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ON A CONSTRUCTION OF RECURRENT MARKOV CHAINS

RYŌJI KONDŌ

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Let S be a denumerable (possibly finite) set and $\boldsymbol{B}$ the space of all real valued and bounded functions defined on S. For a given measure μ, strictly positive at each point of S, we shall denote by $\boldsymbol{N}(\mu)$ the collection of functions f such that the support of f is finite and $\langle\mu, f\rangle = \Sigma_{x \in S}\, \mu(x) f(x) = 0$. A linear operator R from $\boldsymbol{N}(\mu)$ to $\boldsymbol{B}$ is said to satisfy the *semi-complete maximum priniciple* if it has the following property:

(S.C.M) For any $f \in \boldsymbol{N}(\mu)$, if $Rf \leqq m$ on the set $\{f>0\}$, then $Rf \leqq m$ everywhere, where m is a real constant.

We know that if R is a weak potential operator for a recurrent semi-group $(P_t)_{t \geqq 0}$ with an invariant measure μ, it satisfies this maximum principle [7, p. 337]. In this work we shall consider the converse problem: Given a measure μ and a linear operator R satisfying (S.C.M), can we find a recurrent semi-group $(P_t)_{t \geqq 0}$ which has μ as an invariant measure and R as a weak potential operator?

If μ is bounded, this problem has an affiirmative answer, which will be stated in section 2. However, if μ is unbounded, there are several cases, for example, some operators are weak potential operators for transient semi-groups with invariant measure μ and others are never weak potential operators for any Markov semi-group with invariant measure μ. We shall give such examples in section 3. The appropriate conditions under which the problem is solved are not known yet. In section 1 we shall study, for later use, another type of maximum principle which is satisfied by weak potential operators (weak inverses in Orey [10]) for recurrent Markov chains with discrete parameters.

1. Potential operators satisfying the reinforced semi-complete maximum principle

Throughout this work notations and terminology are mainly taken from [7]. We shall denote the collection of all non-empty finite subsets of S by $\mathcal{K}$. Further, for each $E \in \mathcal{K}$, we shall use the following notations:

f_E The function restricted to E.
ν_E The measure restricted to E.
$\boldsymbol{B}_E$ The space of all functions f_E.
$\boldsymbol{N}^E$ The space of functions of $\boldsymbol{N}(\mu)$ with supports in E.

For any function f on S, $f^+=\sup(f, 0)$ and $f^-=\sup(-f, 0)$. The indicator function of a set E will be denoted by χ_E.

A linear operator G from $\boldsymbol{N}(\mu)$ to $\boldsymbol{B}$ is said to satisfy the *reinforced semi-complete maximum principle* if it has the following property:

(R.S.C.M) For any function $f\in\boldsymbol{N}(\mu)$, if $Gf\leqq m$ on the set $\{f>0\}$, then $Gf \leqq m-f^-$ everywhere, where m is a real constant.

Let G be a linear operator from $\boldsymbol{N}(\mu)$ to $\boldsymbol{B}$ satisfying (R.S.C.M).

Lemma 1. *G is non-singular in the sense: If f is a non-zero element of $\boldsymbol{N}(\mu)$, then Gf is never equal to a constant on the support of f. So that $Gf=0$ implies $f=0$.*

Proof. Let f be a non-zero element of $\boldsymbol{N}(\mu)$ and $Gf=m$ on the support of f, where m is a constant. From (R.S.C.M) it follows that $Gf\leqq m-f^-$ everywhere and hence, $m=Gf\leqq m-f^-$ on the set $\{f<0\}$. Therefore $f^-=0$. Similarly we have $f^+=0$, for, $-m=G(-f)\leqq -m-(-f)^-=-m-f^+$ on the set $\{f>0\}$. Thus $f=0$, which is a contradiction.

Lemma 2. *There is a family of (signed) measures $(\lambda^E)_{E\in\mathcal{K}}$ on S such that; (i) the support of each λ^E is contained in E, (ii) $\langle\lambda^E, 1\rangle=1$ and (iii) $\langle\lambda^E, Gf\rangle=0$ for all $f\in\boldsymbol{N}^E$. Such a family is unique.*

Proof. Let $E\in\mathcal{K}$ and the number of elements of E be n. Then the linear dimensions of $\boldsymbol{B}_E$ and $\boldsymbol{N}^E$ are equal to n and $n-1$ respectively. Let us define a linear operator G^E from $\boldsymbol{N}^E$ to $\boldsymbol{B}_E$ by

$$G^E f = (Gf)_E \qquad \text{for} \quad f\in\boldsymbol{N}^E. \tag{1.1}$$

From Lemma 1 it follows that if $G^E f=0$, then $f=0$ and that 1_E, the restriction of the function 1 to E, does not belong to the range $G^E(\boldsymbol{N}^E)$. Therefore, since $\dim G^E(\boldsymbol{N}^E)=\dim \boldsymbol{N}^E=n-1$ and $1_E\notin G^E(\boldsymbol{N}^E)$, we can find exactly one linear functional l_E on $\boldsymbol{B}_E$ such that $l_E(g_E)=0$ if and only if $g_E\in G^E(\boldsymbol{N}^E)$ and $l_E(1_E)=1$. Thus if we define the measure λ^E by $\lambda^E(y)=l_E((\chi_{\{y\}})_E)$ for $y\in E$ and $\lambda^E(y)=0$ for $y\in S\backslash E$, the family $(\lambda^E)_{E\in\mathcal{K}}$ is the desired one. The uniquencess of $(\lambda^E)_{E\in\mathcal{K}}$ is obvious from the above proof.

