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OSCILLATION OF SAMPLE FUNCTIONS IN STATIONARY GAUSSIAN PROCESSES

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1. Introduction

There are many sample function properties of a stationary Gaussian process which satisfy 0–1 law. For example, continuity or unboundedness of sample functions, upper class or lower class and the law of iterated logarithm. In this paper we shall investigate another type of sample property which satisfies 0–1 law.

Let $\{X(t); 0\leq t\leq 1\}$ be a real stationary Gaussian process with the mean $E[X(t)]=0$ which has continuous sample functions with probability one, and let $Q(x)$, $0\leq x<+\infty$, be a continuous increasing function near the origin with $Q(0)=0$. We shall investigate the oscillation of sample functions of $X(t)$ described as follows;

$$(1) \qquad \lim_{||S_n||\to 0} \sum_{i=1}^{m(n)} Q(|X(t_i^{(n)})-X(t_{i-1}^{(n)})|),$$

where $S_n=\{0=t_0^{(n)}<t_1^{(n)}<\cdots<t_{m(n)}^{(n)}=1\}$ is a partition of $[0, 1]$ and $||S_n||=\operatorname*{Max}_{i=1,\dots,m(n)} |t_i^{(n)}-t_{i-1}^{(n)}|$.

In Theorem 1 we shall prove that if $Q(x)$ is suitably chosen, the oscillation (1) satisfies Kolmogorov's 0–1 law for a certain class of stationary Gaussian processes. This class is specified by the conditions on $v(x)=(E[(X(x)-X(0))^2])^{1/2}$ using a regulər increasing function. In Theorem 2 we shall prove that the oscillation (1) has non-zero finite constant with probability one under the stronger conditions of $v(x)$ than those of Theorem 1 and with a nice choice of the seguence of partitions.

In the case that $\{X(t); 0\leq t\leq 1\}$ is the Wiener process, the oscillation (1) for $Q(x)=x^2$ equals 1 with probability one when $\{S_n\}$ is the 2^n equi-partitions (P. Lévy [1]). G. Baxter [2] showed that for the comparatively narrow class of not necessarily stationary Gaussian processes characterized by the conditions on $r(s, t)=E[X(s)X(t)]$, the oscillation (1) for $Q(x)=x^2$ is constant with probability one. E.G. Gladyshev [3] has extended the result of G. Baxter in the direction of the oscillation for $Q(x)=a(n)x^2$, where $a(n)$ is a normalized constant which depends on partitions. Yu.K. Belyaev [8] has treated this problem in a different manner from ours.

In Section 2 we shall state the theorems and in Section 3 we shall prove them. Finally we shall give some examples in Section 4.

2. Theorems

To state the theorems, we shall introduce regular increasing function investigated by J. Karamata [6].

We shall call a real function $P(x)$ defined on $(0, u)$ a *regular increasing function* if $P(x)$ satisfies the following two conditions:

(A–1) $P(x)$ is a strictly increasing positive continuous function.

(A–2) There exists a positive number a such that

$$\lim_{s\downarrow 0}\frac{P(sx)}{P(s)}=x^a$$

holds for every $x>0$.

Set

$$Q(x)=\begin{cases} P^{-1}(x) & \text{if } \ 0<x\leq P(u)=u_0 \\ 0 & \text{if } \ u_0<x\,. \end{cases}$$

Then we have the following theorems.

Theorem 1. *Assume that $v(x)$ is continuous on* $[0, 1]$. *If there exists a regular increasing function $P(x)$ such that*

$$\lim_{s\downarrow 0}\frac{s}{P(s)}=0 \qquad \textit{and} \tag{2}$$

$$\lim_{s\downarrow 0}\frac{v(s)}{P(s)}=1\,, \tag{3}$$

then the oscillation

$$\lim_{||S_n||\to 0}\sum_{i=1}^{m(n)}Q(|X(t_i^{(n)})-X(t_{i-1}^{(n)})|) \tag{4}$$

converges with probability zero or one and if it converges with probability one, the limit is independent of paths.

