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A SELF-SIMILAR TILING GENERATED BY THE PISOT NUMBER WHICH IS THE ROOT OF THE EQUATION $x^3 - x^2 - 1 = 0$

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1. Introduction

Let β be a fixed real number greater than 1 and x be a positive real number. Then the expansion of the form $x = \sum_{-\infty < i \leq N} a_i \beta^i$ is said to be a β*−greedy expansion* if

$$\left| x - \sum_{M \leq i \leq N} a_i \beta^i \right| < \beta^M \tag{1}$$

holds for every M, where a_i's are nonnegative integer with $0 \leq a_i \leq \beta$. In this case, we denote

$$x = a_N a_{N-1} \cdots a_1 a_0 . a_{-1} a_{-2} \cdots \quad \text{(in } \beta\text{)}, \tag{2}$$

where we may omit a cosequence of 0's, if exists, in the tail. We call $.a_{-1}a_{-2}\cdots$ the *fractional part* of x.

A *Pisot number* is an algebraic integer greater than 1 whose conjugates other than itself have modulus smaller than 1.

Let $\mathbb{Q}(\beta)$ denote the smallest field containing the field of rational numbers $\mathbb{Q}$ and $\beta > 1$.

Theorem 1 (A. Bertrand [5], K. Schmidt [4]). *Let β be a Pisot number. Then a positive real x has a periodic greedy expansion in base β if and only if $x \in \mathbb{Q}(\beta)$.*

Let $\mathbf{Fin}(\beta)$ be the set of all elements in $\mathbb{Q}(\beta)$ which have finite greedy expansion in base β, that is, the set of all nonnegative numbers x for which $a_i = 0$ hold except for finitely many i's in (2). Consider the property

$$\mathbf{Fin}(\beta) = \mathbb{Z}[\beta^{-1}]_{\geq 0}, \tag{F}$$

where $\mathbb{Z}[\beta^{-1}]_{\geq 0}$ is the set of all nonnegative elements in $\mathbb{Z}[\beta^{-1}]$. Akiyama [1] studied the property (F) for Pisot numbers β. In Theorem 3, we prove that the Pisot number

β satisfying the equation $\beta^3 - \beta^2 - 1 = 0$ has the property (F).

Let $\beta = \beta^{(1)}, \beta^{(2)}, \cdots, \beta^{(r_1)}$ and $\beta^{(r_1+1)}, \overline{\beta^{(r_1+1)}}, \cdots, \beta^{(r_1+r_2)}, \overline{\beta^{(r_1+r_2)}}$ be respectively the real and the complex conjugates of β. We also denote by $x^{(j)}$ ($j = 1, 2, \cdots, r_1 + 2r_2$) the corresponding conjugates of $x \in \mathbb{Q}(\beta)$. Let p be a nonnegative integer and define $M_j(p)$ ($j = 1, 2, \cdots, r_1 + 2r_2$) as an upper bound of

$$\left| \sum_{i=0}^{p} a_{p-i} (\beta^{(j)})^i \right|,$$

where $\sum_{i=0}^{p} a_i \beta^{-i}$ runs through finite greedy expansions of length at most $p+1$. Let M_j be an upper bound of $M_j(p)$ ($p = 1, 2, \cdots$). One can take $M_j = [\beta]/(1 - |\beta^{(j)}|)$. Here $[x]$ is the greatest integer not exceeding x. Let b_j ($j = 1, 2, \cdots, r_1 + 2r_2$) be the positive real numbers and $C = C(b_1, b_2, \cdots, b_{r_1+2r_2})$ be a set of elements in $\mathbb{Z}[\beta]$ such that

$$|x^{(j)}| \leq b_j.$$

Theorem 2 (S. Akiyama [6]). *Let β be a Pisot number. Then β has the property (F) if and only if every element of $C = C(1, M_2, M_3, \cdots, M_{r_1+2r_2})$ has finite greedy expansion in base β.*

Define a map $\Phi : \mathbb{Q}(\beta) \to \mathbb{R}^{r_1+2r_2-1}$ by

$$\Phi(x) = \left(x^{(2)}, \cdots, x^{(r_1)}, \Re(x^{(r_1+1)}), \Im(x^{(r_1+1)}), \cdots, \Re(x^{(r_1+r_2)}), \Im(x^{(r_1+r_2)}) \right). \tag{3}$$

Proposition 1 (S. Akiyama [6]). *Let β be a Pisot number of degree n. Then $\Phi(\mathbb{Z}[\beta])$ is dense in $\mathbb{R}^{n-1}$.*

Let $A \in \mathbf{Fin}(\beta)$ and $A = a_L a_{L-1} \cdots a_M$ be the greedy expansion of A in β, where $a_L \neq 0$ and $a_M \neq 0$. Put $\deg_\beta(A) = \deg(A) = L$ and $\mathrm{ord}_\beta(A) = \mathrm{ord}(A) = M$. Define $S_A = S_{a_L a_{L-1} \cdots a_M}$ to be the set of all elements in $\mathbf{Fin}(\beta)$ whose greedy expansion has the tail $a_L a_{L-1} \cdots a_M$. It means that each element of S_A has the form:

$$b_k b_{k-1} \cdots b_{L+1} a_L a_{L-1} \cdots a_M.$$

Let $K_A = \overline{\Phi(S_A)}$. A *tile* is a set K_A with $\deg(A) = -1$ and a *subtile* is a set K_A with $\deg(A) \geq -1$. We write by $\mathcal{S} := S_. = \{x \in \mathbf{Fin}(\beta) \mid \mathrm{ord}_\beta(x) \geq 0\}$. Also, let $\mathcal{K} := \overline{\Phi(\mathcal{S})}$ which is called the *central tile*.

For a Pisot unit β of degree n with property (F), we recall some important properties of the tiles due to S. Akiyama [6]:

1. $\mathbb{R}^{n-1} = \bigcup_{\deg(A)=-1} K_A$,

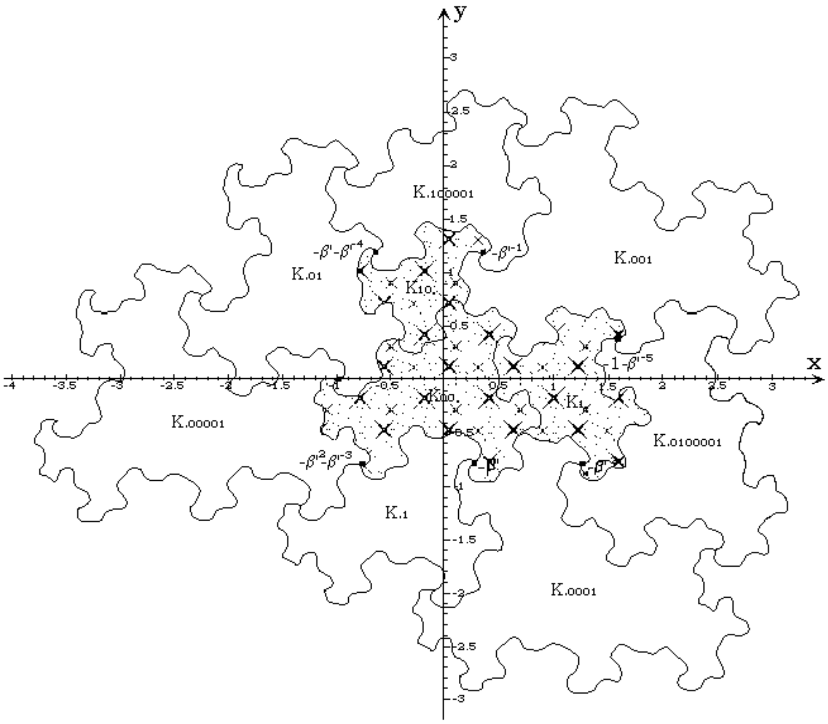

Fig. 1. Space tiling for $x^3 - x^2 - 1 = 0$

2. If $x \in \mathcal{S}$ then $\Phi(x) \in \mathrm{Inn}(\mathcal{K})$ and, especially, the origin is an inner point of the central tile $\mathcal{K}$,
3. If $x \in S_A$ then $\Phi(x) \in \mathrm{Inn}(K_A)$, moreover, $\overline{\mathrm{Inn}(K_A)} = K_A$,
4. The set $\partial(K_A)$ of boundary elements of K_A is closed and nowhere dense in $\mathbb{R}^{n-1}$,
5. Let $\beta^n = a_{-1}\beta^{n-1} + a_{-2}\beta^{n-2} + \cdots + a_{-n}$ be the characteristic equation of β. If $a_{-n} = 1$ then each tile is arcwise connected.

2. Tiling generated by the real root of the equation $x^3 - x^2 - 1 = 0$

In [7] and [2], extensive studies on the tiling generated by the Pisot numbers related to the equations $x^3 - x^2 - x - 1 = 0$ and $x^3 - x - 1 = 0$ were done.

In this paper, we concentrate on the Pisot number β which is the positive root of the equation $x^3 - x^2 - 1 = 0$. It is a unit algebraic integer and $\beta = 1.4655712\cdots$. For a sequence of nonnegative integers

$$a_N a_{N-1} \cdots a_0 . a_{-1} a_{-2} \cdots$$

to be a greedy β-expansion of a nonnegative number in this β is that, $a_i \in \{0, 1\}$ with the condition:

$$a_n = 1 \to a_{n+1} = a_{n+2} = 0 \tag{1}$$

for any $n \in \mathbb{Z}$.

Let $\beta' = \beta^{(2)}$ be one of the complex roots of the equation $x^3 - x^2 - 1 = 0$. The mapping in (3) for our case is $\Phi : \mathbb{Q}(\beta) \to \mathbb{R}^2$. We study the tiling of $\mathbb{C} \simeq \mathbb{R}^2$ for the β (Fig. 1). The central tile is known as tridragon, which is a typical example of a class of fractal sets called Rauzy Fractal.

Proposition 2. *For any distinct tiles T_1 and T_2 it holds that $\mu(T_1 \cap T_2) = 0$, where μ denotes Lebesgue measure.*

Proof. It suffices to show when $T_1 = \mathcal{K}$, $T_2 = K_{.1}$. Recalling the admissibility condition (1), we have that $\mathcal{K} \cup K_{.1} = \beta'^{-1}\mathcal{K}$ and $K_{.1} = \beta'^{-1} + \beta'^2\mathcal{K}$. So it holds that

$$\mu(\mathcal{K} \cup K_{.1}) = |\beta'^{-1}|^2\mu(\mathcal{K}) = \beta\mu(\mathcal{K}) \quad \text{and} \quad \mu(K_{.1}) = |\beta'^2|^2\mu(\mathcal{K}) = \beta^{-2}\mu(\mathcal{K})$$

Since $1 + \beta^{-2} - \beta = 0$ we have that

$$\mu(\mathcal{K} \cap K_{.1}) = \mu(\mathcal{K}) + \mu(K_{.1}) - \mu(\mathcal{K} \cup K_{.1}) = (1 - \beta^{-2} - \beta)\mu(\mathcal{K}) = 0 \qquad \square$$

Theorem 3. $\mathbb{Z}[\beta]_{\geq 0} = \mathbf{Fin}(\beta)$

Proof. We use the Theorem 2. From (1), we can take $M_2 = (1 - |\beta'|^3)^{-1}$. Let

$$C = \{x \in \mathbb{Z}[\beta] \mid 0 < x < 1, |x'| < (1 - |\beta'|^3)^{-1}\}.$$

Each element x of $\mathbb{Z}[\beta]$ can be written as $a + b\beta + c\beta^2$ with $a, b, c \in \mathbb{Z}$. Thus, for $x = a + b\beta + c\beta^2 \in \mathbb{C}$, we have

$$\begin{pmatrix} 1, \beta, \beta^2 \\ 1, \beta', \beta'^2 \\ 1, \overline{\beta'}, \overline{\beta'^2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ x' \\ \overline{x'} \end{pmatrix}$$

with $0 \leq x < 1$, $|x'| = |\overline{x'}| \leq 1/(1 - |\beta'|^3)$. Multiplying by the inverse matrix and considering the absolute value, we get that

$$|a| \leq 2, \quad |b| \leq 2, \quad |c| \leq 1$$

Among them the acceptable possibilities are:

a	b	c	x
-2	0	1	$0.14789\cdots$
-2	2	0	$0.93114\cdots$
-1	1	0	$0.46557\cdots$
0	-1	-1	$0.68232\cdots$
0	2	-1	$0.78324\cdots$
1	-2	1	$0.21675\cdots$
1	1	-1	$0.31767\cdots$
2	-1	0	$0.53442\cdots$

These eight elements have finite greedy expansion in base β:

$$-2+\beta^2=\beta^{-5},\quad -2+2\beta=\beta^{-1}+\beta^{-4}+\beta^{-9},\quad -1+\beta=\beta^{-2},$$
$$-\beta+\beta^2=\beta^{-1},\quad 2\beta-\beta^2=\beta^{-1}+\beta^{-6},\quad 1-2\beta+\beta^2=\beta^{-4},$$
$$1+\beta-\beta^2=\beta^3,\quad 2-\beta=\beta^{-2}+\beta^{-7}$$ □

Let $D(x,r)=\{x\in\mathbb{C}\mid |x-r|\le r\}$.