Let $g\in\boldsymbol{B}$ and $E\in\mathcal{K}$. If we put $h_E=(g-\langle\lambda^E, g\rangle)_E$, then $l_E(h_E)=\langle\lambda^E, g\rangle -\langle\lambda^E, g\rangle=0$, so that we can find unique $f^E\in\boldsymbol{N}^E$ such that $h_E=G^E f^E$. Now let us define the mappings H^E and Π^E from $\boldsymbol{B}$ to $\boldsymbol{B}$ by

$$H^E g = Gf^E + \langle \lambda^E, g \rangle \tag{1.2}$$

and

$$\Pi^E g = Gf^E + \langle \lambda^E, g \rangle - f^E = H^E g - f^E \tag{1.3}$$

respectively. Obviously, H^E and Π^E are linear and $H^E g = \Pi^E g$ on $S \backslash E$.

Lemma 3. *(i) If $g \geqq 0$ on E, then $H^E g \geqq 0$ and $\Pi^E g \geqq 0$ everywhere. (ii) $H^E 1 = 1$ and $\Pi^E 1 = 1$. (iii) If $E, F \in \mathcal{K}$ and $E \subseteq F$, then $H^F H^E g = H^E g$ and $\Pi^F H^E g = \Pi^E g$.*

Proof. Let $g \geqq 0$ on E and $H^E g = Gf^E + \langle \lambda^E, g \rangle$ where $f^E \in \boldsymbol{N}^E$. Since $Gf^E + \langle \lambda^E, g \rangle = g$ on E, $Gf^E \geqq -\langle \lambda^E, g \rangle$ on the support of f^E. Therefore, using (R.S.C.M), we have

$$Gf^E \geqq -\langle \lambda^E, g \rangle + (f^E)^+$$

everywhere, so that

$$H^E g = Gf^E + \langle \lambda^E, g \rangle \geqq (f^E)^+ \geqq 0$$

and

$$\Pi^E g = Gf^E + \langle \lambda^E, g \rangle - f^E \geqq Gf^E + \langle \lambda^E, g \rangle - (f^E)^+ \geqq 0$$

everywhere. Thus, the assertion (i) is true. Next, if $H^E 1 = Gf^E + \langle \lambda^E, 1 \rangle$, then $f^E = 0$ by Lemma 1. Therefore $H^E 1 = \Pi^E 1 = 1$, which implies (ii). Finally, let $E \subseteq F$ and let

$$h = H^E g = Gf^E + \langle \lambda^E, g \rangle \qquad (f^E \in \boldsymbol{N}^E)$$
$$H^F h = Gf^F + \langle \lambda^F, h \rangle \qquad (f^F \in \boldsymbol{N}^F).$$

Since $H^F h = h$ on F, we have

$$Gf^F + \langle \lambda^F, h \rangle = Gf^E + \langle \lambda^E, g \rangle$$

on F. Therefore

$$G(f^F - f^F) = \langle \lambda^E, g \rangle - \langle \lambda^F, h \rangle = \text{const.}$$

on the support of $f^F - f^E$. Using Lemma 1, we have $f^F = f^E$ and $\langle \lambda^E, g \rangle = \langle \lambda^F, h \rangle$, which implies $H^F H^E g = H^E g$ and that

$$\Pi^F H^E g = H^F h - f^F = h - f^E = \Pi^E g.$$

Thus the assertion (iii) was proved.

From this lemma we can see that H^E and Π^E are Markov kernels on S and that for each $x \in S$ the supports of measures $H^E(x,\ )$ and $\Pi^E(x,\ )$ are contained in E.

Corollary. *If $E, F \in \mathcal{K}$, $E \subseteq F$ and g is a non-negative function on S with support in E, then $\Pi^E g \geqq \Pi^F g$ everywhere.*

For,
$$\begin{aligned} &\Pi^E g(x) \\ &= \Pi^F H^E g(x) \\ &= \Sigma_{y\in E}\, \Pi^F(x, y) g(y) + \Sigma_{y\in S\setminus E}\, \Pi^E(x, y) H^E g(y) \\ &\geqq \Sigma_{y\in E}\, \Pi^F(x, y) g(y) \\ &= \Pi^F g(x) \end{aligned}$$
for all $x \in S$.

Theorem 1. *Let μ be a bounded measure which is strictly positive everywhere and G a linear operator from $\boldsymbol{N}(\mu)$ to $\boldsymbol{B}$ satisfying the reinforced semi-complete maximum principle. Then there is a kernel P on S such that*

$$P \geqq 0 \quad \text{and} \quad P1 = 1\,, \tag{1.4}$$

$$\mu P = \mu\,, \tag{1.5}$$

$$(I-P)Gf = f \qquad \text{for all} \quad f \in \boldsymbol{N}(\mu)\,. \tag{1.6}$$

Such a kernel is unique.[1]

Further, P is irreducible recurrent in the sense:

$$\sum_{n=0}^{\infty} P^n(x, y) = \infty \qquad \text{for all} \quad (x, y) \in S \times S\,. \tag{1.7}$$

Proof. Let $(E_n)_{n\geqq 1}$ be an increasing sequence of $\mathcal{K}$ with the union S and $x, y \in S$. Then, there is some n such that $y \in E_k$ for all $k \geqq n$. So that, by Corollary of Lemma 2, we have

$$\Pi^{E_n}(x, y) \geqq \Pi^{E_{n+1}}(x, y) \geqq \cdots \geqq 0\,.$$

Therefore the limit;

$$P(x, y) = \lim_{n\to\infty} \Pi^{E_n}(x, y)\,, \tag{1.8}$$

exists for any $(x, y) \in S \times S$. We shall prove the kernel P defined by (1.8) has