Remark 1. The condition (2) is fulfilled automatically when $0<a<1$, and from the stationarity of the process the condition (3) can be satisfied only if $a\leq 1$.

Remark 2. By the criterion due to X. Fernique [4], the condition (3) implies that $X(t)$ has continuous sample functions with probability one.

Theorem 2. *Suppose that there exists a regular increasing function $P(x)$*

such that the condition (3) *of Theorem* 1 *is satisfied for* $0<a<1$, *and that* $v(x)$ *has twice continuous derivatives on* $(0, 1)$ *which satisfy the following conditions*:

$$(5) \qquad \lim_{s\downarrow 0} \frac{v'(sx)}{v'(s)} = x^{a-1}, \quad \textit{and} \quad \lim_{s\downarrow 0} \frac{v''(sx)}{v''(s)} = x^{a-2}.$$

In addition suppose that the partition $\{S_n\}$ *satisfies the following conditions*:

$$(6) \qquad \|S_n\| = \max_i |t_i^{(n)} - t_{i-1}^{(n)}| \le c_0 2^{-\alpha n}, \quad \textit{and}$$
$$\min_i |t_i^{(n)} - t_{i-1}^{(n)}| \ge c_0 2^{-\beta n},$$

where α, β *satisfy the relation* $2\alpha > \beta \ge \alpha > 0$, *and* $c_0 > 0$.
Then

$$(7) \qquad \lim_{n\to\infty} \sum_{i=1}^{m(n)} Q(|X(t_i^{(n)} - X(t_{i-1}^{(n)})|) = \int_{-\infty}^{\infty} |x|^{1/a} \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)} dx,$$

holds with probability one.

3. Proofs of theorems

First we shall list the properties of regular increasing functions in Lemma 1.

Lemma 1. *Let* $Q(x)$ *be the function defined in* §2. *Then we have*
(i) $Q(x)$ *is a regular increasing function on* $(0, u_0]$.
(ii) *It holds that*

$$\lim_{s\downarrow 0} \frac{Q(sx)}{Q(s)} = x^{1/a}, \qquad \textit{for any} \quad x>0,$$

uniformly in x on any compact set contained in the half line $(0, \infty)$.
(iii) $Q(x)$ *can be expressed in the form*

$$Q(x) = \begin{cases} x^{1/a} q(x), & 0<x\le u_0, \\ 0, & u_0<x, \end{cases}$$

where $q(x)$ *is a function satisfying the following condition* (S).

$$(S) \qquad \lim_{s\downarrow 0} \frac{q(sx)}{q(s)} = 1 \qquad \textit{for any} \quad x>0,$$

(iv) *For any* $\varepsilon>0$, *it holds that*

$$\lim_{s\downarrow 0} s^{\varepsilon} q(s) = 0.$$

(v) *For any* $\varepsilon>0$, *it holds that*

$$\lim_{s\downarrow 0} s^{-\varepsilon} q(s) = +\infty.$$

(vi) *For any* $\varepsilon>0$, *there exists* $c(\varepsilon)=c_1>0$ *such that*

$$\frac{q(sx)}{q(s)}\leq c_1 x^{\varepsilon}$$

holds for any $x\geq 1$ *and any* $s>0$ *with* $0<sx\leq u_0$.
The proof of Lemma 1 is omitted. (Many properties of regular increasing functions are found in [7].)

Let $\{X(t); 0\leq t\leq 1\}$ be a stationary Gaussian process with continuous paths. consider the spectral decomposition of $X(t)$;

$$X(t)=\int_{-\infty}^{\infty} e^{i\lambda t}dM(\lambda).$$

Set

$$Z_n(t)=\int_{n-1\leq|\lambda|<n} e^{i\lambda t}dM(\lambda).$$

Then we have

Lemma 2. (K. Itô-M. Nisio [5]). *The sum*

$$\sum_{n=1}^{N} Z_n(t)$$

converges to $X(t)$ *uniformly in* t *with probability one.*

For the proof of Theorem 1 we proceed as follows. By virtue of Lemma 2 and Remark 2 of Theorem 1, $X(t)$ can be expanded as a series of independent random variables $Z_n(t)$ which converges uniformly in t with probability one;

$$X(t)=\sum_{n=1}^{\infty} Z_n(t) \text{ uniformly in } t. \tag{8}$$

We can therefore choose such a basic probability space $(\Omega, \mathcal{B}, P)$ that for every $\omega\in\Omega$ the series $\sum_{n=1}^{\infty} Z_n(t, \omega)$ converges uniformly in t.

