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ON MULTIPLY TRANSITIVE GROUPS II

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Let G be a 4-fold transitive group on $\Omega=\{1, 2, \cdots, n\}$ and $H=G_{1,2,3,4}$ the subgroup of G consisting of all the elements leaving the four letters 1, 2, 3 and 4 invariant.

Jordan ([2]) proved in 1872 that if $H=1$ then G must be one of the following groups: S_4, S_5, A_6 or M_{11}, where M_{11} is the Mathieu group of degree 11. The Jordan's theorem was extended by M. Hall ([1], Theorem 5.8.1) in the following way: If H is of odd order then G must be one of the following groups: S_4, S_5, A_6, A_7 or M_{11}.

In this paper, we shall treat the case in which H is of even order and prove the following two theorems.

Theorem 1. *Let G and H be as above, P a Sylow 2-subgroup of H and let $\Delta=\{1, 2, 3, 4, \cdots\}$ be the totality of the latters left invariant by P. If H is of even order, P is an elementary abelian group and transitive on $\Omega-\Delta$, then G must be one of the following groups*: S_6, A_8 *or* M_{23}.

Theorem 2. *Let G, H, P and Δ be as in Theorem 1. If H is of even order, P is a normal subgroup of H and transitive on $\Omega-\Delta$, then G must be one of the following groups*: S_6, A_8, M_{12} *or* M_{23}.

In the proofs of these theorems a central involution of P is important. From the theorem of M. Hall it follows that the number of letters in Δ is not greater than 11. On the other hand, by the transitivity of P on $\Omega-\Delta$, the set of all letters left invariant by a central involution of P coincides with Δ. This shows that a central involution of P leaves only small number of letters invariant. This property of a central involution is essential in the following arguments.

DEFINITIONS AND NOTATIONS. A permutation a is called *semi-regular* if there is no letter left invariant by a. A permutation group S is called *regular* if S is transitive and each element of S, which is different from 1, is semi-regular.

For a set X let $|X|$ denote the number of elements of X. For a set S of permutations on Ω, the totality of the letters left invariant by

S is denoted by $I(S)$. If a subset Δ of Ω is a fixed block of S, i.e. if $\Delta^S=\Delta$, then the restriction of S on Δ is a set of permutations on Δ. We denote it by S^Δ. If S is a permutation group then we have a natural homomorphism $S \rightarrow S^\Delta$. The kernel of this homomorphism consists of all the elements of S leaving each letter in Δ invariant. We shall denote it by S_Δ.

1. Proof of Theorem 1

In this and the next sections, we assume that G is a 4-fold transitive group on $\Omega=\{1, 2, \cdots, n\}$ and $H=G_{1,2,3,4}$ is of even order. We denote a Sylow 2-subgroup of H by P, $I(P)$ by Δ, and the normalizer of P in G by N.

Now it is known that a 4-fold transitive group of degree less than 35 is one of the four Mathieu groups M_{11}, M_{12}, M_{23}, M_{24} or a symmetric or alternating group ([1], p. 80). Therefore our theorems are true for $n<35$. (For the Mathieu groups, see [5].) In the following, we assume that

(*) *n is not less than* 35

and we shall show that there are no groups satisfying the assumptions of our theorems.

The following lemma is an easy consequence of the theorem of M. Hall.

Lemma 1. *The restriction N^Δ of N on Δ is one of the following groups*: S_4, S_5, A_6, A_7 *or* M_{11}.

Proof. By a theorem of Witt ([5]; [1], Theorem 5.7.1) N^Δ is 4-fold transitive. Since $(N^\Delta)_{1,2,3,4} \simeq (N \cap H)/N_\Delta$ and $N_\Delta \supset P$, $(N^\Delta)_{1,2,3,4}$ is of odd order. Then the lemma follows from the theorem of M. Hall.