1) Precisely speaking, a Markov kernel satisfying (1. 6), if it exists, is unique, even if μ is unbounded. We can see this in the proof of the theorem. Similar circumstance occurs in Lemma 5 and Theorem 2 in the next section.

all the properties stated in the theorem. Since Π^{E_n} are Markov kernels, P is obviously sub-Markov kernel, that is, $P\geqq 0$ and $P1\leqq 1$, by Fatou's inequality. From the definition of the kernel H^E, we can find $f^{E_n}\in \boldsymbol{N}^{E_n}$ such that

$$H^{E_n}(x, y) = Gf^{E_n}(x)+\lambda^{E_n}(y)\,.$$

Since,

$$\Pi^{E_n}(x, y) = H^{E_n}(x, y)-f^{E_n}(x)\,,$$

we have,

$$\begin{aligned}&\Sigma_{x\in E_n}\,\mu(x)\,\Pi^{E_n}(x, y)\\ &\quad= \Sigma_{x\in E_n}\,\mu(x)\,H^{E_n}(x, y)-\Sigma_{x\in E_n}\,\mu(x)f^{E_n}(x)\\ &\quad= \mu(y)\,,\end{aligned}$$

whenever $y\in E_n$. On the other hand, since $0\leqq \chi_{E_n}(x)\,\Pi^{E_n}(x, y)\leqq 1$, $\lim\limits_{n}\chi_{E_n}(x)$ $\Pi^{E_n}(x, y)=P(x, y)$ and μ is a bounded measure, we have

$$\begin{aligned}\mu P(y) &= \Sigma_{x\in S}\,\mu(x)\,(\lim_{n\to\infty}\chi_{E_n}(x)\,\Pi^{E_n}(x, y))\\ &= \lim_{n\to\infty}\Sigma_{x\in S}\,\mu(x)\,\chi_{E_n}(x)\,\Pi^{E_n}(x, y)\\ &= \mu(y)\end{aligned}$$

for all $y\in S$. Thus (1.5) was proved. From (1. 5) it follows that $\langle\mu, P1\rangle$ $=\langle\mu P, 1\rangle=\langle\mu, 1\rangle$. Since $0\leqq P1\leqq 1$, we have $P1=1$ almost everywhere with respect to μ. However, since μ is strictly positive everywhere, we have $P1=1$. That is, (1.4) is true. Let $f\in \boldsymbol{N}(\mu)$ and $g=Gf+\|Gf\|$, where $\|\ \|$ denotes the uniform norm in $\boldsymbol{B}$. If n is so large that the support of f is contained in E_n, we have

$$\begin{aligned}\Pi^{E_n}g(x) &= \Pi^{E_n}Gf(x)+\|Gf\|\\ &= Gf(x)-f(x)+\|Gf\|\end{aligned}$$

for all $x\in S$ and hence, noting that $g\geqq 0$, we have

$$Pg(x)\leqq \liminf_{n\to\infty}\Pi^{E_n}g(x)=Gf(x)-f(x)+\|Gf\|\,,$$

which implies

$$PGf\leqq Gf-f\,. \tag{1.9}$$

Similarly, by replacing f to $-f$ in (1.9), we have $PGf\geqq Gf-f$, so that $PGf=Gf-f$ which proves (1.6). If $\tilde{P}$ is any kernel satisfying (1.4) and (1.6), then for any $g\in\boldsymbol{B}$

$$\begin{aligned}\tilde{P}g &= \lim_{n\to\infty}\tilde{P}H^{E_n}g = \lim_{n\to\infty}\tilde{P}(Gf^{E_n}+\langle\lambda^{E_n}, g\rangle)\\ &= \lim_{n\to\infty}(Gf^{E_n}-f^{E_n}+\langle\lambda^{E_n}, g\rangle) = \lim_{n\to\infty}PH^{E_n}g = Pg\,,\end{aligned}$$

where $H^{E_n}g = Gf^{E_n} + \langle \lambda^{E_n}, g \rangle$ and $f^{E_n} \in N^{E_n}$. Thus the uniqueness of P is proved. Finally we shall prove (1.7). If there is some $y \in S$ such that

$$\sum_{n=0}^{\infty} P^n(y, y) < \infty ,$$

then

$$\sum_{n=0}^{\infty} P^n(x, y) \leqq \sum_{n=0}^{\infty} P^n(y, y) < \infty$$

for all $x \in S$. Consequently $\lim_n P^n(x, y) = 0$ for all $x \in S$.

Therefore, using (1.5), we have

$$\mu(y) = \sum_{x \in S} \mu(x) (\lim_n P^n(x, y)) = 0 ,$$

which contradicts the assumption that μ is strictly positive everywhere. Thus (1.7) is true when $x = y$. To show (1.7) in the case $x \neq y$, it is sufficient that we prove there is some n such that $P^n(x, y) > 0$. Let us introduce the function e_y in $N(\mu)$ by

$$e_y(z) = \begin{cases} 1 & z = x \\ -\mu(x)/\mu(y) & z = y \\ 0 & \text{otherwise} . \end{cases}$$

If $P^n(x, y) = 0$ for all $n \geqq 0$, we have

$$\begin{aligned} \sum_{k=0}^{n} P^k(x, x) &= \sum_{k=0}^{n} P^k e_y(x) \\ &= Ge_y(x) - P^{n+1} Ge_y(x) \\ &= [Ge_y(x) - Ge_y(y)] - P^{n+1}[Ge_y - Ge_y(y)](x) \\ &\leqq Ge_y(x) - Ge_y(y) , \end{aligned}$$

because $Ge_y \geqq Ge_y(y)$ everywhere. Consequently we have

$$\sum_{k=0}^{\infty} P^k(x, x) \leqq Ge_y(x) - Ge_y(y) < \infty$$

which is a contradiction. Thus the theorem was proved.