Let $\mathcal{B}_N$ be the smallest σ-algebra generated by $\{Z_i(t); i=N, N+1, \cdots\}$. An event which is measurable with respect to $\mathcal{B}_\infty=\bigcap_{N=1}\mathcal{B}_N$ is called a tail event.

Set

$$X_N(t)=\sum_{i=N}^{\infty} Z_i(t), \quad Y_N(t)=\sum_{i=1}^{N-1} Z_i(t),$$

and

$$\Delta_i^n X(t)=X(t_i^{(n)})-X(t_{i-1}^{(n)}), \quad \Delta_i^n t=t_i^{(n)}-t_{i-1}^{(n)}.$$

Then we shall prove that the event $A=\{\omega; \lim_{\|S_n\|\to 0}\sum_{i=1}^{m(n)} Q(|\Delta_i^n X(t)|) \text{ converges}\}$ is a tail event.

Set $\Omega_\delta^n=\{\omega; |\Delta_i^k Y_N(\mathbf{t})|\leq\delta^2 v(\Delta_i^k t) \text{ for all } k\geq n, \text{ and } i=1, 2, \cdots, m(k)\}$. Then we have

(9) $$\Omega_\delta^n \subset \Omega_\delta^{n+1} \subset \cdots .$$

Since the path $Y_N(t, \omega)$ is differentiable and $\lim_{x \downarrow 0} x/v(x) = 0$ from the conditions (2) and (3), we have

(10) $$\Omega = \bigcup_{n=1}^{\infty} \Omega_\delta^n ,$$

for any $\delta > 0$ and any N.

Fix positive integer N arbitrarily. There exists positive $\delta_0 \left(\leq \frac{1}{2}\right)$ such that for any positive $\delta < \delta_0$ and any $s \leq u$ we have

(11) $$2(\delta + \delta^2) P(s) \leq u_0 .$$

From the condition (3) of Theorem 1 follows the assertion: For any ε and any δ with $0 < \varepsilon < 1$ and $0 < \delta < \delta_0$ respectively, there exist s_0 with $0 < s_0 < \frac{u_0}{2}$ such that for any $s < s_0$ we have

(12) $$v(s) < (1 + \varepsilon) p(s) .$$

By Lemma 1 (ii), for any x such that $\min\left(\delta + \delta^2, \frac{1}{2}\right) \leq x \leq \frac{3}{2}$ and for any $s \leq s_0$ we have

(13) $$\left| \frac{Q(sx)}{Q(s)} - x^{1/a} \right| < \varepsilon .$$

There exists n_0 such that for any $n \geq n_0$ we have

(14) $$P(\|S_n\|) \leq s_0 .$$

For any fixed $\omega \in \Omega$, there exists n_1 and for any $n \geq n_1$ we have

(15) $$|\max_i \Delta_i^n X_N(\omega)| \leq s_0 .$$

Set $n_2 = \max(n_0, n_1)$. Then the above ω satisfies

(16) $$\Omega_\delta^n \ni \omega \qquad \text{for any} \quad n \geq n_2 .$$

For such an ω, satisfying (15) and (16), set

(17) $$K_n = \{i;\ |\Delta_i^n X_N(t)| \leq \delta\, v(\Delta_i^n t)\} , \qquad n \geq n_2 .$$