Lemma 2. *An element c of the center of P, which is different from* 1, *is semi-regular on $\Gamma=\Omega-\Delta$, i.e. $I(c)=\Delta$ if P is transitive on Γ.*

Proof. Let c be a central element of P. Cleary $\Delta \subset I(c)$. Since $\Gamma \cap I(c)$ is a fixed block of P, $\Gamma \cap I(c)=\phi$ or Γ. If $\Gamma \cap I(c)=\Gamma$ then $c=1$. Therefore $\Gamma \cap I(c)=\phi$, i.e. $I(c) \subset \Omega-\Gamma=\Delta$. Thus we have $\Delta=I(c)$ and c is semi-regular on Γ.

Proof of Theorem 1. We assume that P is an elementary abelian group and transitive on $\Gamma=\Omega-\Delta$. Then by Lemma 2 P is regular on Γ.

Since G is 4-fold transitive, there is an involution

$$a = (1)(2)(3, 4) \cdots$$

in G which is conjugate to an involution of P. Then $|I(a)|=|\Delta|$. An involutive automorphism α of H is induced by a and then α fixes at least one Sylow 2-subgroup of H. We may assume that α fixes P. Let P_0 be the subgroup of P consisting of all the elements left invariant by α. Since $|I(a)|=|\Delta|$ and a takes the letter 3 in Δ to the letter 4, there is a letter i in $\Gamma\cap I(a)$. By the regularity of P on Γ, for any j in Γ there exists only one element b in P such that $b=(i, j)\cdots$ and then $j\in I(a)$ if and only if $b\in P_0$. Therefore $|P_0|=|\Gamma\cap I(a)|\leq|I(a)|-2=|\Delta|-2$. By Lemma in [3] we have $|P|\leq|P_0|^2\leq(|\Delta|-2)^2$. Now by Lemma 1 $|\Delta|$ is one of the following number: 4, 5, 6, 7 or 11. If $|\Delta|\leq 7$ then $|P|=|\Gamma|\leq 25$. Therefore $n=|\Delta|+|\Gamma|\leq 32$. This contradicts the assumption (*). Therefore we may assume $|\Delta|=11$. Then $|\Gamma|=|P|\leq 9^2=81$. Since $|P|$ is a power of 2, $|\Gamma|=64$, 32, 16 or 8. For $|\Gamma|=16$ or 8, $n<35$. Therefore we may treat the two cases in which $|\Gamma|=64$, $n=75$ and $|\Gamma|=32$, $n=43$.

Since $|N^\Delta|$ is divisible by 11 and $n<11^2$ in our cases, there is an element x of order 11 in N such that x^Δ is a cycle of length 11. If x is a cycle of length 11, then by a theorem of Jordan ([4]) G must be a symmetric or alternating group. Therefore by the assumption of the theorem G must be S_6 or A_8. This contradicts the assumption (*). Thus x must have at least two cycles of length 11. Let σ be the automorphism of P induced by x and let P^* be the subgroup of P consisting of all the elements left invariant by σ. In our cases $n\not\equiv 0 \pmod{11}$, therefore $I(x)\neq\phi$. Now, since $I(x)\subset\Gamma$ and P is regular on Γ, we have $|I(x)|=|P^*|$ by the same argument as is used above for α. Therefore $|I(x)|$ is a power of 2 and n must be of the form $11k+2^l$ with $k\geq 2$. But this is impossible for $n=75$ or 43.

2. Proof of Theorem 2

Let G, H, P, N and Δ be as in § 1 and assume (*).

We assume here that P is a normal subgroup of H and transitive on $\Gamma=\Omega-\Delta$. By Lemma 1 we have one of the following cases:

CASE I. $N^\Delta=M_{11}$, $|\Delta|=11$,

CASE II. $N^\Delta=A_7$, $|\Delta|=7$,

CASE III. $N^\Delta=A_6$, $|\Delta|=6$,

CASE IV. $N^\Delta=S_5$, $|\Delta|=5$,

CASE V. $N^\Delta=S_4$, $|\Delta|=4$.