In the proof of this theorem, we have used essentially the boundedness of the measure μ. Examples of operators G for unbounded measures will be given and discussed in section 3.

2. The potential operators satisfying the semi-complete maximum principle.

Let μ be a measure on S, strictly positive everywhere, and R a linear operator from $N(\mu)$ to $\boldsymbol{B}$ which satisfies the semi-complete maximum principle.

In this section we shall assume always that μ is bounded. For each positive number α, we put $G_\alpha = I + \alpha R$, where I is the identity operator. Evidently G_α is a linear operator from $\boldsymbol{N}(\mu)$ to $\boldsymbol{B}$.

Lemma 4. *G_α satisfies the reinforced semi-complete maximum principle.*

Proof. Let $G_\alpha f \leqq m$ on the set $\{f>0\}$, where m is a real constant. Then $\alpha Rf \leqq G_\alpha f \leqq m$ on the set $\{f>0\}$, so that $\alpha Rf \leqq m$ everywhere by (S. C. M). Therefore $-f^- + \alpha Rf \leqq m - f^-$ everywhere. Hence we have $G_\alpha f = -f^- + \alpha Rf \leqq m - f^-$ on the set $\{f \leqq 0\}$, which implies $G_\alpha f \leqq m - f^-$ everywhere.

Since G_α satisfies (R. S. C. M), we can apply Theorem 1 to G_α, so that there is a kernel Q_α on S which has all the properties in Theorem 1. Put $R_\alpha = Q_\alpha / \alpha$, then

Lemma 5. *The family of kernels $(R_\alpha)_{\alpha>0}$ satisfies the following conditions:*

$$\alpha R_\alpha \geqq 0 \text{ and } \alpha R_\alpha 1 = 1 , \tag{2.1}$$

$$\alpha \mu R_\alpha = \mu , \tag{2.2}$$

$$R_\alpha - R_\beta + (\alpha - \beta) R_\alpha R_\beta = 0 , \tag{2.3}$$

$$(I - \alpha R_\alpha) Rf = R_\alpha f \qquad \text{for all} \quad f \in \boldsymbol{N}(\mu) . \tag{2.4}$$

Such a family is unique.

Further

$$\lim_{\alpha \to 0} R_\alpha(x, y) = \infty \qquad \text{for all} \quad (x, y) \in S \times S . \tag{2.5}$$

Proof. (2.1), (2.2) and (2.4) are the same as (1.4), (1.5) and (1.6) of Theorem 1 respectively and the uniqueness of such a family is a consequence of Theorem 1, too. So we have only to prove (2.3) and (2.5). Let us denote by $(\lambda_\alpha^E)_{E \in \mathcal{K}}$ the family of measures satisfying (i), (ii) and (iii) of Lemma 2 for G_α and by H_α^E the kernel defined by (1.2) with respect to G_α and λ_α^E. If $g \in \boldsymbol{B}$ and $H_\beta^E g = G_\beta f^E + \langle \lambda_\beta^E, g \rangle$, where $f^E \in \boldsymbol{N}^E$, then, noting the relation

$$H_\beta^E g = G_\alpha f^E + (\beta - \alpha) Rf^E + \langle \lambda_\beta^E, g \rangle ,$$

we have

$$R_\alpha H_\beta^E g = Rf^E + (\beta - \alpha) R_\alpha Rf^E + \langle \lambda_\beta^E, g \rangle / \alpha .$$

Since

$$R_\beta H_\beta^E g = Rf^E + \langle \lambda_\beta^E, g \rangle / \beta ,$$

we have

$$\begin{aligned} & R_\alpha H_\beta^E g - R_\beta H_\beta^E g \\ & \quad = (\beta - \alpha) [R_\alpha Rf^E + \langle \lambda_\beta^E, g \rangle / \alpha\beta] . \end{aligned}$$

We can easily verify that the last term is equal to $(\beta-\alpha)\ R_\alpha R_\beta H_\beta^E g$, so that

$$R_\alpha H_\beta^E g - R_\beta H_\beta^E g = (\beta-\alpha) R_\alpha R_\beta H_\beta^E g \tag{2.6}$$

for all $g\in \boldsymbol{B}$, $E\in \mathcal{K}$ and α, $\beta>0$. Let $(E_n)_{n\geqq 1}$ be an increasing sequence of sets in $\mathcal{K}$ with the union S. Since $||H_\beta^{E_n}g||\leqq ||g||$ and $\lim_n H_\beta^{E_n}g(x)=g(x)$ for all $x\in S$, we have

$$\begin{aligned} R_\alpha g - R_\beta g &= \lim_{n\to\infty} [R_\alpha H_\beta^{E_n}g - R_\beta H_\beta^{E_n}g] \\ &= (\beta-\alpha) \lim_n R_\alpha R_\beta H_\beta^{E_n} g \\ &= (\beta-\alpha)\ R_\alpha R_\beta g\ , \end{aligned}$$

which proves (2.4). Finally we shall prove (2.5). First we prove the inequality

$$R_\alpha(x, y)\leqq R_\alpha(y, y)\ . \tag{2.7}$$

Since $\beta R_{\alpha+\beta}$ is a sub-Markov kernel on S and $I+\beta R_\alpha=\sum_{n=0}^{\infty}(\beta R_{\alpha+\beta})^n$, we have