Then it follows that

$$\begin{aligned} \sum_{i \in K_n} Q(|\Delta_i^n X_1(t)|) &= \sum_{i \in K_n} Q(|\Delta_i^n X_N(t) + \Delta_i^n Y_N(t)|) \\ &\leq \sum_{i \in K_n} Q((\delta + \delta^2) v(\Delta_i^n t)) \end{aligned}$$

$$\leq \sum_{i\in K_n} Q((\delta+\delta^2)(1+\varepsilon)P(\Delta_i^n(t)))$$
$$\leq \sum_{i\in K_n} \Delta_i^n t\{(\delta+\delta^2)^{1/a}(1+\varepsilon)^{1/a}+\varepsilon\}$$
$$\leq (\delta+\delta^2)^{1/a}(1+\varepsilon)^{1/a}+\varepsilon$$

and

$$\sum_{i\notin K_n} Q(|\Delta_i^n X_1(t)|) = \sum_{i\notin K_n} Q\left(|\Delta_i^n X_N(t)|\left|1+\frac{\Delta_i^n Y_N(t)}{\Delta_i^n X_N(t)}\right|\right)$$
$$\leq \sum_{i\notin K_n} Q(|\Delta_i^n X_N(t)|(1+\delta))$$
$$\leq \sum_{i\notin K_n} Q(|\Delta_i^n X_N(t)|)\{(1+\delta)^{1/a}+\varepsilon\}\,.$$

Analogously we have

$$\sum_{i\notin K_n} Q(|\Delta_i^n X_1(t)|) \geq \sum_{i\notin K_n} Q(|\Delta_i^n X_N(t)|)\{(1-\delta)^{1/a}-\varepsilon\}\,.$$

Hence it follows that

$$\begin{aligned}\overline{\lim_{\|S_n\|\to 0}} \sum_{i=1}^{m(n)} Q(|\Delta_i^n X_1(t)|) &= \overline{\lim_{\|S_n\|\to 0}} \sum_{i=1}^{m(n)} Q(|\Delta_i^n X_N(t)|)\\ \underline{\lim_{\|S_n\|\to 0}} \sum_{i=1}^{m(n)} Q(|\Delta_i^n X_1(t)|) &= \underline{\lim_{\|S_n\|\to 0}} \sum_{i=1}^{m(n)} Q(|\Delta_i^n X_N(t)|)\end{aligned} \tag{18}$$

Since the equation (18) is true for any N and since the ΔZ_n are independent we can concluded that the event $A=\{\omega$; the oscillation (4) converges$\}$ is a tail event. By virtue of Kolmogorov's 0–1 law, the probability of the event A is 0 or 1.

(q. e. d.)

In order to prove Theorem 2, we shall prove the following lemma.

Lemma 3. *If $v(x)$ satisfies the conditions of Theorem 2, then for any $\varepsilon>0$, there exists $c_2(\varepsilon)=c_2>0$ and for any s, h_1, h_2 such that $s/4\geq h_1$, $h_2>0$ it holds*

$$|r(s+h_1-h_2)+r(s)-r(s-h_2)-r(s+h_1)| \leq c_2\frac{h_1^{1-a-(\varepsilon/2)}h_2^{1-a-(\varepsilon/2)}}{s^{2-2a+\varepsilon}}v(h_1)v(h_2)\,,$$

where $r(t)=E[X(t)X(0)]$.

Proof. By the assumption of Theorem 2, we have

$$\frac{d^2v^2(x)}{dx^2} = x^{2a-2}f(x)\,, \tag{19}$$

where $f(x)$ satisfies the property (S).

