^r=0$ follow from

$$\mathrm{ch}\, g'\alpha = 2b(a/2!+\cdots+a^r/(2r)!) = 2\,\mathrm{ch}\,\beta \quad \text{and} \quad \mathrm{ch}\,\beta\alpha^r = 0$$

respectively. The relation $g'\nu_1=\beta\nu_1=0$ is trivial for $n=2r$.

In case of $n=2r-1$, since the elements α, ν_1, g' and β of $\tilde{K}^*_U(D(2t+1, 2r-1))$ are induced from the elements α, ν_1, g' and β of $\tilde{K}^*_U(D(2t+1, 2r))$ by the inclusion map i: $D(2t+1, 2r-1) \subset D(2t+1, 2r)$, multiplicativity of the homomorphism $i^!$ shows the relations $g'\nu_1=\beta\nu_1=0$ and $g'\alpha=2\beta$ for $n=2r-1$. Also, the element $\beta\alpha^{r-1} \in \tilde{K}^1_U(D(2t+1, 2r-1))$ is the image of $\beta\alpha^{r-1} \in \tilde{K}^1_U(D(2t+1, 2r))$ by $i^!$. On the other hand, consider the exact sequence

$$\tilde{K}^1_U(D(2t+1, 2r)) \xrightarrow{i^!} \tilde{K}^1_U(D(2t+1, 2r-1)) \to \tilde{K}^2_U(D(2t+1, 2r)/D(2t+1, 2r-1)) .$$

In virtue of Proposition 2, i), and [1], Theorem 7.3, we have

$$\tilde{K}^2_U(D(2t+1, 2r)/D(2t+1, 2r-1)) \cong \tilde{K}^{-2r+2}_U(RP(2t+2r+1)/RP(2r-1)) \cong Z+Z_{2^t} ,$$

so that we have $i^!\beta\alpha^{r-1}=2^t\omega$ for $\beta\alpha^{r-1}\in\tilde{K}_U^1(D(2t+1, 2r)$. Therefore we have the relation $\beta\alpha^{r-1}=2^t\omega$ in $\tilde{K}_U^1(D(2t+1, 2r-1))$.

Since $\mathrm{ch}\,\alpha^{r+1}=0$ and $\mathrm{ch}\,\gamma\alpha^r=0$ (for $n=2r$) ($\mathrm{ch}\,\gamma\alpha^{r+1}=0$ (for $n=2r+1$)), the elements α^{r+1} and $\gamma\alpha^r$ (for $n=2r$) ($\gamma\alpha^{r+1}$ (for $n=2r+1$)) lie in $p^!\tilde{K}_U^0(RP(m))$. Therefore the relation $r^!\alpha=0$ implies $\alpha^{r+1}=p^!r^!\alpha^{r+1}=0$ and $\gamma\alpha^r=p^!r^!(\gamma\alpha^r)=p^!((r^!\gamma)(r^!\alpha^r))=0$ (for $n=2r$) ($\gamma\alpha^{r+1}=p^!r^!(\gamma\alpha^{r+1})=0$ (for $n=2r+1$)), where r is the cross section defined in [4], Lemma (3.4).

Since $\gamma\nu_1\in p^!\tilde{K}_U^0(RP(2t))$ and $r^!\gamma=0$, we have $\gamma\nu_1=p^!r^!(\gamma\nu_1)=p^!((r^!\gamma)(r^!\nu_1))=0$. The relation $\alpha\nu_1=0$ was showed in [4].

The elements $g'\omega$ and $\beta\omega$ lie in $p^!\tilde{K}_U^0(RP(2t+1))$. Since the diagram

$$\begin{array}{ccc}
\tilde{K}_U^1(D(2t+1,\ 2r+1)/D(2t+1,2r)) & \xrightarrow{\pi^!} & \tilde{K}_U^1(D(2t+1,\ 2r+1)) \\
r^!\downarrow\uparrow p^! & & r^!\downarrow\uparrow p^! \\
\tilde{K}_U^1(*) & \xrightarrow{\pi^!} & \tilde{K}_U^1(RP(2t+1))
\end{array}$$

is commutative, we have $r^!\omega=\pi^!r^!(h^!g^{[r]}\nu^{(r+1)})=0$. Therefore we have $g'\omega=p^!r^!(g'\omega)=p^!((r^!g')(r^!\omega))=0$ and $\beta\omega=p^!r^!(\beta\omega)=p^!((r^!\beta)(r^!\omega))=0$.

Finally we show the relations $\omega\alpha=0$, $\omega\gamma=0$ and $\omega\nu_1=-2\omega$ in $\tilde{K}_U^1(D(m, 2r+1))$. For simplicity we put $Y_1=RP(m+2r+1)$, $Y_2=RP(2r)$, $X_1=D(m, 2r+1)$, $X_2=D(m, 2r)$ and $Z=D(m+2r+1, 2r+1)$.