$$I(x, y)+\beta R_\alpha(x, y)\leqq I(y, y)+\beta R_\alpha(y, y) \tag{2.8}$$

for all $(x, y)\in S\times S$. Hence, dividing both side of (2.8) by β, and letting $\beta\to\infty$, we obtain (2.7). If there is some $y\in S$ such that $\lim_{\alpha\to 0} R_\alpha(y, y)<\infty$, then $\lim_{\alpha\to 0} \alpha R_\alpha(x, y)=0$ for all $x\in S$ by (2.7). Therefore

$$\mu(y) = \lim_{\alpha\to 0} \alpha\mu R_\alpha(y) = \mu\ (\lim_{\alpha\to 0} \alpha R_\alpha)(y) = 0\ ,$$

which is a contradiction. Thus (2.5) is true when $x=y$. Let $r_\beta(x)=R_\beta(x, y)/R_\beta(y, y)$ and $r(x)=\liminf_{\beta\to 0} r_\beta(x)$. From (2.7) it follows that $0\leqq r(x)\leqq 1$ for all $x\in S$. Since the resolvent equation (2.3) implies

$$\alpha R_\alpha r_\beta(x) = \beta R_\alpha r_\beta(x)+r_\beta(x)-R_\alpha(x, y)/R_\beta(y, y)$$

and since

$$\begin{aligned} &0\leqq R_\alpha(x, y)/R_\beta(y, y)\leqq 1/\alpha R_\beta(y, y)\ , \\ &0\leqq \beta R_\alpha r_\beta(x)\leqq \beta/\alpha\ , \end{aligned}$$

we have

$$\alpha R_\alpha r(x)\leqq \liminf_{\beta\to 0} \alpha R_\alpha r_\beta(x)\leqq \liminf_{\beta\to 0} r_\beta(x) = r(x)$$

for all $x\in S$, which implies the function r is excessive with respect to the kernel $Q_\alpha=\alpha R_\alpha$. By Theorem 1, Q_α is irreducible recurrent, so that r should be a constant function, which is proved in [5, p. 226]. Since $r(y)=1$, we have

$$r(x) = \lim_{\alpha\to 0} R_\alpha(x, y)/R_\alpha(y, y) = 1 \tag{2.9}$$

for all $x \in S$, which implies $\lim_{\alpha \to 0} R_\alpha(x, y) = \infty$ for all $(x, y) \in S \times S$. Thus the theorem was proved.

Using (2.9), we can obtain easily the following corollaries:

Corollary 1. $\lim_{\alpha \to 0} \alpha R_\alpha(x, y) = \mu(y)/\langle \mu, 1 \rangle$ *for all* $(x, y) \in S \times S$.

Corollary 2. *For each* $f \in \mathbf{N}(\mu)$ *there exists the limit* $R_0 f = \lim_{\alpha \to 0} R_\alpha f$ *and*

$$R_0 f = Rf - \langle \mu, Rf \rangle / \langle \mu, 1 \rangle \qquad \text{for all } f \in N(\mu)$$

and hence, the linear operator R_0 *satisfies* (S.C.M), *too.*

Let $a \in S$ and define the function f_y by

$$f_y(x) = \begin{cases} 1 & x = y \\ -\mu(y)/\mu(a) & x = a \\ 0 & \text{otherwise.} \end{cases}$$

If we put ${}^aR(x, y) = Rf_y(x) - Rf_y(a)$, then ${}^aR$ is a non-negative kernel on S with ${}^aR(a, y) = {}^aR(x, a) = 0$ for all $x, y \in S$.

Corollary 3. *Put*

$${}^aR_\alpha(x, y) = R_\alpha(x, y) - R_\alpha(x, a) R_\alpha(a, y) / R_\alpha(a, a)$$

then $({}^aR_\alpha)_{\alpha > 0}$ *is a sub-Markov resolvent with* $\lim_{\alpha \to 0} {}^aR_\alpha = {}^aR$.

The meaning of these corollaries will be made clear later.

Theorem 2. *Let* μ *be a bounded measure on* S, *strictly positive everywhere, and* R *a linear operator from* $N(\mu)$ *to* $\boldsymbol{B}$ *which satisfies the semi-complete maximum principle. Then there exists a family of kernels* $(P_t)_{t>0}$ *such that:*

(2.9) $\qquad P_t \geqq 0$ *and* $P_t 1 = 1$ *for all* $t > 0$.

(2.10) $\qquad P_t P_s = P_{t+s}$ *for all* $s, t > 0$.

(2.11) $\qquad \mu P_t = \mu$ *for all* $t > 0$.

(2.12) *The functions* $t \to P_t(x, y)$ *are continuous in the open interval* $(0, \infty)$ *for all* $(x, y) \in S \times S$.

(2.13) $\qquad (I - P_t) Rf(x) = \int_0^t P_s f(x)\, ds$ *for all* $f \in N(\mu)$, $x \in S$ *and* $t > 0$.[2)]

Such a family is unique.

2) If a linear operator R from $\boldsymbol{N}(\mu)$ to $\boldsymbol{B}$ satisfies (2.13) for a Markov semi-group $(P_t)_{t>0}$, it will be called a *weak potential operator* for $(P_t)_{t>0}$.