We now treat these cases separately.

CASE I. $N^{\Delta}=M_{11}$, $|\Delta|=11$.

Since $N_{1,2,3,4}=N\cap H=H$ and $(N^{\Delta})_{1,2,3,4}=(N_{1,2,3,4})^{\Delta}=1$, we have $I(H)=\Delta$.

Now, since G is 4-fold transitive, there is an involution $a=(1,2)(3,4)\cdots$ which is conjugate to a central involution of P. Then $|I(a)|=11$ by Lemma 2, $a\in N$ and a^{Δ} is an involution of M_{11}. Therefore $|I(a^{\Delta})|=|I(a)\cap I(H)|=3$ (see [5]). For any subset $\{i_1, i_2\}$ of $I(a)$ consisting of two letters, $G_{1,2,i_1,i_2}$ is normalized by a. From the same reason as above we have $|I(a)\cap I(G_{1,2,i_1,i_2})|=3$. Let $I(a)\cap I(G_{1,2,i_1,i_2})=\{i_1, i_2, i_3\}$. Then $G_{1,2,i_1,i_2}=G_{1,2,i_1,i_3}=G_{1,2,i_2,i_3}$. Now consider the mapping $\varphi:\{i_1, i_2\}\rightarrow G_{1,2,i_1,i_2}$ from the set of all the subsets of $I(a)$ consisting of two letters into the set of subgroups of G. Then each $\varphi^{-1}(G_{1,2,i_1,i_2})$ consists of three subsets, since if $G_{1,2,i_1,i_2}=G_{1,2,j_1,j_2}$ for some $\{j_1, j_2\}\subset I(a)$ then $\{j_1, j_2\}\subset I(a)\cap I(G_{1,2,i_1,i_2})=\{i_1, i_2, i_3\}$. Therefore we have ${}_{11}C_2=55\equiv 0 \pmod 3$. This is a contradiction.

CASE II. $N^{\Delta}=A_7$, $|\Delta|=7$.

Let $a=(1,2)\cdots$ be an involution which is conjugate to a central involution of P. Then $|I(a)|=7$.

For a subset $\{i_1, i_2\}$ of $I(a)$ consisting of two letters, let P' be a Sylow 2-subgroup of $G_{1,2,i_1,i_2}$ and let $\Delta'=I(P')$. Then a normalizes P' and $a^{\Delta'}$ is an even permutation. Therefore we have

$$a = (1,2)(i_1)(i_2)(i_3)(k,l)\cdots,$$

where $\Delta'=\{1, 2, i_1, i_2, i_3, k, l\}$. P' is then the common Sylow 2-subgroup of $G_{1,2,i_\mu,i_\nu}$ and $G_{1,2,k,l}$. Thus a subset $\{i_1, i_2\}$ of $I(a)$ determines uniquely a 2-cycle (k, l) of a. If a subset $\{j_1, j_2\}$ of $I(a)$ determines the same 2-cycle (k, l), then the Sylow 2-subgroup P' of $G_{1,2,k,l}$ is contained in $G_{1,2,j_1,j_2}$. Therefore $\{j_1, j_2\}\subset I(P')\cap I(a)=\{i_1, i_2, i_3\}$. Thus just three subsets $\{i_\mu, i_\nu\}$ of $I(P')\cap I(a)$ determine the same 2-cycle (k, l) of a.

Now suppose that a 2-cycle (k, l) of a other than $(1, 2)$ is given. Let P'' be a Sylow 2-subgroup of $G_{1,2,k,l}$ and let $\Delta''=I(P'')=\{1,2,i_1,i_2,i_3,k,l\}$. Then a normalizes $G_{1,2,k,l}$ and $a^{\Delta''}$ is an even permutation. Therefore $\{i_1, i_2, i_3\}\subset I(a)$. Since $P''\subset G_{1,2,i_1,i_2}$, $\{i_1, i_2\}$ determines (k, l) in the above sense.