Lemma 1. *We have the homotopy-commutative diagram*

$$\begin{array}{ccccccc}
X_1\wedge X_1 & \xrightarrow{\pi\wedge 1} & (X_1/X_2)\wedge X_1 & \underset{\approx}{\xrightarrow{h\wedge 1}} & S^{2r+1}\wedge(Y_1/Y_2)\wedge X_1 & \xrightarrow{1\wedge 1\wedge i} & S^{2r+1}\wedge(Y_1/Y_2)\wedge Z \\
\uparrow d_1 & & \uparrow \bar{d}_1 & & \uparrow 1\wedge 1\wedge r & & \uparrow 1\wedge 1\wedge r \\
 & & & & S^{2r+1}\wedge(Y_1/Y_2)\wedge RP(m) & & S^{2r+1}\wedge(Y_1/Y_2)\wedge Y_1 \\
 & & & & \uparrow q & & \uparrow 1\wedge d_2 \\
X_1 & \xrightarrow{\pi} & X_1/X_2 & \underset{\approx}{\xrightarrow{h}} & S^{2r+1}\wedge(Y_1/Y_2) & = & S^{2r+1}\wedge(Y_1/Y_2),
\end{array}$$

where i is the inclusion map $X_1\subset Z$, h is the homeomorphism of Proposition 2, i), *d_1 is the diagonal map, $\bar{d}_1$ and d_2 are the maps induced by the diagonal maps, r is the cross section of* [4], *Lemma* (3.4), *and q is the map given by*

$$q([a]\wedge[b, \sqrt{1-||b||^2}x])=[a]\wedge[b, \sqrt{1-||b||^2}x]\wedge[x]\,.$$

Proof. It is sufficient to show the followings:

i) the maps $u=(1\wedge 1\wedge i)\circ(h\wedge 1)\circ\bar{d}_1\circ h^{-1}$ and $v=(1\wedge 1\wedge r)\circ(1\wedge d_2)$ are homotopic,

ii) the map q is well defined and the maps $\bar{u}=(h\wedge 1)\circ\bar{d}_1\circ h^{-1}$ and $w=(1\wedge 1\wedge r)\circ q$ are homotopic.

For this purpose we investigate the details of the homeomorphism h. If we identify $D(m, 0)$ with $RP(m)$, the canonical real 2-plane bundle η_1 over $D(m, 0)$ is the 2-plane bundle $1\oplus\xi$ over $RP(m)$. The homeomorphism

$$h_1\colon T((2r+1)\oplus(2r+1)\xi) \to S^{2r+1}\wedge(Y_1/Y_2)$$

is induced from the map

$$\begin{aligned} f_1\colon (S^m\times D^{2r+1}\times D^{2r+1},\ & S^m\times(S^{2r}\times D^{2r+1}\cup D^{2r+1}\times S^{2r})) \\ & \to (D^{2r+1}\times S^{m+2r+1},\ S^{2r}\times S^{m+2r+1}\cup D^{2r+1}\times S^{2r}) \end{aligned}$$

given by

$$f_1(x, a, b) = (a, (b, \sqrt{1-||b||^2}x)),$$

and the homeomorphism

$$h_2\colon T((2r+1)\eta_1) \to X_1/X_2$$

is induced from the map

$$\begin{aligned} f_2\colon (S^m\times S^1\times D^{2(2r+1)},\ & S^m\times S^1\times S^{2(2r)+1}) \\ & \to (S^m\times S^{2(2r+1)+1},\ S^m\times S^{2(2r)+1}) \end{aligned}$$

given by

$$f_2(x, z, v) = (x, (v, \sqrt{1-||v||^2}z)),$$

where D^{2r+1} and $D^{2(2r+1)}$ are unit disks of R^{2r+1} and C^{2r+1} respectively.

We define a map

$$\begin{aligned} \phi\colon (S^m\times D^{2r+1}\times D^{2r+1},\ & S^m\times(S^{2r}\times D^{2r+1}\cup D^{2r+1}\times S^{2r})) \\ & \to (S^m\times S^1\times D^{2(2r+1)},\ S^m\times S^1\times S^{2(2r)+1}) \end{aligned}$$

by

$$\phi(x, a, b) = (x, 1, \theta(a, b)),$$

where θ is the standard homeomorphism $D^{2r+1}\times D^{2r+1}\to D^{2(2r+1)}$ given by

$$\theta(a, b) = \max(||a||, ||b||)(||a||^2+||b||^2)^{-1/2}(a+bi).$$

Since

$$\phi(-x, a, -b) = (-x, 1, \overline{\theta(a, b)}),$$

the map ϕ defines a quotient map

$$\psi\colon T((2r+1)\oplus(2r+1)\xi) \to T((2r+1)\eta_1)$$

which is a homeomorphism. The homeomorphism h is the composition $h_1\circ\psi^{-1}\circ h_2^{-1}$ of the three homeomorphisms.