Further $(P_t)_{t>0}$ *is irreducible recurrent in the sense:*

$$(2.14) \qquad \int_0^\infty P_t(x, y)\,dt = \infty \qquad \text{for all} \quad (x, y) \in S \times S\,.$$

Proof. Let $(R_\alpha)_{\alpha>0}$ be the family constructed in Lemma 6. Since it satisfies (2.1) and (2.3), using the result of Reuter [12], we can find $(P_t)_{t>0}$ which satisfies (2.9), (2.10), (2.11) and

$$(2.15) \qquad R_\alpha(x, y) = \int_0^\infty e^{-\alpha t} P_t(x, y)\,dt \qquad \text{for all} \quad (x, y) \in S \times S\,.$$

Since the functions $t \to \mu P_t(y)$ are continuous in $(0. \infty)$ and

$$\int_0^\infty e^{-\alpha t} \mu P_t(y)\,dt = \mu R_\alpha(y)/\alpha = \int_0^\infty e^{-\alpha t} \mu(y)\,dt\,,$$

we have (2.11) by the uniqueness of the inverse Laplace transform. We remark here that, for any $f \in \boldsymbol{B}$ and $x \in S$, the function $t \to P_t f(x)$ is continuous in $(0, \infty)$. In fact, if $0 \leqq f \leqq 1$, the functions $t \to P_t f(x)$ and $t \to P_t(1-f)(x) = 1 - P_t f(x)$ are lower-semi-continuous in $(0, \infty)$ and hence, the function $t \to P_t f(x)$ is continuous in $(0, \infty)$. The general case is reduced to this case by the usual procedure. From this remark we know that the both sides of (2.13) are continuous with respect to t in $(0, \infty)$. Since the Laplace transform of (2.13) is equal to (2.4), (2.13) is true by the property of the Laplace transform. Similarly the uniqueness of $(P_t)_{t>0}$ is followed from Lemma 6 and the uniqueness of the inverse Laplace transform. Relation (2.14) is evident by

$$\int_0^\infty P_t(x, y)\,dt = \lim_{\alpha \to 0} R_\alpha(x, y) = \infty\,.$$

Thus the theorem was proved.

Corollary 1 of Lemma 6 implies the *ergodic property* of $(P_t)_{t>0}$; $\lim_{t\to\infty} P_t(x, y) = \mu(y)/\langle \mu, 1 \rangle$, and Corollary 2 implies the *normality* of $(P_t)_{t>0}$; for any $f \in \boldsymbol{N}(\mu)$ and $x \in S$, there exists the limit; $R_0 f(x) = \lim_{t\to\infty} \int_0^t P_s f(x)\,ds$, and which satisfies the equation (2.13), too.

Now we discuss the continuity of $(P_t)_{t>0}$ at $t=0$.

Theorem 3. *Under the same conditions of Theorem* 1, *the relation*

$$(2.16) \qquad \lim_{t\to 0} P_t(x, y) = I(x, y) \qquad \text{for all } (x, y) \in S \times S$$

holds if and only if R is non-singular.

Proof. First let us assume that $(P_t)_{t>0}$ satisfies (2.16). Let f be a non-zero element of $N(\mu)$ and $Rf=m$ on the support of f, where m is a constant. Since R satisfies (S.C.M), $Rf=m$ everywhere, so that $\int_0^t P_s f(x)\,ds=0$ for all $x\in S$. Therefore, from (2.15) it follows that

$$f(x)=\lim_{t\to 0}\,[\int_0^t P_s f(x)\,ds]/t=0$$

for all $x\in S$, which is a contradiction. Therefore if f is a non-zero element of $N(\mu)$, Rf is never equal to a constant on the support of f, which is the meaning of that R is non-singular. Conversely we assume that R is non-singular. In this case we can define a family of measures $(\lambda^E)_{E\in\mathcal{K}}$ and a family of Markov kernels $(H^E)_{E\in\mathcal{K}}$ corresponding to R in the same way as stated in Lemma 2 and Lemma 3 of section 1 respectively. Let $(E_n)_{n\geqq 1}$ be an increasing sequence of $\mathcal{K}$ with the union S and further let $g=\chi_{\{y\}}$ and

$$H^{E_n}g=Rf^{E_n}+\langle\lambda^{E_n},g\rangle\,,$$

where $f^{E_n}\in N^{E_n}$. Then, using (2.9) and (2.13), we have

$$\begin{aligned}(2.17)\qquad P_tH^{E_n}g&=P_tRf^{E_n}+\langle\lambda^{E_n},g\rangle\\&=Rf^{E_n}-\int_0^t P_s f^{E_n}\,ds+\langle\lambda^{E_n},g\rangle\\&=H^{E_n}g-\int_0^t P_s f^{E_n}\,ds\end{aligned}$$

for each n and $t>0$. On the other hand, we know that, for each $(x,y)\in S\times S$, there exists the limit

$$(2.18)\qquad W(x,y)=\lim_{t\to 0}P_t(x,y)$$

and the kernel W is a sub-Markov kernel with $W^2=W$ [1, p. 118]. Therefore, using Fatou's inequality, we have

$$\begin{aligned}(2.19)\qquad WH^{E_n}g(x)&\leqq\liminf_{t\to 0}\,[H^{E_n}g(x)-\int_0^t P_s f^{E_n}(x)\,ds]\\&=H^{E_n}g(x)\end{aligned}$$

for each n and $x\in S$. Noting that $0\leqq H^{E_n}g\leqq 1$ and $\lim_n H^{E_n}g(x)=\chi_{\{y\}}(x)$ $=I(x,y)$ for all $x\in S$, we have from (2.19)

$$(2.20)\qquad W(x,y)\leqq I(x,y)\qquad\text{for all}\quad (x,y)\in S\times S\,.$$

Thus $W(x,y)=w(x)I(x,y)$, where w is a function on S which takes only two values 0 or 1, for $W^2=W$. However, since

$$\mu(y)w(y) = \mu W(y) = \lim_{t\to 0} \mu P_t(y) = \mu(y)$$

for all $y\in S$ and since μ is strictly positive everywhere, we have $w=1$ on S. Therefore.

$$I = W = \lim_{t\to 0} P_t\,.$$

Thus the theorem was proved.