Lemma 1. 1. *There exists a positive constant c_1 and a nonnegative integer p such that for any a_i $(i=0,1,\cdots,p)$, which admits greedy expansion $\sum_{i=0}^p a_i\beta^i$ with $a_0=1$, we have that $|\sum_{i=0}^p a_i(\beta')^i|-|\beta'|^{p+1}M_2>c_1|\beta'|$.*
2. *$D(0,c_1)\subset\mathcal{K}$.*

Proof. 1. It suffices to show that for any positive constant ϵ there exists a nonnegative integer p, such that for any a_i $(i=0,1,\cdots,p)$, which admits a greedy expansion $\sum_{i=0}^p a_i\beta^i$ with $a_0=1$, we have

$$\left|\sum_{i=0}^{p} a_i(\beta')^i\right| > (1+\epsilon)|\beta'|^{p+1}M_2$$

because we can take $c_1=\epsilon|\beta'|^pM_2$. Assuming the contrary, then there exists a positive constant ϵ, such that for any nonnegative integer p there exist a_i $(i=0,1,\cdots,p)$ satisfying the above conditions and

$$\left|\sum_{i=0}^{p} a_i(\beta')^i\right| \le (1+\epsilon)|\beta'|^{p+1}M_2.$$

So, we have

$$\left|\sum_{i=0}^{p} a_i(\beta')^{i-p}\right| \le (1+\epsilon)|\beta'|M_2.$$

Then, we see that the algebraic integers

$$\sum_{i=0}^{p} a_i(\beta)^{i-p} = \sum_{i=-p}^{0} a_{i+p}(\beta)^i$$

must lie in $C' = C(|\beta|, (1+\epsilon)|\beta'|M_2) \cap \mathbb{Z}[\beta]$. As $a_0 = 1$, the integers of $\mathbb{Q}(\beta)$ expressed by greedy expansion $\sum_{i=-p}^{0} a_{i+p}(\beta)^i$ must be distinct. Since we can take infinitely many p, this contradicts with the fact that C' is a finite set.

2. It suffices to show that $x \in \mathbb{Q}(\beta)$ and $|x'| < c_1$ implies that $x' \in \mathcal{K}$. We first note that there exist infinitely many N such that $|x|(\beta^N - 1) \in \mathbb{Z}[\beta]_{\geq 0}$. From Theorem 1 we have that

$$|x| = \sum_{i=p_0}^{p_1} a_i \beta^{-i} + \frac{\sum_{i=p_1+1}^{p_2} a_i \beta^{-i}}{\beta^{p_2-p_1} - 1},$$

which shows that for every N that is a multiple of $p_2 - p_1$, it holds that $|x|(\beta^N - 1) \in \mathbb{Z}[\beta]_{\geq 0}$. Then, from Property (F) we have that

$$(\beta^N - 1)|x| = \sum_{i=0}^{n} a_i \beta^i + \sum_{i=-m}^{-1} a_i \beta^i$$

If we suppose that $a_{-m} \neq 0$ (it means that $x' \notin \mathcal{K}$) then, by conjugating both sides, we get

$$\begin{aligned}((\beta')^N - 1)|x'| &= \sum_{i=0}^{n} a_i(\beta')^i + \sum_{i=-m}^{-1} a_i(\beta')^i \\ &= |\beta'|^{-m} \left| \sum_{i=0}^{n+m} a_{i-m}(\beta')^i \right| \\ &\geq |\beta'|^{-1} \left(\left| \sum_{0 \leq i \leq p} a_{i-m}(\beta')^i \right| - |\beta'|^{p+1} M_2 \right) \\ &> c_1\end{aligned}$$

(Here we can take sufficiently large N such that $n > p$.) Taking the limit when $N \to \infty$, this inequality implies that $|x'| \geq c_1$, which is a contradiction. $\square$

By computer calculation, the minimum of $|\sum_{i=0}^{31} a_i(\beta')^i|$ under the condition $a_0 = 1$ and (1) is

$$|1 + (\beta')^5 + (\beta')^9 + (\beta')^{12} + (\beta')^{15} + (\beta')^{18} + (\beta')^{22} + (\beta')^{25} + (\beta')^{29}| \approx 0.342683.$$

On the other hand, since $M_2 = 1/(1 - |\beta'|^3)$, we obtain c_1:

$$c_1 = \frac{|1 + (\beta')^5 + (\beta')^9 + (\beta')^{12} + (\beta')^{15} + (\beta')^{18} + (\beta')^{22} + (\beta')^{25} + (\beta')^{29}| - |\beta'|^{32} M_2}{|\beta'|}$$
$$\approx 0.4087313$$

Proposition 3. 1. *Every point of $\mathcal{K}$ is an inner point of $\mathcal{K} \cup (K_{.1} \cup K_{.00001} \cup K_{.01} \cup K_{.100001} \cup K_{.001} \cup K_{.0100001} \cup K_{.0001})$.*
2. *Consider seven tiles $K_{.1}, K_{.00001}, K_{.01}, K_{.100001}, K_{.001}, K_{.0100001}, K_{.0001}$ in this order 'cyclically', so we consider that $K_{.1}$ and $K_{.0001}$ are also adjacent.*
Then, $\mathcal{K}$ has infinitely many common points with any of these tiles. Also, two adjacent tiles have infinitely many points in common, while two tiles, which are not adjacent, have no points in common.

Proof. 1. Let $.a_{-1}a_{-2}\cdots a_{-m}$ be an admissible word of length m. We say also that the tile $K_{.a_{-1}a_{-2}\cdots a_{-m}}$ has length m. Since we have that $c_1|\beta'|^{-9} < M_2$ and $c_1|\beta'|^{-10} > M_2$, it holds that every point of $\mathcal{K}$ is an inner point of $(\beta')^{-10}\mathcal{K}$. Let us see first only the tiles of length 10. There are 19 tiles of this type. (See Fig. 2. Remark that there exist subtiles in this figure.)

$K_{.0000000001}$, $K_{.0000001001}$, $K_{.0001001001}$, $K_{.1001001001}$, $K_{.0010001001}$,
$K_{.0100001001}$, $K_{.1000001001}$, $K_{.0000010001}$, $K_{.0010010001}$, $K_{.0100010001}$,
$K_{.1000010001}$, $K_{.0000100001}$, $K_{.0100100001}$, $K_{.1000100001}$, $K_{.0001000001}$,
$K_{.1001000001}$, $K_{.0010000001}$, $K_{.0100000001}$, $K_{.1000000001}$.