Now, the homeomorphism h is given by

$$h^{-1}([a]\wedge[b, \sqrt{1-||b||^2}x]) = [x, (\theta(a, b), \sqrt{1-||\theta(a, b)||^2})]\,.$$

Therefore we have

$$u([a]\wedge[b, \sqrt{1-||b||^2}x]) = [a]\wedge[b, \sqrt{1-||b||^2}x]\wedge[(x, 0), (\theta(a, b), \sqrt{1-||\theta(a, b)||^2})]$$

and

$$v([a]\wedge[b, \sqrt{1-||b||^2}x]) = [a]\wedge[b, \sqrt{1-||b||^2}x]\wedge[(b, \sqrt{1-||b||^2}x), (1, 0)]\,.$$

We define two maps F_t^1 and F_t^2, for $0\leq t\leq 1$,

$$\begin{aligned}&(S^m\times D^{2r+1}\times D^{2r+1}, S^m\times(S^{2r}\times D^{2r+1}\cup D^{2r+1}\times S^{2r}))\\ &\quad\to(D^{2r+1}\times S^{m+2r+1}\times(S^{m+2r+1}\times S^{2(2r+1)+1}), (S^{2r}\times S^{m+2r+1}\cup D^{2r+1}\times S^{2r})\\ &\qquad\times(S^{m+2r+1}\times S^{2(2r+1)+1}))\end{aligned}$$

by

$$F_t^1(x, a, b) = (a, (b, \sqrt{1-||b||^2}x))\times((x, 0), (t\theta(a, b), \sqrt{1-||t\theta(a, b)||^2}))$$

and

$$F_t^2(x, a, b) = (a, (b, \sqrt{1-||b||^2}x)\times((tb, \sqrt{1-||tb||^2}x), (1, 0))\,.$$

Then the maps F_t^1 and F_t^2 are compatible with the identification, so that they define maps G_t^1 and G_t^2 respectively

$$\begin{aligned}&(D((2r+1)\oplus(2r+1)\xi), S((2r+1)\oplus(2r+1)\xi))\\ &\quad\to(D^{2r+1}\times RP(m+2r+1)\times D(m+2r+1, 2r+1), (S^{2r}\times RP(m+2r+1)\\ &\qquad\cup D^{2r+1}\times RP(2r))\times D(m+2r+1, 2r+1))\,.\end{aligned}$$

Therefore, they define quotient maps H_t^1 and H_t^2 respectively

$$T((2r+1)\oplus(2r+1)\xi)\to S^{2r+1}\wedge(Y_1/Y_2)\wedge Z\,,$$

and we have

$$u = H_1^1\circ h_1^{-1} \quad\text{and}\quad v = H_1^2\circ h_1^{-1}\,.$$

Since the maps F_0^1 and F_0^2 are homotopic, the maps H_0^1 and H_0^2 are homotopic. Therefore the maps H_1^1 and H_1^2 are homotopic, so that the maps u and v are homotopic. This shows i).

The map q is defined as follows: We define a map

$$\begin{aligned}f\colon&(S^m\times D^{2r+1}\times D^{2r+1}, S^m\times(S^{2r}\times D^{2r+1}\cup D^{2r+1}\times S^{2r}))\\ &\quad\to(D^{2r+1}\times S^{m+2r+1}\times S^m, (S^{2r}\times S^{m+2r+1}\cup D^{2r+1}\times S^{2r})\times S^m)\end{aligned}$$

by

$$f(x,\ a,\ b) = (a,\ (b,\ \sqrt{1-\|b\|^2}x),\ x)\,.$$

Since

$$f(-x,\ a,\ -b) = (a,\ (-b,\ -\sqrt{1-\|b\|^2}x),\ -x)\,,$$

the map f defines a quotient map

$$g\colon T((2r+1)\oplus(2r+1)\xi) \to S^{2r+1}\wedge(Y_1/Y_2)\wedge RP(m)\,,$$

and we have $q=g\circ h_1^{-1}$.

Now, we can define a map, for $0\leq t\leq 1$,

$$H_t\colon T((2r+1)\oplus(2r+1)\xi) \to S^{2r+1}\wedge(Y_1/Y_2)\wedge X_1$$

by

$$H_t([x,\ a,\ b]) = [a]\wedge[b,\ \sqrt{1-\|b\|^2}x]\wedge[x,\ (t\theta(a,\ b),\ \sqrt{1-\|t\theta(a,\ b)\|^2}]\,,$$

and we have $\bar{u}=H_1\circ h_1^{-1}$. Since the maps w and $H_0\circ h_1^{-1}$ are homotopic, the maps $\bar{u}$ and w are homotopic. This shows ii).