Thus we have that the number of 2-cycles of a other than $(1, 2)$ is equal to $\frac{1}{3}{}_7C_2=7$. Hence $n=2+7+2\times 7=23$. This contradicts the assumption $(*)$.

CASE III. $N^{\Delta}=A_6$, $|\Delta|=6$.

Since $N_{1,2,3,4}=H$ and $(N^{\Delta})_{1,2,3,4}=(N_{1,2,3,4})^{\Delta}=1$, we have $I(H)=\Delta$.

Let $a=(1,2)\cdots$ be an involution which is conjugate to a central involution of P. Then $|I(a)|=6$. For a subset $\{i_1, i_2\}$ of $I(a)$ consisting of two letters let P' be a Sylow 2-subgroup of $G_{1,2,i_1,i_2}$ and $\Delta'=I(P')=\{1, 2, i_1, i_2, k, l\}$. Then a normalizes P' and $a^{\Delta'}$ is an even permutation. Therefore we have

$$a^{\Delta'} = (1, 2)(i_1)(i_2)(k, l)\,.$$

Since $I(G_{1,2,i_1,i_2})=\Delta'$ we have $G_{1,2,i_1,i_2}=G_{1,2,k,l}$. In this way, any given subset $\{i_1, i_2\}$ of $I(a)$ determines uniquely a 2-cycle (k, l) of a in such a way that $G_{1,2,i_1,i_2}=G_{1,2,k,l}$. Coversely for any given 2-cycle (k, l) of a other than $(1, 2)$, there is a subset $\{i_1, i_2\}$ in $I(a)$ such as $G_{1,2,k,l}=G_{1,2,i_1,i_2}$, and then $I(G_{1,2,k,l})=I(G_{1,2,i_1,i_2})=\{1, 2, i_1, i_2, k, l\}$. Thus we have that the number of 2-cycles of a other than $(1,2)$ is equal to ${}_6C_2=15$ and hence $n=2+6+2\times 15=38$.

Now consider the mapping φ: $i\rightarrow G_{1,2,3,i}$ from $\{4, 5, \cdots, 38\}$ to the set of subgroups $\{G_{1,2,3,i}\}$. Since $G_{1,2,3,j}=G_{1,2,3,i}$ if and only if $j\in I(G_{1,2,3,i})$ and $|I(G_{1,2,3,i})|=6$, $\varphi^{-1}(G_{1,2,3,i})$ consists of three letters. Therefore we have $35\equiv 0 \pmod 3$. This is a contradiction.

CASE IV. $N^{\Delta}=S_5$, $|\Delta|=5$.

As in CASE III we have $\Delta=I(H)$. Let $a=(1,2)\cdots$ be an involution which is conjugate to a central involution of P. Then $|I(a)|=5$.

For any two letters i, j in $I(a)$, let $\Delta'=I(G_{1,2,i,j})=\{1, 2, i, j, k\}$. Then a normalizes $G_{1,2,i,j}$ and hence $\Delta'^{a}=\Delta'$ and $k\in I(a)$. Clearly $G_{1,2,i,j}=G_{1,2,j,k}=G_{1,2,k,i}$. On the other hand, if $G_{1,2,i',j'}=G_{1,2,i,j}$ for some $\{i', j'\}\subset I(a)$ then $\{i', j'\}\subset I(G_{1,2,i,j})$.

Now consider the mapping φ: $\{i, j\}\rightarrow G_{1,2,i,j}$ from the set of all the subsets $\{i, j\}$ of $I(a)$ consisting of two letters into the set of subgroups of G. As is shown above, $\varphi^{-1}(G_{1,2,i,j})$ $(i, j\in I(a))$ consists of three subsets. Therefore we have ${}_5C_2=10\equiv 0 \pmod 3$. This is a contradiction.

CASE V. $N^{\Delta}=S_4$, $|\Delta|=4$.