Since we have the inclusion $\mathcal{K} \subset D(0, M_2)$ with $M_2 = 1/(1 - |\beta'|^3)$, we can write:

$$\mathcal{K} \cap K_{.0000000001} \subset D(0, M_2) \cap D((\beta')^{-10}, M_2) = \emptyset$$
$$\mathcal{K} \cap K_{.0000001001} \subset D(0, M_2) \cap D((\beta')^{-10} + (\beta')^{-7}, M_2) = \emptyset$$
$$\mathcal{K} \cap K_{.0001001001} \subset D(0, M_2) \cap D((\beta')^{-10} + (\beta')^{-7} + (\beta')^{-4}, M_2) = \emptyset$$
$$\mathcal{K} \cap K_{.1001001001} \subset D(0, M_2) \cap D((\beta')^{-10} + (\beta')^{-7} + (\beta')^{-4} + (\beta')^{-1}, |\beta'|^2 M_2) = \emptyset$$
$$\mathcal{K} \cap K_{.0010001001} \subset D(0, M_2) \cap D((\beta')^{-10} + (\beta')^{-7} + (\beta')^{-3}, M_2) = \emptyset$$
$$\mathcal{K} \cap K_{.0100001001} \subset D(0, M_2) \cap D((\beta')^{-10} + (\beta')^{-7} + (\beta')^{-2}, |\beta'| M_2) = \emptyset$$
$$\mathcal{K} \cap K_{.1000001001} \subset D(0, M_2) \cap D((\beta')^{-10} + (\beta')^{-7} + (\beta')^{-1}, |\beta'|^2 M_2) = \emptyset$$
$$\mathcal{K} \cap K_{.0000010001} \subset D(0, M_2) \cap D((\beta')^{-10} + (\beta')^{-6}, M_2) = \emptyset$$
$$\mathcal{K} \cap K_{.0010010001} \subset D(0, M_2) \cap D((\beta')^{-10} + (\beta')^{-6} + (\beta')^{-3}, M_2) = \emptyset$$
$$\mathcal{K} \cap K_{.0100010001} \subset D(0, M_2) \cap D((\beta')^{-10} + (\beta')^{-6} + (\beta')^{-2}, |\beta'| M_2) = \emptyset$$
$$\mathcal{K} \cap K_{.1000010001} \subset D(0, M_2) \cap D((\beta')^{-10} + (\beta')^{-6} + (\beta')^{-1}, |\beta'|^2 M_2) = \emptyset$$
$$\mathcal{K} \cap K_{.0001000001} \subset D(0, M_2) \cap D((\beta')^{-10} + (\beta')^{-4}, M_2) = \emptyset$$
$$\mathcal{K} \cap K_{.1001000001} \subset D(0, M_2) \cap D((\beta')^{-10} + (\beta')^{-4} + (\beta')^{-1}, |\beta'|^2 M_2) = \emptyset$$

$$\mathcal{K} \cap K_{.0010000001} \subset D(0, M_2) \cap D((\beta')^{-10} + (\beta')^{-3}, M_2) = \emptyset$$
$$\mathcal{K} \cap K_{.0100000001} \subset D(0, M_2) \cap D((\beta')^{-10} + (\beta')^{-2}, |\beta'|M_2) = \emptyset$$
$$\mathcal{K} \cap K_{.1000000001} \subset D(0, M_2) \cap D((\beta')^{-10} + (\beta')^{-1}, |\beta'|^2 M_2) = \emptyset$$

because, by computer calculations, we have:

$$\begin{aligned}
|(\beta')^{-10}| = 6.7613\cdots &> 4.5832\cdots = 2M_2\\
|(\beta')^{-10} + (\beta')^{-7}| = 9.9596\cdots &> 4.5832\cdots = 2M_2\\
|(\beta')^{-10} + (\beta')^{-7} + (\beta')^{-4}| = 10.9695\cdots &> 4.5832 = 2M_2\\
|(\beta')^{-10} + (\beta')^{-7} + (\beta')^{-4} + (\beta')^{-1}| = 10.8722\cdots &> 3.8552\cdots = (1 + |\beta'|^2)M_2\\
|(\beta')^{-10} + (\beta')^{-7} + (\beta')^{-3}| = 11.3918\cdots &> 4.5832 = 2M_2\\
|(\beta')^{-10} + (\beta')^{-7} + (\beta')^{-2}| = 8.79182\cdots &> 4.1845\cdots = (1 + |\beta'|)M_2\\
|(\beta')^{-10} + (\beta')^{-7} + (\beta')^{-1}| = 9.6507\cdots &> 3.8552\cdots = (1 + |\beta'|^2)M_2\\
|(\beta')^{-10} + (\beta')^{-6}| = 8.5613\cdots &> 4.5832\cdots = 2M_2\\
|(\beta')^{-10} + (\beta')^{-6} + (\beta')^{-3}| = 10.3299\cdots &> 4.5832\cdots = 2M_2\\
|(\beta')^{-10} + (\beta')^{-6} + (\beta')^{-2}| = 8.1416\cdots &> 4.1845\cdots = (1 + |\beta'|)M_2\\
|(\beta')^{-10} + (\beta')^{-6} + (\beta')^{-1}| = 7.64046\cdots &> 3.8552\cdots = (1 + |\beta'|^2)M_2\\
|(\beta')^{-10} + (\beta')^{-4}| = 7.38089\cdots &> 4.5832\cdots = 2M_2\\
|(\beta')^{-10} + (\beta')^{-4} + (\beta')^{-1}| = 7.1437\cdots &> 3.8552\cdots = (1 + |\beta'|)M_2\\
|(\beta')^{-10} + (\beta')^{-3}| = 8.4072\cdots &> 4.5832\cdots = 2M_2\\
|(\beta')^{-10} + (\beta')^{-2}| = 5.90557\cdots &> 4.1845\cdots = (1 + |\beta'|)M_2\\
|(\beta')^{-10} + ((\beta')^{-1}| = 6.19203\cdots &> 3.8552\cdots = (1 + |\beta'|^2)M_2.
\end{aligned}$$

Hereafter we call this type of arguments as ‘encircling method’. For the remaining tiles we cannot confirm that the intersection is empty by simple encircling method. We subdivide the tiles into subtiles, and use a *refined* version of the encircling method. For simplicity, we call also this version as ‘encircling method’. So, we have:

$$\mathcal{K} \cap K_{.0000100001} = (K_{0.} \cup K_{1.}) \cap (K_{0.0000100001} \cup K_{1.0000100001}) = \emptyset$$

because

$$\begin{aligned}
K_{0.} \cap K_{0.0000100001} &\subset D(0, |\beta'|M_2) \cap D((\beta')^{-10} + (\beta')^{-5}, |\beta'|M_2) = \emptyset\\
K_{0.} \cap K_{1.0000100001} &\subset D(0, |\beta'|M_2) \cap D((\beta')^{-10} + (\beta')^{-5} + 1, |\beta'|^3 M_2) = \emptyset\\
K_{1.} \cap K_{0.0000100001} &\subset D(1, |\beta'|^3 M_2) \cap D((\beta')^{-10} + (\beta')^{-5}, |\beta'|M_2) = \emptyset\\
K_{1.} \cap K_{1.0000100001} &\subset D(1, |\beta'|^3 M_2) \cap D((\beta')^{-10} + (\beta')^{-5} + 1, |\beta'|^3 M_2) = \emptyset
\end{aligned}$$

since we have that

$$\begin{aligned}
|(\beta')^{-10}+(\beta')^{-5}| &= 4.2036\cdots > 3.7858\cdots = 2|\beta'|M_2\\
|(\beta')^{-10}+(\beta')^{-5}+1| &= 5.1479\cdots > 3.1845\cdots = (|\beta'|+|\beta'|^3)M_2\\
|(\beta')^{-10}+(\beta')^{-5}-1| &= 3.2924\cdots > 3.1845\cdots = (|\beta'|^3+|\beta'|)M_2\\
|(\beta')^{-10}+(\beta')^{-5}| &= 4.2036\cdots > 2.5832\cdots = 2|\beta'|^3M_2.
\end{aligned}$$