This completes the proof of Lemma 1.

Lemma 2. *We have the commutative diagram*

$$\begin{tikzcd}
\tilde{K}^1_U(S^{2r+1}\wedge(Y_1/Y_2))\otimes\tilde{K}^0_U(Z) \arrow[r, "1\otimes i^!"] \arrow[d] & \tilde{K}^1_U(S^{2r+1}\wedge(Y_1/Y_2))\otimes\tilde{K}^0_U(X_1) \arrow[r, "h^!\otimes 1"] \arrow[d] & \tilde{K}^1_U(X_1/X_2)\otimes\tilde{K}^0_U(X_1) \arrow[r, "\pi^!\otimes 1"] \arrow[d] & \tilde{K}^1_U(X_1)\otimes\tilde{K}^0_U(X_1) \arrow[d] \\
\tilde{K}^1_U(S^{2r+1}\wedge(Y_1/Y_2)\wedge Z) \arrow[r, "(1\wedge i)^!"] \arrow[d, "(1\wedge r)^!"] & \tilde{K}^1_U(S^{2r+1}\wedge(Y_1/Y_2)\wedge X_1) \arrow[r, "(h\wedge 1)^!"] \arrow[d, "(1\wedge r)^!"] & \tilde{K}^1_U((X_1/X_2)\wedge X_1) \arrow[r, "(\pi\wedge 1)^!"] \arrow[dd, "\bar{d}_1^!"] & \tilde{K}^1_U(X_1\wedge X_1) \arrow[dd, "d_1^!"] \\
\tilde{K}^1_U(S^{2r+1}\wedge(Y_1/Y_2)\wedge Y_1) \arrow[d, "(1\wedge d_2)^!"] & \tilde{K}^1_U(S^{2r+1}\wedge(Y_1/Y_2)\wedge RP(m)) \arrow[d, "q^!"] & & \\
\tilde{K}^1_U(S^{2r+1}\wedge(Y_1/Y_2)) \arrow[r, equal] & \tilde{K}^1_U(S^{2r+1}\wedge(Y_1/Y_2)) \arrow[r, "h^!"] & \tilde{K}^1_U(X_1/X_2) \arrow[r, "\pi^!"] & \tilde{K}^1_U(X_1)\,.
\end{tikzcd}$$

Proof. It follows from Lemma 1 by naturality.

Proposition. *We have the relations* $\omega\alpha=0$, $\omega\gamma=0$ *and* $\omega\nu_1=-2\omega$ *in* $\tilde{K}_U^1(D(m, 2r+1))$.

Proof. Since $r^!\alpha=r^!\gamma=0$ in $\tilde{K}_U^0(RP(m))$, we have $q^!(1\wedge r)^!(g^{[r]}\nu^{(r+1)}\wedge\alpha) =q^!(g^{[r]}\nu^{(r+1)}\wedge r^!\alpha)=0$ and $q^!(1\wedge r)^!(g^{[r]}\nu^{(r+1)}\wedge\gamma)=q^!(g^{[r]}\nu^{(r+1)}\wedge r^!\gamma)=0$ in $\tilde{K}_U^1(S^{2r+1}\wedge(Y_1/Y_2))$. In virtue of definition of ω and Lemma 2, these show $\omega\alpha=0$ and $\omega\gamma=0$ in $\tilde{K}_U^1(X_1)$.

Since the element ν_1 of $\tilde{K}_U^0(X_1)$ is induced from the element ν_1 of $\tilde{K}_U^0(Z)$ by the inclusion map $X_1\subset Z$, in order to show the relation $\omega\nu_1=-2\omega$ in $\tilde{K}_U^1(X_1)$, in virtue of definition of ω and Lemma 2, it is sufficient to show that we have the relation $\nu^{(r+1)}\cdot r^!\nu_1=-2\nu^{(r+1)}$ in $\tilde{K}_U^0(Y_1/Y_2)$ for ν_1 of $\tilde{K}_U^0(Z)$.

Since $r^!\nu_1=\nu$ in $\tilde{K}_U^0(Y_1)$, we have the relation $\nu^{r+1}\cdot r^!\nu_1=-2\nu^{r+1}$ in $\tilde{K}_U^0(Y_1)$. The homomorphism, induced by the projection j: $Y_1\to Y_1/Y_2$,

$$j^!: \tilde{K}_U^0(Y_1/Y_2)\to\tilde{K}_U^0(Y_1)$$

is monomorphism, so that we have $\nu^{(r+1)}\cdot r^!\nu_1=-2\nu^{(r+1)}$ in $\tilde{K}_U^0(Y_1/Y_2)$. The proof is complete.

This completes the proof of Theorem 3.

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