The proof in this case is a little complicated. Since P is transitive on $\Gamma=\{5, 6, \cdots, n\}$, H is also transitive on Γ and hence G is 5-fold transitive. Let $K=G_{1,2,3,4,5}$. Then $Q=K\cap P$ is a normal Sylow 2-subgroup of K. Let N' be the normalizer of Q in G, and let $\Delta'=I(Q)$.

We first remark that a central involution c of P is contained in N' and semi-regular on $\Delta'-\Delta$, therefore $|\Delta'|$ must be even.

By the same argument as in the proof of Lemma 1, it is shown that $(N')^{\Delta'}$ must be one of the following groups: S_5, S_6, A_7, A_8 or M_{12}. Thus, from the remark above, we have one of the following cases:

CASE (V.A.). $N^{\Delta}=S_4$, $(N')^{\Delta'}=M_{12}$, $|\Delta|=4$, $|\Delta'|=12$, CASE (V.B.). $N^{\Delta}=S_4$, $(N')^{\Delta'}=A_8$, $|\Delta|=4$, $|\Delta'|=8$,

CASE (V.C.). $N^{\Delta}=S_4$, $(N')^{\Delta'}=S_6$, $|\Delta|=4$, $|\Delta'|=6$.

We shall treat three cases separately.

CASE (V.A). $N^{\Delta}=S_4$, $(N')^{\Delta'}=M_{12}$, $|\Delta|=4$, $|\Delta'|=12$.

Let $\Delta'=\{1, 2, \cdots, 12\}$. Since G is 5-fold transitive, there is an involution $a=(1,2)(3,4)(5)\cdots$ which is conjugate to a central involution of P. Then $|I(a)|=4$. Now $a\in N'$ and $a^{\Delta'}$ is an involution in M_{12}. Therefore we may assume that

$$a = (1,2)(3,4)(5)(6)(7)(8)(9,10)(11,12)\cdots$$

and then $I(a)=\{5,6,7,8\}$. Hence $I(a)\subset I(Q)$, where Q is a Sylow 2-subgroup of $G_{1,2,3,4,5}$. From the same reason, for any 2-cycle (k, l) of a other than $(1,2)$, we have $I(a)\subset I(Q')$, where Q' is a Sylow 2-subgroup of $G_{1,2,5,k,l}$. Assume that $k, l>12$. Since $\{6,7\}\subset I(a)\subset I(Q')$, $Q'\subset G_{1,2,5,6,7}$. Therefore Q and Q' are both normal Sylow 2-subgroup of $G_{1,2,5,6,7}$ and hence $Q=Q'$. On the other hand, k, l are in $I(Q')$ but not in $I(Q)$. This is a contradiction. Hence we have $n=12$, which contradicts the assumption (*).

CASE (V.B). $N^{\Delta}=S_4$, $(N')^{\Delta'}=A_8$, $|\Delta|=4$, $|\Delta'|=8$.

Let $\Delta'=\{1, 2, \cdots, 8\}$ and let $a=(1,2)(3,4)(5)\cdots$ be an involution which is conjugate to a central involution of P. Then $|I(a)|=4$. Now $a\in N'$ and $a^{\Delta'}$ is an even permutation. Therefore we have

$$a = (1,2)(3,4)(5)(6)(7)(8)\cdots$$

and $I(a)=\{5,6,7,8\}\subset I(Q)$. Then by the same argument as in CASE (V.A.) we have a contradiction.

CASE (V.C.). $N^{\Delta}=S_4$, $(N')^{\Delta'}=S_6$, $|\Delta|=4$, $|\Delta'|=6$.

Since $K=N'_{1,2,3,4,5}$ and $(N')^{\Delta'}_{1,2,3,4,5}=(N'_{1,2,3,4,5})^{\Delta'}=1$, we have $K^{\Delta'}=1$, $\Delta'=I(K)$.