From an easy calculation, it follows that

$$\begin{aligned}&r(s+h_1-h_2)+r(s)-r(s-h_2)-r(s+h_1)\\ &\quad= \frac{1}{2}(v^2(s+h_1)+v^2(s-h_2)-v^2(s+h_1-h_2)-v^2(s))\,.\end{aligned} \tag{20}$$

Without loss of generality, we can assume $h_1 \leq h_2$. Then we have

$$\begin{aligned}(21)\qquad &|r(s+h_1-h_2)-r(s-h_2)-(r(s+h_1)-r(s))| \\ &= \frac{h_1}{2}\left|\left\{\frac{dv^2}{dt^2}(s-h_2+\theta_1(h_1))-\frac{dv^2}{dt^2}(s+\theta_2(h_1))\right\}\right| \\ &\leq h_1 h_2 \left|\left\{\frac{d^2v^2}{dt^2}(s+\theta_3(h_1, h_2))\right\}\right| ,\end{aligned}$$

where $0 \leq \theta_1(h_1)$, $\theta_2(h_1) \leq h_1$, and $0 \leq \theta_3(h_1, h_2) \leq 2h_1+h_2$. Hence it follows from (19), (20), (21) and Lemma 1 that

$$\begin{aligned}&|r(s+h_1-h_2)+r(s)-r(s-h_2)-r(s+h_1)| \\ &= \left|\frac{d^2v^2(t)}{dt^2}\right|_{t=s+\theta_3(h_1, h_2)}\left|\frac{h_1 h_2}{v(h_1)v(h_2)}\right| v(h_1)v(h_2) \\ &\leq c_2 \frac{h_1^{1-a-(\varepsilon/2)} h_2^{1-a-(\varepsilon/2)}}{s^{2-2a+\varepsilon}} v(h_1)v(h_2) . \qquad \text{(q.e.d.)}\end{aligned}$$

We now come to the proof of Theorem 2. First we show that

$$(22)\qquad \lim_{n\to\infty} E[\sum_{i=1}^{m(n)} Q(|\Delta_i^n X(t)|)] = \int_{-\infty}^{\infty} |x|^{1/a} \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)} dx .$$

From the condition (3) of Theorem 1, for any $0<\varepsilon<1$, there exists $s_0>0$ and for any $s<s_0$ we have

$$(23)\qquad P(s)(1-\varepsilon) \leq v(s) \leq P(s)(1+\varepsilon)$$

For any $0<\delta<1$ there exists $s_1>0$ and for any $s<s_1$ and any $x\in[\delta, \delta^{-1}(1+\varepsilon)]$, we have from Lemma 1 (ii)

$$(24)\qquad \left|\frac{Q(xs)}{Q(s)}-x^{1/a}\right| \leq \varepsilon .$$

There exists n_0 such that for any $n \geq n_0$, we have

$$(25)\qquad 2P(||S_n||) \leq \min(s_0, s_1),\ P(||S_n||)(1+\varepsilon) < \delta u_0 .$$

We can therefore estimate the expectation $E[Q(|\Delta_i^n X(t)|)]$ as follows:

$$\begin{aligned}&\int_{|x|\leq u_1} Q(|x|v(\Delta_i^n t))\frac{1}{\sqrt{2\pi}} e^{-(x^2/2)} dx, \ \left(u_1 = \frac{u_0}{v(\Delta_i^n t)}\right) \\ &= \left(\int_{|x|<\delta} + \int_{\delta\leq|x|\leq\delta^{-1}}\right) Q(|x|P(\Delta_i^n t)(1+\varepsilon))\frac{1}{\sqrt{2\pi}} e^{-(x^2/2)} dx \\ &\quad + \int_{\delta^{-1}<|x|\leq u_1} Q(|x|v(\Delta_i^n t))\frac{1}{\sqrt{2\pi}} e^{-(x^2/2)} dx \\ &= I_1 + I_2 + I_3 .\end{aligned}$$