In the same way we divide

$$\mathcal{K}\cap K_{.0100100001} = (K_{0.}\cup K_{1.})\cap(K_{00.0100100001}\cup K_{10.0100100001}) = \emptyset$$

because

$$\begin{aligned}
K_{0.}\cap K_{00.0100100001} &\subset D(0,|\beta'|M_2)\cap D((\beta')^{-10}+(\beta')^{-5}+(\beta')^{-2},|\beta'|^2M_2)=\emptyset\\
K_{0.}\cap K_{10.0100100001} &\subset D(0,|\beta'|M_2)\cap D((\beta')^{-10}+(\beta')^{-5}+(\beta')^{-2}+\beta',|\beta'|^4M_2)=\emptyset\\
K_{1.}\cap K_{00.0100100001} &\subset D(1,|\beta'|^3M_2)\cap D((\beta')^{-10}+(\beta')^{-5}+(\beta')^{-2},|\beta'|^2M_2)=\emptyset\\
K_{1.}\cap K_{10.0100100001} &\subset D(1,|\beta'|^3M_2)\cap D((\beta')^{-10}+(\beta')^{-5}+(\beta')^{-2}+\beta',|\beta'|^4M_2)=\emptyset
\end{aligned}$$

since we have that

$$\begin{aligned}
|(\beta')^{-10}+(\beta')^{-5}+(\beta')^{-2}| &= 3.5484\cdots > 3.4565\cdots = (|\beta'|+|\beta'|^2)M_2\\
|(\beta')^{-10}+(\beta')^{-5}+(\beta')^{-2}+\beta'| &= 3.9632\cdots > 2.9598\cdots = (|\beta'|+|\beta'|^4)M_2\\
|(\beta')^{-10}+(\beta')^{-5}+(\beta')^{-2}-1| &= 2.8682\cdots > 2.8552\cdots = (|\beta'|^3+|\beta'|^2)M_2\\
|(\beta')^{-10}+(\beta')^{-5}+(\beta')^{-2}+\beta'-1| &= 4.2036\cdots > 2.3585\cdots = (|\beta'|^3+|\beta'|^4)M_2.
\end{aligned}$$

Also

$$\mathcal{K}\cap K_{.1000100001} = (K_{00.}\cup K_{1.}\cup K_{10.})\cap(K_{000.1000100001}\cup K_{100.1000100001}) = \emptyset$$

because

$$\begin{aligned}
K_{00.}\cap K_{000.1000100001} &\subset D(0,|\beta'|^2M_2)\cap D((\beta')^{-10}+(\beta')^{-5}+(\beta')^{-1},|\beta'|^3M_2)=\emptyset\\
K_{00.}\cap K_{100.1000100001} &\subset D(0,|\beta'|^2M_2)\cap D((\beta')^{-10}+(\beta')^{-5}+(\beta')^{-1}+(\beta')^2,|\beta'|^5M_2)=\emptyset\\
K_{1.}\cap K_{000.1000100001} &\subset D(1,|\beta'|^3M_2)\cap D((\beta')^{-10}+(\beta')^{-5}+(\beta')^{-1},|\beta'|^3M_2)=\emptyset\\
K_{10.}\cap K_{000.1000100001} &\subset D(\beta',|\beta'|^4M_2)\cap D((\beta')^{-10}+(\beta')^{-5}+(\beta')^{-1},|\beta'|^3M_2)=\emptyset\\
K_{10.}\cap K_{100.1000100001} &\subset D(\beta',|\beta'|^4M_2)\cap D((\beta')^{-10}+(\beta')^{-5}+(\beta')^{-1}+(\beta')^2,|\beta'|^5M_2)=\emptyset
\end{aligned}$$

since we have that

$$\begin{aligned}
|(\beta')^{-10}+(\beta')^{-5}+(\beta')^{-1}| = 3.5929\cdots &> 2.8552\cdots = (|\beta'|^2+|\beta'|^3)M_2\\
|(\beta')^{-10}+(\beta')^{-5}+(\beta')^{-1}+(\beta')^2| = 3 &> 2.4449\cdots = (|\beta'|^2+|\beta'|^5)M_2\\
|(\beta')^{-10}+(\beta')^{-5}+(\beta')^{-1}-1| = 2.6002\cdots &> 2.5832\cdots = (|\beta'|^3+|\beta'|^3)M_2
\end{aligned}$$

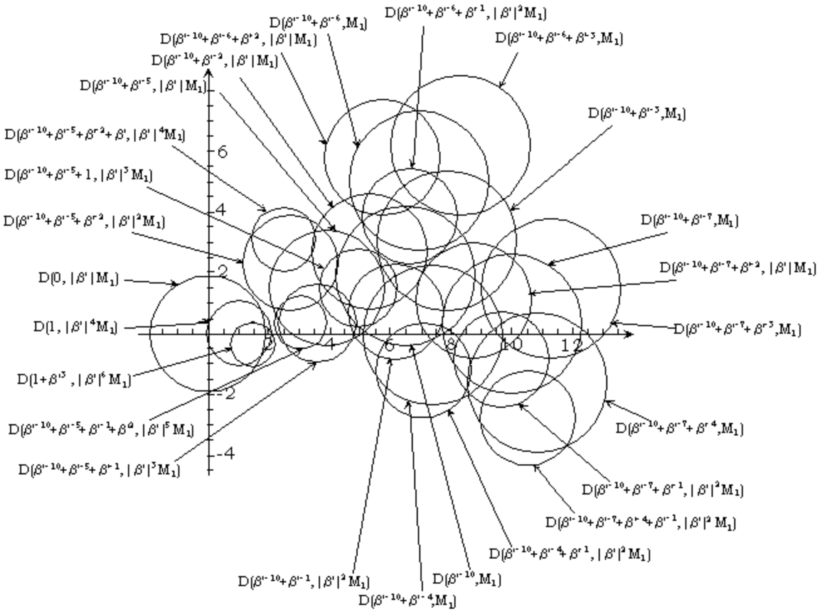

Fig. 2. Encircling method for $\mathcal{K}$ and tiles of length 10.