Let C be the center of P. By Lemma 2 C is semi-regular on $\Omega-\Delta=\{5, 6, \cdots, n\}$. Therefore $C\cap K=1$. On the other hand, $N'_{1,2,3,4}\supset CK$ and $N'_{1,2,3,4}/K=(N')^{\Delta'}_{1,2,3,4}$ is of order 2. Therefore the order of C is 2 and P has the unique central involution c. Then c is also the unique central involution of H, since P is a normal Sylow 2-subgroup of H.

Let c_i $(i\geq 4)$ be the unique central involution of $G_{1,2,3,i}$. Then $I(c_i)=\{1, 2, 3, i\}$. Let X be the subgroup of $G_{1,2,3}$ generated by $\{c=c_4, c_5, \cdots, c_n\}$. Then X is a normal subgroup of $G_{1,2,3}$. In the following we shall show that X is a Frobenius group as a permutation group on $\{4, 5, \cdots, n\}$. If this has been done, the Frobenius kernel of X is a normal subgroup which is regular on $\{4, 5, \cdots, n\}$. Then by [3], Theorem 2 G must be one of the following groups: S_5, S_6, S_7, A_7 or M_{12}. This contradicts the assumption (*).

We first remark that if $c_i=(1)(2)(3)(i)(k, l)\cdots$ then $G_{1,2,3,k,l}=G_{1,2,3,i,k}=G_{1,2,3,i,l}$. Since c_i normalizes $G_{1,2,3,k,l}$ and $|I(G_{1,2,3,k,l})|=6$, a letter j in $I(G_{1,2,3,k,l})$, which is different from $1, 2, 3, k, l$, is left invariant by c_i. Therefore $j=i$ and $I(G_{1,2,3,k,l})=\{1, 2, 3, i, k, l\}$. The remark above follows now easily.

(a) We first show that the order of $c_i c_j$ is 3 and $I(c_i c_j)=\{1, 2, 3\}$ if $i\neq j$.

Let

$$c_i = (1)(2)(3)(i)(j, j')\cdots ,$$
$$c_j = (1)(2)(3)(j)(i, i')\cdots .$$

Then from the remark above $I(G_{1,2,3,i,j})=\{1, 2, 3, i, j, j'\}=\{1, 2, 3, i, j, i'\}$. Therefore we have $i'=j'$ and

$$c_i c_j = (1)(2)(3)(i, i', j)\cdots .$$

Now $(c_i c_j)^{-1}c_j(c_i c_j)=(1)(2)(3)(i)(i', j)\cdots$ and it is the central involution of $G_{1,2,3,i}$. Hence $c_i=(c_i c_j)^{-1}c_j(c_i c_j)$, $(c_i c_j)^3=1$. Thus $c_i c_j$ is of order 3.

Now suppose that a letter $k>3$ is left invariant by $c_i c_j$. Then $k^{c_i}=k^{c_j}$, and c_i and c_j have a 2-cycle (k, l) $(l=k^{c_i}=k^{c_j})$ in common. But by the remark above we have $I(G_{1,2,3,k,l})=\{1, 2, 3, k, l, i\}=\{1, 2, 3, k, l, j\}$, which is a contradiction. Thus we have $I(c_i c_j)=\{1, 2, 3\}$.

(b) We next show that $c_i c_j c_k$ is the central involution of $G_{1,2,3,l}$, i.e. $c_i c_j c_k=c_l$ for some l. If two of $\{i, j, k\}$ are the same, this is clear. We assume that i, j, k are all different. For the simpicity, let $i=4$, $j=5$.