Now the meaning of Corollary 3 of Lemma 5 is the following. Assume that R is non-singular, then the corresponding semi-goup $(P_t)_{t>0}$ in Theorem 3 is continuous at $t=0$. In this case we can find a Markov process $X=(\Omega, \mathcal{M}, (X_t)_{t\geq 0}(\theta_t)_{t\geq 0}, (P_x)_{x\in S})$ with an enlarged state space $\bar{S}$ such that

$$P_x(X_t = y) = P_t(x, y) \qquad \text{for all} \quad (x, y)\in S\times S \quad \text{and} \quad t>0$$

(for precise definitions, see [7]). For any $a\in S$, if we define the family of kernels $({}^aP_t)_{t>0}$ by

$${}^aP_t(x, y) = P_x(X_t = y,\, t<T^a) \qquad \text{for} \quad (x, y)\in S\times S\,,$$

where T^a denotes the first hitting time of the set $\{a\}$, then $({}^aP_t)_{t>0}$ is a sub-Markov semi-group which is continuous at $t=0$. Corollary 3 shows that $({}^aP_t)_{t>0}$ is transient and its potential kernel is ${}^aR$.

3. Examples

In this section we shall give examples of operators satisfying (R.S.C.M) with unbounded measures. Since (R.S.C.M) implies (S.C.M), these are also examples of non-singular operators satisfying (S.C.M).

EXAMPLE 1. Let S be the set of all integers and $\mu(x)=1$ for all $x\in S$. Define a linear operator G by

$$Gf(x) = -\Sigma_{y\in S}|\,y-x\,|f(y) \qquad \text{for} \quad f\in \boldsymbol{N}(\mu)\,. \tag{3.1}$$

Then, by simple calculations, we have the following formulae;

$$Gf(x) = Gf(x-1)+2\,\Sigma_{y\geq x}f(y)\,, \tag{3.2}$$

$$Gf(x) = Gf(x+1)+2\,\Sigma_{x\geq y}f(y)\,, \tag{3.3}$$

$$Gf(x) = \frac{1}{2}\,[Gf(x-1)+Gf((x+1)]+f(x) \tag{3.4}$$

for all $x\in S$. If the support of f is contained in $\{a, a+1\,,\cdots, b\}$, by (3.3) and (3.2), $Gf(x)=Gf(a)$ for $x<a$ and $Gf(x)=Gf(b)$ for $x>b$, respectively. Therefore Gf is bounded on S, that is, G mapps $\boldsymbol{N}(\mu)$ into $\boldsymbol{B}$. To show that G satisifes

(R.S.C.M) we assume $Gf \leqq m$ on the set $\{f>0\}=\{a_1, a_2, \cdots, a_p\}$, where $a_1<a_2<\cdots<a_p$. For each $x<a_1$, using (3.3), we have $Gf(x)\leqq Gf(x+1)+f(x)$ and $Gf(x+1)\leqq Gf(a_1)$, so that $Gf(x)\leqq Gf(a_1)+f(x)\leqq m-f^-(x)$. Similarly, for each $x>a_p$, using (3.2), we have $Gf(x)\leqq m-f^-(x)$. For $a_k<x<a_{k+1}$, using (3.4), we have $Gf(x)\leqq \sup (Gf(a_k), Gf(a_{k+1}))+f(x)\leqq m-f^-(x)$, $k=1, 2, \cdots, p-1$. Therefore $Gf\leqq m-f^-$ everywhere, so that G satisfies (R.S.C.M). Let us introduce a Markov kernel P on S by

$$P(x, y) = \begin{cases} 1/2 & y = x\pm 1 \\ 0 & \text{otherwise}, \end{cases}$$

then μ is an invariant measure for P and the relation (1.6) of Theorem 1 holds for all $f\in \mathbf{N}(\mu)$, for, (1.6) is equivalent to (3.4) in this case. Further, since P is the transition function of (simple) symmetric random walk of dimension one, it is irreducible recurrent. Thus, Theorem 1 is valid for G, though μ is unbounded. If we define the Markov semi-group $(P_t)_{t>0}$ by $P_t=e^{t(P-I)}$, that is,

$$P_t(x, y) = e^{-t}\sum_{n=0}^{\infty}\frac{1}{n!}(tP)^n(x, y) \qquad \text{for} \quad (x, y)\in S\times S,$$

it has an invariant measure μ and a weak potential operator G. Obviously $(P_t)_{t>0}$ is irreducible recurrent, so that Theorem 2 is valid for G, too.

EXAMPLE 2. Let S and μ be the same in Example 1. Define a linear operator G from $\boldsymbol{N}(\mu)$ to $\boldsymbol{B}$ by

$$Gf(x) = \Sigma_{y\geqq x}f(y) \qquad \text{for all} \quad f\in \boldsymbol{N}(\mu). \tag{3.5}$$

To show that G satisfies (R.S.C.M) we assume that $Gf\leqq m$ on the set $\{f>0\}=\{a_1, a_2, \cdots, a_p\}$, where $a_1<a_2<\cdots<a_p$. Since $0\leqq Gf(a_1)\leqq m$, m should be non-negative. If $a_{k-1}<x<a_k$,

$$\begin{aligned} Gf(x) &= Gf(x+1)+f(x)\leqq Gf(a_k)+f(x) \\ &\leqq m-f^-(x), \end{aligned}$$

$k=1, 2, \cdots, p$ (we regard a_0 as $-\infty$). If $x>a_p$,

$$\begin{aligned} Gf(x) &= Gf(x+1)+f(x)\leqq \sup (Gf(a_p), 0)+f(x) \\ &\leqq m-f^-(x). \end{aligned}$$

Consequently $Gf\leqq m-f^-$ everywhere, which shows that G satisfies (R.S.C.M).