Hence we have from (23), (24), (25) and Lemma 1 (vi),

$$I_1 \leq Q(\delta(1+\varepsilon)P(\Delta_i^n t)) \int_{|x|<\delta} \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)} dx$$
$$\leq \Delta_i^n t\{\delta^{1/a}(1+\varepsilon)^{1/a}+\varepsilon\}$$
$$I_2 \leq \int_{\delta\leq\delta|x|\leq\delta^{-1}} \Delta_i^n t\{|x|^{1/a}(1+\varepsilon)^{1/a}+\varepsilon\} \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)} dx$$
$$I_3 \leq \int_{\delta^{-1}<|x|\leq u_1} c_1 Q(v(\Delta_i^n t))(1+\varepsilon)^{(1/a)+\varepsilon'} x^{(1/a)+\varepsilon'} \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)} dx$$
$$\leq \int_{\delta^{-1}<|x|} c_1 Q((1+\varepsilon)P(\Delta_i^n t))(1+\varepsilon)^{(1/a)+\varepsilon'} x^{(1/a)+\varepsilon'} \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)} dx$$
$$\leq \int_{\delta^{-1}<|x|} \Delta_i^n t \cdot c_1^2 (1+\varepsilon)^{(2/a)+2\varepsilon'} |x|^{(1/a)+\varepsilon'} \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)} dx .$$

Hence it follows that

$$\lim_{n\to\infty} \sum_{i=1}^{m(n)} E[Q(|\Delta_i^n X(t)|)] \leq \int_{-\infty}^{\infty} |x|^{1/a} \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)} dx .$$

Analogously we have

$$\lim_{n\to\infty} \sum_{i=1}^{m(n)} E[Q(|\Delta_i^n X(t)|)] \geq \int_{-\infty}^{\infty} |x|^{1/a} \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)} dx .$$

Thus the equation (22) is proved.

Next we shall estimate

$$B_n = E[(\sum_{i=1}^{m(n)} Q(|\Delta_i^n X(t)|) - \sum_{i=1}^{m(n)} E(Q(|\Delta_i^n X(t)|))^2] .$$

Fix $b>0$ such that

$$b < \frac{(2\alpha-\beta)(2-2a)}{\beta} ,$$

then we can choose $\mu>0$ such that

$$\text{(26)} \qquad \frac{2-2a-b}{2-2a+b} \alpha > \mu > \beta - \alpha .$$

By virtue of (12), for $b>0$ fixed above, there exist $c_3(b)=c_3$ and n_0 such that for any $n\geq n_0$ we have

$$\text{(27)} \qquad Q(|x|v(\Delta_i^n t)) \leq c_3 \Delta_i^n t(|x|^{(1/a)+b}+1)$$

For fixed ω such that (26) is satisfied, set

$$B_n = \Big(\sum_{|t_{i-1}^{(n)}-t_{j-1}^{(n)}|<2^{-\mu n}} + \sum_{|t_{i-1}^{(n)}-t_{j-1}^{(n)}|\geq 2^{-\mu n}}\Big)\{E[Q(|\Delta_i^n X(t)|)Q(|\Delta_j^n X(t)|)]$$
$$-E[Q(|\Delta_i^n X(t)|)]E[Q(|\Delta_j^n X(t)|)]\} = B_n^{(1)} + B_n^{(2)} .$$

By Schwartz's inequality and (27), we have

$$(28) \qquad B_n^{(1)} \leq \sum_{|t_{i-1}^{(n)} - t_{j-1}^{(n)}| \leq 2^{-\mu n}} \sqrt{E[Q(|\Delta_i^n X(t)|)^2] E(Q(|\Delta_j^n X(t)|)^2]} + E[Q(|\Delta_i^n X(t)|)] E[Q(|\Delta_j^n X(t)|)]$$
$$\leq c_4 \sum_{|t_{i-1}^{(n)} - t_{j-1}^{(n)}| \leq 2^{-\mu n}} \Delta_i^n t \Delta_j^n t$$
$$\leq c_4 2^{-(\mu+\alpha-\beta)n},$$

where c_4 is a constant independent of n.