Now let

$$c_4 = (1)(2)(3)(4)(5, 6)(7, 8)(9, 10)(11, 12)\cdots .$$

Then as is shown in (a)

$$c_5 = (1)(2)(3)(5)(4, 6)\cdots .$$

Since $c_4c_5c_4=c_6$ and $(c_4c_5)^3=1$, we have $c_4c_5c_6=c_5$. Therefore we have only to prove the assertion for $k\geq 7$. Let $k=7$ and assume that c_5 takes 7 to 9. Then $8^{c_5}\neq 10$ and hence $8^{c_5}\geq 11$ since $(c_4c_5)^3=1$. We may assume that c_5 takes 8 to 11. Then $7^{c_5c_4c_5}=10^{c_5}=7^{c_4c_5c_4}=12$, i.e. c_5 takes 10 to 12. Thus we have

$$c_5 = (1)(2)(3)(5)(4, 6)(7, 9)(8, 11)(10, 12)\cdots .$$

Using the same arguments as in (a) and as above for c_4, c_7 and c_5, c_7, we have easily

$$c_7 = (1)(2)(3)(7)(4, 8)(5, 9)(6, 12)(10, 11)\cdots$$

and then

$$c_4c_5c_7 = (1)(2)(3)(11)(4, 12)(5, 8)(6, 9)(7, 10)\cdots .$$

Now we shall show that $c_4c_5c_7=c_{11}$. Since $c_{11}=(c_4c_5)^{-1}c_7(c_4c_5)$, c_{11} and c_4c_5 induce the same permutation on $\Sigma=\{1, 2, \cdots, 12\}$. Since $c_4c_5c_7$ is in $G_{1,2,3,11}$, it commutes with c_{11}. Hence $(c_4c_5c_7)c_{11}=c_{11}(c_4c_5c_7)$, $c_4c_5c_7c_{11}c_7=c_{11}c_4c_5$. Since $c_4c_5c_7c_{11}$ is in G_Σ, it commutes with c_7. Hence $c_7c_4c_5c_7c_{11}=c_{11}c_4c_5$, $(c_7c_{11})^{-1}(c_4c_5)(c_7c_{11})=c_4c_5$. Thus c_4c_5 and c_7c_{11} commute with each other. Therefore we have

$$(c_4c_5c_7c_{11})^3 = (c_4c_5)^3(c_7c_{11})^3 = 1 .$$

Now $c_4c_5c_7$ commutes with c_{11}, hence $(c_4c_5c_7c_{11})^3=(c_4c_5c_7)^3c_{11}=1$ and we have

$$(c_4c_5c_7)^3 = c_{11} .$$

This shows that $c_4c_5c_7$ is of order 2 or 6 and $I(c_4c_5c_7)=\{1, 2, 3, 11\}$. If $c_4c_5c_7$ is of order 2, we have $c_4c_5c_7=c_{11}$. We assume that $c_4c_5c_7$ is of order 6, and let $a=(c_4c_5c_7)^2$. Then $|I(a)|=4+2r$, where r is the number of 2-cycles of $c_4c_5c_7$. On the other hand, if (i, j, k) is a 3-cycle of a, then a normalizes $G_{1,i,j,k}$ and hence a commutes with the central involution c of $G_{1,i,j,k}$. Therefore c induces a semi-regular permutation on $I(a)-\{1\}$, and hence $|I(a)-\{1\}|=3+2r$ must be even. This is a contradiction.

(c) From (b), it follows that X consists of the elements c_i's and (c_ic_j)'s, and $I(c_i)=\{1, 2, 3, i\}$, $I(c_ic_j)=\{1, 2, 3\}$ if $i\neq j$. Therefore X is a Frobenius group as a permutation group on $\{4, 5, \cdots, n\}$.

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References

[1] M. Hall: The Theory of Groups, Macmillan (1959), New York.
[2] C. Jordan: *Recherches sur les substitutions*, J. Math. Pures Appl. (2) **17** (1872), 351–363.
[3] H. Nagao: *On multiply transtive groups I*, to appear in Nagoya Math. J.
[4] H. Wielandt: Finite permutation groups, Academic Press (1964), New York.
[5] E. Witt: *Die 5-fach transitiven Gruppen von Mathieu*, Abh. Math. Sem. Univ. Hamburg **12** (1937), 256–264.