$$|(\beta')^{-10}+(\beta')^{-5}+(\beta')^{-1}-\beta'| = 3.83023\cdots > 2.3585\cdots = (|\beta'|^4+|\beta'|^3)M_2$$
$$|(\beta')^{-10}+(\beta')^{-5}+(\beta')^{-1}+(\beta')^2-\beta'| = 4.2036\cdots > 3.3285\cdots = (|\beta'|^4+|\beta'|^5)M_2$$

and

$$K_{1.}\cap K_{100.1000100001} = (K_{0001.}\cup K_{1001.})\cap K_{100.1000100001}$$
$$\subset (D(1,|\beta'|^4M_2)\cup D(1+(\beta')^3,|\beta'|^6M_2))\cap D((\beta')^{-10}+(\beta')^{-5}+(\beta')^{-1}+(\beta')^2,|\beta'|^5M_2) = \emptyset$$

because

$$|(\beta')^{-10}+(\beta')^{-5}+(\beta')^{-1}+(\beta')^2-1| = 2 > 1.9482\cdots = (|\beta'|^4+|\beta'|^5)M_2$$
$$|(\beta')^{-10}+(\beta')^{-5}+(\beta')^{-1}+(\beta')^2-1-(\beta')^3| = 1.6166 > 1.6092\cdots = (|\beta'|^6+|\beta'|^5)M_2$$

So we showed, actually, that each element of $\mathcal{K}$ is an inner point of $(\beta')^{-9}\mathcal{K}$. In the same way, we show that the intersection of $\mathcal{K}$ with the tiles of length 9 and 8 is which means that each element of $\mathcal{K}$ is an inner point of $(\beta')^{-7}\mathcal{K}$.

Using the encircling method we have that $\mathcal{K}$ does not intersect each of the tiles of length 7,6,5,4,3,2 except for the tiles:

$$K_{.0100001},\ K_{.100001},\ K_{.00001},\ K_{.0001},\ K_{.001},\ K_{.01},\ K_{.1}.$$

Since we have that:

$$-(\beta')^{-2} = \frac{1}{1-(\beta')^3} = 1 + (\beta')^3 + (\beta')^6 + (\beta')^9 + \cdots \in K_{1.} \tag{2}$$

and that

$$\begin{aligned} -(\beta')^{-2} &= (\beta')^{-7} + (\beta')^{-2} - 1 \\ &= (\beta')^{-7} + (\beta')^{-2} + \frac{(\beta')^2}{1-(\beta')^3} \\ &= (\beta')^{-7} + (\beta')^{-2} + (\beta')^2 + (\beta')^5 + (\beta')^8 + \cdots \\ &\in K_{.0100001} \end{aligned} \tag{3}$$

we get that $-(\beta')^{-2} \in \mathcal{K} \cap K_{.0100001}$ so $\mathcal{K} \cap K_{.0100001} \neq \emptyset$.

Multiplying by β' in (2) and (3) we get:

$$-(\beta')^{-1} = \frac{\beta'}{1-(\beta')^3} = \beta' + (\beta')^4 + (\beta')^7 + (\beta')^{10} + \cdots \in K_{10.} \tag{4}$$

and that

$$\begin{aligned} -(\beta')^{-1} &= (\beta')^{-6} + (\beta')^{-1} + \frac{(\beta')^3}{1-(\beta')^3} \\ &= (\beta')^{-6} + (\beta')^{-1} + (\beta')^3 + (\beta')^6 + (\beta')^9 + \cdots \\ &\in K_{.100001} \end{aligned} \tag{5}$$

which shows that $-(\beta')^{-1} \in \mathcal{K} \cap K_{.100001}$ so $\mathcal{K} \cap K_{.100001} \neq \emptyset$.

Multiplying by β' in (4) and (5) we get:

$$-1 = \frac{(\beta')^2}{1-(\beta')^3} = (\beta')^2 + (\beta')^5 + (\beta')^8 + (\beta')^{11} + \cdots \in K_{100.} \tag{6}$$

and that

$$-1 = (\beta')^{-5} + 1 + (\beta')^4 + (\beta')^7 + (\beta')^{10} + \cdots \in K_{.00001} \tag{7}$$

which shows that $-1 \in \mathcal{K} \cap K_{.00001}$ so $\mathcal{K} \cap K_{.00001} \neq \emptyset$.

Multiplying by β' in (6) and (7) we get:

$$-\beta' = \frac{(\beta')^3}{1-(\beta')^3} = (\beta')^3 + (\beta')^6 + (\beta')^9 + (\beta')^{12} + \cdots \in K_{1000.} \tag{8}$$

and that

$$-\beta' = (\beta')^{-4} + \beta' + (\beta')^5 + (\beta')^8 + (\beta')^{11} + \cdots \in K_{.0001} \tag{9}$$

which shows that $-\beta' \in \mathcal{K} \cap K_{.0001}$ so $\mathcal{K} \cap K_{.0001} \neq \emptyset$.

We have another greedy expansion for $-(\beta')^{-1}$

$$
\begin{aligned}
-(\beta')^{-1} &= (\beta')^{-3} - 1 = (\beta')^{-3} + \frac{(\beta')^2}{1-(\beta')^3} \\
&= (\beta')^{-3} + (\beta')^2 + (\beta')^5 + (\beta')^8 + \cdots \in K_{.001}.
\end{aligned} \tag{10}
$$

From (10) and (4), we have that $-(\beta')^{-1} \in \mathcal{K} \cap K_{.001}$, so $\mathcal{K} \cap K_{.001} \neq \emptyset$. Multiplying by β' in (10) we get:

$$
\begin{aligned}
-1 &= (\beta')^{-2} - \beta' = (\beta')^{-2} + \frac{(\beta')^3}{1-(\beta')^3} \\
&= (\beta')^{-2} + (\beta')^3 + (\beta')^6 + (\beta')^9 + \cdots \in K_{.01}.
\end{aligned} \tag{11}
$$

From (11) and (6), we have that $-1 \in \mathcal{K} \cap K_{.01}$, so $\mathcal{K} \cap K_{.01} \neq \emptyset$. Multiplying by β' in (11) we get:

$$
-(\beta') = (\beta')^{-1} - (\beta')^2 = (\beta')^{-1} + (\beta')^4 + (\beta')^7 + (\beta')^{10} + \cdots \in K_{.1}. \tag{12}
$$

From (12) and (8), we have that $-\beta' \in \mathcal{K} \cap K_{.1}$, so $\mathcal{K} \cap K_{.1} \neq \emptyset$.