$$B_n^{(2)} = \sum_{|t_{i-1}^{(n)} - t_{j-1}^{(n)}| \geq 2^{-\mu n}} \int_{-\infty}^{\infty} Q(|x_1| v(\Delta_i^n t)) Q(|x_2| v(\Delta_j^n t)) \frac{1}{2\pi} e^{-\frac{(x_1^2+x_2^2)}{2}}$$
$$\times \left\{ \frac{v(\Delta_i^n t) v(\Delta_j^n t)}{\sqrt{v^2(\Delta_i^n t)^2 v^2(\Delta_j^n t) - \Delta_{ij}^n r^2}} \exp\left[-\frac{\Delta_{ij}^n r^2 (x_1^2+x_2^2) - 2\Delta_{ij}^n r v(\Delta_i^n t) v(\Delta_j^n t) x_1 x_2}{2(v^2(\Delta_i^n t) v^2(\Delta_j^n t) - \Delta_{ij}^n r^2)} \right] - 1 \right\} dx_1 dx_2,$$

where $\Delta_{ij}^n r = E[\Delta_i^n X(t) \Delta_j^n X(t)]$.

Making use of the relation

$$(29) \qquad (Bx, x) = \frac{1}{2}(x, x) + \frac{1}{2}(Cx, Cx),$$

where

$$B = \frac{1}{v_1^2 v_2^2 - r^2} \begin{pmatrix} v_1^2 v_2^2, & -r v_1 v_2 \\ -r v_1 v_2, & v_1^2 v_2^2 \end{pmatrix}$$
$$C = \frac{1}{v_1^2 v_2^2 - r^2} \begin{pmatrix} r, & -v_1 v_2 \\ -v_1 v_2, & r \end{pmatrix} \quad \text{and} \quad (x, y) = x_1 y_1 + x_2 y_2,$$

we have

$$e^{-\frac{(x_1^2+x_2^2)}{2}} \left\{ \frac{v(\Delta_i^n t) v(\Delta_j^n t)}{\sqrt{v^2(\Delta_i^n t) v^2(\Delta_j^n t) - \Delta_{ij}^n r^2}} \exp\left[\frac{-\Delta_{ij}^n r^2 (x_1^2+x_2^2) - 2\Delta_{ij}^n r v(\Delta_i^n t) v(\Delta_j^n t) x_1 x_2}{2(v^2(\Delta_i^n t) v^2(\Delta_j^n t) - \Delta_{ij}^n r^2)} \right] - 1 \right\}$$
$$= e^{-(x, x)/4} \left\{ \frac{v(\Delta_i^n t) v(\Delta_j^n t)}{\sqrt{v^2(\Delta_i^n t) v^2(\Delta_j^n t) - \Delta_{ij}^n r^2}} e^{-(Cx, Cx)/4} - e^{-(x, x)/4} \right\}$$
$$\leq \frac{|\Delta_{ij}^n r|}{v(\Delta_i^n t) v(\Delta_j^n t)} e^{-(x, x)/4} (1 + x_1^2 + x_2^2).$$

Therefore, it follows from (27) that

$$B_n^{(2)} \leq c_3 \sum_{|t_{i-1}^{(n)} - t_{j-1}^{(n)}| \geq 2^{-\mu n}} \Delta_i^n t \Delta_j^n t \frac{|\Delta_{ij}^n r|}{v(\Delta_i^n t) v(\Delta_j^n t)} \times$$

$$\cdot \int_{-\infty}^{\infty} (|x|^{(1/a)+b}+1)(|x_2|^{(1/a)+b}+1)(1+x_1^2+x_2^2) e^{-\frac{(x_1^2+x_2^2)}{4}} dx_1 dx_2$$

$$\leq c_5 \max_{|t_{i-1}^{(n)} - t_{j-1}^{(n)}| \geq 2^{-\mu n}} \frac{|\Delta_{ij}^n r|}{v(\Delta_i^n t) v(\Delta_j^n t)},$$

where c_5 is a constant independent on i, j, n.