Let us now define a Markov kernel P on S by

$$P(x, y) = \begin{cases} 1 & y = x+1 \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, P has μ as an invariant measure and satisfies the relation (1.6) of Theorem 1. However, since $\sum_{n=0}^{\infty} P^n(x, y)=0$ or $=1$ according as $x>y$ or $x\leqq y$, P is not irreducible recurrent. If we define a Markov semi-group $(P_t)_{t>0}$ by $P_t=e^{t(P-I)}$, it has μ as an invariant measure and G as a weak potential opeator. But it is transient in the sense:

$$\int_0^{\infty} P_t(x, y)\,dt<\infty \qquad \text{for all } (x, y)\in S\times S\,.$$

EXAMPLE 3. Let $S=\{0, 1, \cdots\}$ and $\mu(x)=1$ for all $x\in S$. Define a linear operator G from $\boldsymbol{N}(\mu)$ to $\boldsymbol{B}$ by

$$Gf(x) = \textstyle\sum_{y\geqq x} f(y) \qquad \text{for all} \quad f\in \boldsymbol{N}(\mu)\,. \tag{3.6}$$

That G satisfies (R.S.C.M) is proved in the same way as stated in Example 2. Let us introduce a Markov kernel P on S by

$$P(x, y) = \begin{cases} 1 & y = x+1 \\ 0 & \text{otherwise.} \end{cases}$$

Then P satisfies (1.6) of Theorem 1. Since a Markov kernel satisfying (1.6) is unique, P is only such a kernel. However, the relation; $1=\mu(0)>\mu P(0)=0$, shows that μ is not an invariant measure for P. If a Markov semi-group $(P_t)_{t>0}$ with a weak potential operator G exists, it should be equal to that defined by $P_t=e^{t(P-I)}$. Since μ is not an invariant measure for $(P_t)_{t>0}$, there is never Markov semi-group which has μ as an invariant measure and G as a weak potential operator.

Finally we notice some remarks on our problem. We shall assume again that S is any denumerable set and μ is any measure on S, strictly positive everywhere. Let R be a non-singular operator from $\boldsymbol{N}(\mu)$ to $\boldsymbol{B}$ satisfying (R. C. M.), for example, an operator satisfying (R. S. C. M.). Take a function g on S which is strictly positive everywhere and $\langle\mu, g\rangle<\infty$. Define a measure $\tilde{\mu}$ on S by $\tilde{\mu}(x)=g(x)\,\mu(x)$ for all $x\in S$. Then, $f\in \boldsymbol{N}(\tilde{\mu})$ if and only if $gf\in \boldsymbol{N}(\mu)$, so that we may define a linear operator $\tilde{R}$ from $\boldsymbol{N}(\tilde{\mu})$ to $\boldsymbol{B}$ by $\tilde{R}f=R(gf)$. We can easily verify that $\tilde{R}$ is also a non-singular operator satisfying (R. C. M.). Since $\tilde{\mu}$ is bounded, by Theorem 2 and 3, we can find a Markov semi-group $(\tilde{P}_t)_{t>0}$ which is continuous at $t=0$ and has $\tilde{\mu}$ and $\tilde{R}$ as its own invariant measure and weak potential operator, respectively. Let $\tilde{X}=(\Omega, \mathcal{M}, (\tilde{X}_t)_{t\geqq 0}\, (\tilde{\theta}_t)_{t\geqq 0}, (P_x)_{x\in S})$ be a Markov process with a state space $\bar{S}$, some metric completion of S, such that

$$\tilde{P}_t(x, y) = P_x(\tilde{X}_t = y) \qquad \text{for all} \quad (x, y)\in S\times S\,.$$

Let us introduce an additive functional $(A_t)_{t\geqq 0}$ for $\tilde{X}$ by

$$A_t = \begin{cases} \int_0^t [1/g(X_s)]\, ds & \text{for} \quad t < T \\ \infty & \text{for} \quad t \geqq T\,, \end{cases}$$

where $T=\sup\{t\colon \int_0^t [1/g(X_s)]\,ds<\infty\}$. Further we put $C_t=s$ if and only if $A_s=t$ for $s\in[0, T)$. If we denote $X_t=\tilde{X}_{C_t}$ and $\theta_t=\tilde{\theta}_{C_t}$, $X=(\Omega, \mathcal{M}, (X_t)_{t\geqq 0}, (\theta_t)_{t\geqq 0}, (P_x)_{x\in S})$ is a Markov process with a state space $\bar{S}$, too. Using properties of $\tilde{X}$, we can prove that a family of kernels $(P_t)_{t>0}$ on S defined by; $P_t(x, y)=P_x(X_t=y)$ for all $(x, y)\in S\times S$, is a sub-Markov semi-group on S, continuous at $t=0$. If the condition;

$$P_x(T=\infty) = 1 \qquad \text{for all } x\in S\,, \tag{3.7}$$

is satisfied, we can prove that $(P_t)_{t>0}$ is an irreducible recurrent Markov semi-group with an invariant measure μ and a weak potential operator R. In Example 1, condition (3.7) is true, however, in Example 2 and 3, (3.7) is not true. Unwillingly, we could not express these facts as analytic conditions on R.

SHIZUOKA UNIVERSITY

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