2. Consider the map $\eta'(x) = \beta' x + (\beta')^{-4}$. Then we can show that

$$
\mathcal{K} \xmapsto{\eta'} K_{.0001} \xmapsto{\eta'} K_{.0100001} \xmapsto{\eta'} \mathcal{K}.
$$

by showing that

$$
\mathcal{S} \xmapsto{\eta} S_{.0001} \xmapsto{\eta} S_{.0100001} \xmapsto{\eta} \mathcal{S}.
$$

where $\eta(x) = \beta x + \beta^{-4}$. For example, if $x = \cdots .0001 \in S_{.0001}$ then

$$
\eta(x) = \cdots 0.0011 = \cdots 0.0010101 = \cdots 0.0100001 \in S_{.0100001}.
$$

Since

$$
\begin{aligned}
-(\beta')^{-2} &= (\beta')^{-4} + \frac{\beta'}{1-(\beta')^3} \\
&= (\beta')^{-4} + \beta' + (\beta')^4 + (\beta')^7 + (\beta')^{10} + \cdots \in K_{.0001}
\end{aligned} \tag{13}
$$

and (2), (8), (9), we have $\{-(\beta')^{-2}, -\beta'\} \subset \mathcal{K} \cap K_{.0001}$. As

$$
(\eta')^3(x) = (\beta')^3 x + 1
$$

is a contraction map with fixed point $-(\beta')^{-2}$, $\{(\eta')^{3n}(-\beta') \mid n = 0, 1, 2, \cdots\}$ is an infinite set contained in $\mathcal{K} \cap K_{.0001}$. Also we have that

$$(\mathcal{K} \cap K_{.1}) + 1 = \mathcal{K} \cap K_{.0001} \tag{17a}$$

$$(\mathcal{K} \cap K_{.100001}) + (\beta')^{-4} = \mathcal{K} \cap K_{.0001}, \tag{17b}$$

$$(\mathcal{K} \cap K_{.0100001})\beta' = \mathcal{K} \cap K_{.100001} \tag{17c}$$

$$(\mathcal{K} \cap K_{.00001})(\beta')^{-2} = \mathcal{K} \cap K_{.0100001} \tag{17d}$$

$$(\mathcal{K} \cap K_{.001}) - (\beta')^{-3} = \mathcal{K} \cap (K_{.00001} \cup K_{.1}) \tag{17e}$$

$$(\mathcal{K} \cap K_{.01}) + 1 - (\beta')^{-2} = \mathcal{K} \cap K_{.0100001} \tag{17f}$$

which show that $\mathcal{K} \cap K_{.1}$, $\mathcal{K} \cap K_{.100001}$, $\mathcal{K} \cap K_{.0100001}$, $\mathcal{K} \cap K_{.00001}$, $\mathcal{K} \cap K_{.001}$, and $\mathcal{K} \cap K_{.01}$ are also infinite sets. So we showed that the intersection of $\mathcal{K}$ with each of 7 tiles is an infinite set.

Since

$$K_{.01} \cap K_{.100001} = (\mathcal{K} \cap K_{.001}) + (\beta')^{-2} \tag{18a}$$

$$K_{.100001} \cap K_{.001} = (\mathcal{K} \cap K_{.01}) + (\beta')^{-3} \tag{18b}$$

$$K_{.001} \cap K_{.0100001} = (\mathcal{K} \cap K_{.100001}) + (\beta')^{-2} + (\beta')^{-7} \tag{18c}$$

$$K_{.0100001} \cap K_{.0001} = (\mathcal{K} \cap K_{.001}) + (\beta')^{-4} \tag{18d}$$

$$K_{.0001} \cap K_{.1} = (\mathcal{K} \cap K_{.01}) + (\beta')^{-4} \tag{18e}$$

$$K_{.1} \cap K_{.00001} = (K_{00.} \cap K_{.01}) + (\beta')^{-1} \tag{18f}$$

$$K_{.00001} \cap K_{.01} = (\mathcal{K} \cap K_{.001}) + (\beta')^{-5} \tag{18g}$$

$\mathcal{K} \cap K_{.1}$, $\mathcal{K} \cap K_{.100001}$, $K_{.01} \cap K_{.001}$, $K_{.001} \cap K_{.0100001}$, $K_{.0100001} \cap K_{.0001}$, $K_{.0001} \cap K_{.1}$, $K_{.1} \cap K_{.00001}$, and $K_{.00001} \cap K_{.01}$ are also infinite sets. Using the encircling method we can show that

$$K_{.01} \cap (K_{.001} \cup K_{.0100001} \cup K_{.0001} \cup K_{.1}) = \emptyset \tag{19a}$$

$$K_{.100001} \cap (K_{.0100001} \cup K_{.0001} \cup K_{.1} \cup K_{.00001}) = \emptyset \tag{19b}$$

$$K_{.001} \cap (K_{.0001} \cup K_{.1} \cup K_{.00001} \cup K_{.01}) = \emptyset \tag{19c}$$

$$K_{.0100001} \cap (K_{.1} \cup K_{.00001} \cup K_{.01} \cap K_{.100001}) = \emptyset \tag{19d}$$

$$K_{.0001} \cap (K_{.00001} \cup K_{.01} \cup K_{.100001} \cup K_{.001}) = \emptyset \tag{19e}$$

$$K_{.1} \cap (K_{.01} \cup K_{.100001} \cup K_{.001} \cup K_{.0100001}) = \emptyset \tag{19f}$$

$$K_{.00001} \cap (K_{.100001} \cup K_{.001} \cup K_{.0100001} \cup K_{.0001}) = \emptyset \tag{19g}$$

□

A common point of at least two tiles is called an element of the *boundary* of the tiling. A common point of at least three tiles is called an element of the *vertex* of the tiling. We define $\delta(K_{.x_{-1}x_{-2}\cdots x_{-N}})$ to be the set of all boundary points of a tile $K_{.x_{-1}x_{-2}\cdots x_{-N}}$ and $V(K_{.x_{-1}x_{-2}\cdots x_{-N}})$ to be the set of all vertices in $K_{.x_{-1}x_{-2}\cdots x_{-N}}$.

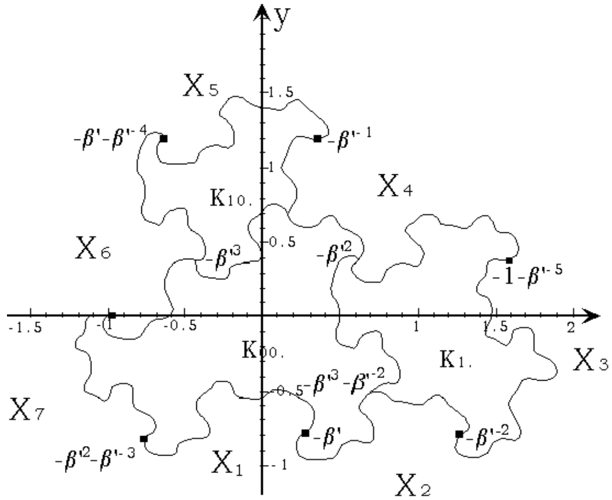

Fig. 3. Subset of $\mathcal{K} \cap (K_{.01} \cup K_{.00001} \cup K_{.1} \cup K_{.0001} \cup K_{.0100001} \cup K_{.001} \cup K_{.100001})$.