On the other hand by Lemma 3, it follows that

$$\frac{|\Delta_{ij}^n r|}{v(\Delta_i^n t) v(\Delta_j^n t)} = \frac{|r(|t_i^{(n)} - t_j^{(n)}|) + r(|t_{i-1}^{(n)} - t_{j-1}^{(n)}|) - r(|t_i^{(n)} - t_{j-1}^{(n)}|) - r(|t_{j-1}^{(n)} - t_j^{(n)}|)}{v(\Delta_i^n t) v(\Delta_j^n t)}$$

$$\leq c_2 \frac{(\Delta_j^n t)^{1-a-(b/2)} (\Delta_j^n t)^{1-a-(b/2)}}{|t_{i-1}^{(n)} - t_{j-1}^{(n)}|^{2-2a+b}}.$$

Hence we have

$$B_n^{(2)} \leq c_2 c_5 2^{-n\{\alpha(2+a-b) - \mu(2-2a+b)\}}. \tag{30}$$

Set
$$\Omega_d^{(n)} = \left\{\omega;\ \left|\sum_{i=1}^{m(n)} Q(|\Delta_i^n X(t)|) - d\right| > \frac{1}{n}\right\},$$
where
$$d = \int_{-\infty}^{\infty} |x|^{1/a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

Then by Chebyshev's inequality and from (28) and (30) we have

$$P(\Omega_d^{(n)}) \leq n^2 (c_4 2^{-(\mu+\alpha-\beta)n} + c_2 c_5 2^{-n\{\alpha(2+a-b) - \mu(2-2a+b)\}}).$$

Since μ satisfies (26), it follows that

$$\sum_{n=1}^{\infty} P(\Omega_d^{(n)}) < +\infty.$$

Thus Theorem 2 can be proved immediately by Borel-Cantelli lemma. (q.e.d.)

Remark. It is obvious that $Q(x)$ can be a extended arbitrarily to $x > u_0$ so far as $Q(x)$ satisfies the inequality $Q(x) \leq cx^b$ $(b > 0)$.

4. Examples

Example 1. Let $\{X(t):\ 0 \leq t \leq 1\}$ be a stationary Gaussian process with $E[X(t)] = 0$ having the spectral density $f(\lambda)$ such that

$$f(\lambda) = \begin{cases} c\lambda^{-a} & \text{if } \lambda \geq 1 \\ 0 & \text{if } 0 \leq \lambda < 1, \end{cases}$$

where $\quad 1 < a < 3, \quad \text{and} \quad c = \left(\int_1^{\infty} \lambda^{-a} d\lambda\right)^{-1}.$

Then $$v^2(x) = 2\int_0^\infty \sin^2\frac{x\lambda}{2} f(\lambda)d = c_1|x|^{a-1}.$$

Put $P(x)=c_1 x^{(a-1)/2}$, then it follows that

$$\lim_{x\downarrow 0}\frac{P(x)}{v(x)} = 1.$$

Thus $Q(x)$ is defined as follows;

$$Q(x) = c_1^{-2/(a-1)} x^{2/(a-1)}.$$

From an easy calculation, it follows that $v(x)$ is twice differentiable at $x\neq 0$, and satisfies the conditions of Theorem 2. Therefore for the sequence of partitions which satisfies the conditions of Theorem 2, we have

$$\lim_{n\to\infty}\sum_{i=1}^{m(n)} |X(t_i^{(n)})-X(t_{i-1}^{(n)})|^{2/(a-1)} = c_1^{2/(a-1)}\int_{-\infty}^{\infty} |x|^{2/(a-1)}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx,$$

with probability one.

EXAMPLE 2. Let $\{x(t);\ 0\leq t\leq 1\}$ be a stationary Markov Gaussian process with $E[x(t)]=0$ and $E[X(t)^2]=1$. Then there exists some constant $k>0$ and

$$E[(X(t)-X(s))^2] = 1-e^{-k|t-s|}.$$

Thus we have for $P(x)=\sqrt{kx}$

$$\lim_{x\downarrow 0}\frac{v(x)}{P(x)} = 1.$$

Hence $Q(x)$ is defined as follows;

$$Q(x) = k^{-1}x^2.$$

Therefore for the sequence of partitions which satisfies the conditions of Theorem 2, we have

$$\lim_{n\to\infty}\sum_{i=1}^{m(n)} |X(t_i^{(n)})-X(t_{i-1}^{(n)})|^2 = k,$$

with probability one.

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