Theorem 4. $V(\mathcal{K}) = \{-1, -(\beta')^2 - (\beta')^{-3}, -\beta', -(\beta')^{-2}, -1-(\beta')^{-5}, (\beta')^{-1}, -\beta' - (\beta')^{-4}\}$

Proof. If we prove that the intersection of each 3 adjacent tiles in Proposition 3 is only one point, then there are exactly 7 vertices in $\mathcal{K}$. From what we proved in Proposition 3, we have:

$$\begin{aligned} -1 &\in \mathcal{K} \cap K_{.01} \cap K_{.00001} \\ -\beta' &\in \mathcal{K} \cap K_{.1} \cap K_{.0001} \\ -(\beta')^{-1} &\in \mathcal{K} \cap K_{.001} \cap K_{.100001} \\ -(\beta')^{-2} &\in \mathcal{K} \cap K_{.0001} \cap K_{.0100001}. \end{aligned}$$

From (6), (7) and (11) we have that

$$\begin{aligned} -1-(\beta')^{-5} &= (\beta')^{-7} + (\beta')^{-2} + \beta' + (\beta')^5 + (\beta')^8 + \cdots \\ &= 1 + (\beta')^4 + (\beta')^7 + (\beta')^{10} + (\beta')^{13} + \cdots \\ &= (\beta')^{-3} + (\beta')^3 + (\beta')^6 + (\beta')^9 + \cdots \end{aligned} \tag{14}$$

so it holds

$$-1-(\beta')^{-5} = K_{.0100001} \cap \mathcal{K} \cap K_{.001}.$$

Multiplying by β' and $(\beta')^2$ in (14) we get:

$$-\beta' - (\beta')^{-4} = K_{.100001} \cap K_{0.} \cap K_{.01}$$

$$-(\beta')^2 - (\beta')^{-3} = K_{.00001} \cap K_{00.} \cap K_{.1}.$$

The only thing left to be proved is that the intersection of 2 adjacent tiles with $\mathcal{K}$ is only one point. So let us show, for example, that $\mathcal{K} \cap K_{.0001} \cap K_{.0100001}$ is only one point. Since we know that $-(\beta')^{-2} \in \mathcal{K} \cap K_{.0001} \cap K_{.0100001}$ it is enough to show that $\mathcal{K} \cap K_{.0001} \cap K_{.0100001} = \{-(\beta')^{-2}\}$. We have that

$$\mathcal{K} \cap K_{.0001} \cap K_{.0100001} = (\mathcal{K} \cap K_{.0001}) \cap (\mathcal{K} \cap K_{.0100001})$$

For the contraction $(\eta')^3(x) = (\beta')^3(x) + 1$ we showed that

$$\mathcal{K} \overset{\eta'}{\longmapsto} K_{.0001} \overset{\eta'}{\longmapsto} K_{.0100001} \overset{\eta'}{\longmapsto} \mathcal{K}.$$

We have that $(\eta')^3(\mathcal{K} \cap K_{.0001}) = K_{1.} \cap K_{.0001}$ and $(K_{00.} \cap K_{.0001}) \cap K_{.0100001} = \emptyset$. Also, $(\eta')^3(\mathcal{K} \cap K_{.0100001}) = \mathcal{K} \cap K_{00.0100001}$ and $(\mathcal{K} \cap K_{10.0100001}) \cap K_{.0001} = \emptyset$. So we have that

$$(\eta')^3(\mathcal{K} \cap K_{.0001} \cap K_{.0100001}) = \mathcal{K} \cap K_{.0001} \cap K_{.0100001}$$

Since $(\eta')^3$ is a contraction map with fixed point $-(\beta')^{-2}$, we have that

$$\mathcal{K} \cap K_{.0001} \cap K_{.0100001} = \{-(\beta')^{-2}\} \qquad \square$$

Theorem 5. *The boundary of $\mathcal{K}$ is a union of* 7 *self-affine sets.* (*See* Fig. 3) *The Hausdorff dimension of the boundary is* $1.47131\cdots$.

According to this theorem we say that X_1 is an edge between $-(\beta')^2 - (\beta')^{-3}$ and $-\beta'$, which is denoted by $E(-(\beta')^2 - (\beta')^{-3}, -\beta')$, and so on.

Proof. In the proof of Proposition 3, we already showed that

$$\delta(\mathcal{K}) = \mathcal{K} \cap (K_{.1} \cup K_{.00001} \cup K_{.01} \cup K_{.100001} \cup K_{.001} \cup K_{.0100001} \cup K_{.0001}).$$

Let us denote by

$$\begin{aligned} X_1 &:= \mathcal{K} \cap K_{.1} \\ X_2 &:= \mathcal{K} \cap K_{.0001} \\ X_3 &:= \mathcal{K} \cap K_{.0100001} \\ X_4 &:= \mathcal{K} \cap K_{.001} \\ X_5 &:= \mathcal{K} \cap K_{.100001} \\ X_6 &:= \mathcal{K} \cap K_{.01} \\ X_7 &:= \mathcal{K} \cap K_{.00001} \end{aligned}$$

From (17) of Proposition 3 we get

$$X_1 + 1 = X_2 \tag{21a}$$

$$X_5 + (\beta')^{-4} = X_2 \tag{21b}$$

$$\beta' X_3 = X_5 \tag{21c}$$

$$(\beta')^{-2} X_7 = X_3 \tag{21d}$$

$$X_4 - (\beta')^{-3} = X_7 \cup X_1 \tag{21e}$$

$$X_6 + (1 - (\beta')^2) = X_3 \tag{21f}$$

so all the edges are self-similar. First consider X_2. By using the encircling method we have that

$$X_2 = (K_{00.} \cap K_{10.0001}) \cup (K_{1.} \cap K_{10.0001}).$$

For the transformation $(\eta')^3(x) = (\beta')^3 x + 1$ we have that

$$(\eta')^3(X_2) = K_{1.} \cap K_{10.0001}.$$

Since

$$K_{00.} \cap K_{10.0001} = (\beta')^2 X_5 = (\beta')^2 (X_2 - (\beta')^{-4}) = (\beta')^2 X_2 - (\beta')^{-2},$$

then

$$X_2 = ((\beta')^2 X_2 - (\beta')^{-2}) \cup (((\beta')^3 X_2 + 1).$$

We use here the criterion of Exercise 3.3 in [3] to show that the Hausdorff dimension s of X_2 coincides with the upper and lower box counting dimension and Hausdorff measure $\mathcal{H}^s(X_2)$ is positive. It is also proved in Corollary 3.3 of [3], that $\mathcal{H}^s(X_2) < \infty$. Noting that

$$((\beta')^2 X_2 - (\beta')^{-2}) \cap ((\beta')^3 X_2 + 1) = \{-(\beta')^3 - (\beta')^{-2}\}$$

we get

$$\mathcal{H}^s(X_2) = \mathcal{H}^s((\beta')^2 X_2 - (\beta')^{-2}) + \mathcal{H}^s((\beta')^3 X_2 + 1) = |\beta'|^{2s} \mathcal{H}^s(X_2) + |\beta'|^{3s} \mathcal{H}^s(X_2)$$

and

$$1 = |\beta'|^{2s} + |\beta'|^{3s}$$

So, the Hausdorff dimension s of X_2 is

$$s = \frac{\lambda}{\log|\beta'|} = 1.47131\cdots$$

where λ is the real root of the equation $x^3 + x^2 - 1 = 0$. □

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