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EXISTENCE OF GREEN FUNCTION AND BOUNDED HARMONIC FUNCTIONS ON GALOIS COVERS OF RIEMANNIAN MANIFOLDS

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1. Existence of Green functions on transient covers

Let $p : X \to M$ be a Galois covering of an orientable Riemannian manifold M, with T as the deck transformation group. In the case where M is compact, it is known since Royden [14] that X and T share same properties. Namely, the canonical Brownian motion on X induces a Brownian motion on T and vice versa. Moreover, if one of them is transient, then so is the other one. It is also well known that the transience of the canonical Brownian motion on X is equivalent to the existence of Green functions on X (See [1] and references therein). However, much less is known in the case where the base M is a non-compact manifold. In this paper, we prove the following result in this direction:

Theorem 1.1. *Let $p : X \to M$ be a Galois covering of a Riemannian manifold M whose deck transformation group T is an extension of a finitely generated transient group. Then X is transient also, i.e., X carries Green functions.*

Question. Is Theorem 1.1 valid for an arbitrary transient group T?

Before passing to the proof, let us note that if M is compact, then T is a finitely generated group. For such groups, there is a nice characterization of transience given by Guivarch [4], Lyons-Sullivan [8] and Varopoulos [17] : A finitely generated group is recurrent (i.e. not transient) if and only if it is a finite extension of one of the groups $\{0\}$, $\mathbb{Z}$, or $\mathbb{Z}^2$.

In what follows, $X \in P_G$ means that X has Green functions (equivalently, X is transient) and $X \in O_G$ means that X do not carry Green functions (equivalently, X is recurrent).

Theorem 1.1 is based on the following generalization of a theorem of Kusunoki-Mori [5], which was originaly stated for Riemann surfaces (See also [15] III.1.2G). The proof for Riemannian manifolds given below is a mere repetition of their proof, which involves the concept of Royden compactification. Existence of the Royden compactification for Riemannian manifolds and all the related results that we use in the proof have been established by Glasner-Katz [3].

Theorem 1.2. *Let X be a Riemannian manifold of class O_G, and let Y be an open submanifold of X with a smooth boundary ∂Y. Then the double* dbl Y *of Y along ∂Y is also of class O_G.*

Recall that the double dbl Y of Y is obtained by gluing together two copies of Y along their boundaries, metric structure on dbl Y being the natural one.

NOTATION. For a non-compact orientable Riemannian manifold X, a continuous function $f : X \to \mathbb{R}$ with locally integrable first partial derivatives is called a *Tonelli function*. The set $\mathbb{M}(X)$ of all bounded Tonelli functions f on X with finite Dirichlet integrals $D(f)$ is an algebra, the *Royden algebra* of X. Let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence of functions in $\mathbb{M}(X)$ converging uniformly to f on compact subsets of X. We say that $f = BD\text{-}\lim f_n$ if $\lim D(f - f_n) = 0$. The Royden algebra is complete under this topology. There is a unique compact Hausdorff space X^*, which is called the *Royden compactification* of X, satisfying (i) X is dense in X^*, (ii) Every $f \in \mathbb{M}(X)$ extends continuously to X^*, (iii) $\mathbb{M}(X)$ separates the points of X^*. The *Royden boundary* is the set $\Gamma := X^*\backslash X$. The ideal of $\mathbb{M}(X)$ consisting of functions with compact support is denoted by $\mathbb{M}_0(X)$, and the closure of $\mathbb{M}_0(X)$ in the BD-topology is denoted by $\mathbb{M}_\Delta(X)$. The subset of Γ consisting of points p with the property that $f(p) = 0$ for all $f \in \mathbb{M}_\Delta(X)$ is called the *harmonic boundary* of X, and denoted by Δ. Existence of Green functions on X implies that the harmonic boundary is nonempty, in fact, $X \in P_G \iff \Delta \neq \emptyset$.

Proof of Theorem 1.2. Let X^* be the Royden compactification of X, Δ be its harmonic boundary, and let $\overline{Y}$ be the closure of Y in X^*. Since $X \in O_G$, we know that $\Delta = \emptyset$. Hence, for all $y \in \overline{Y}$ one can find an $f_y \geq 0$ in $\mathbb{M}_\Delta(X)$ with $f_y(y) > 1$. The set $\overline{Y}$ being compact in X^*, one can choose a finite number of points $y_1, y_2, \ldots, y_n$ such that

$$\overline{Y} \subset \bigcup_{i=1}^{n} \{y : y \in X^*, f_{y_i}(y) > 1\}.$$

Thus, the function $f = \sum_{i=1}^{n} f_{y_i} \in \mathbb{M}_\Delta(X)$ satisfies $f > 1$ on $\overline{Y}$. Let $\{g_i\}_{i\in\mathbb{N}}$ be a sequence of functions in $\mathbb{M}_0(R)$ such that $f = BD\text{-}\lim g_n$. Put $\hat{f}$, $\hat{g}_i$ for the symmetric extensions of $f|_{\overline{Y}}$, $g_i|_{\overline{Y}}$ to dbl Y. Then one has $\hat{f} \in \mathbb{M}(\text{dbl } Y)$, $\hat{g}_i \in \mathbb{M}_0(\text{dbl } Y)$, and $\hat{f} = BD\text{-}\lim \hat{g}_i$. This implies that $\hat{f} \in \mathbb{M}_\Delta(\text{dbl } Y)$. Since $f > 1$ on dbl Y, $1 = (1/\hat{f})\hat{f} \in \mathbb{M}_\Delta(\text{dbl } Y)$, which shows that the harmonic boundary of dbl Y is empty, i.e., dbl $Y \in O_G$. □

Proof of Theorem 1.1. The proof will be achieved in two steps. First assume that T is finitely generated. Set $H := p_\star\pi_1(X) \triangleleft \pi_1(M)$. As T is finitely generated, one can choose a finite number of loops $\gamma_1, \ldots, \gamma_n$ in M such that $[\gamma_1], \ldots, [\gamma_n] \in \pi_1(M)$ generate $T = \pi_1(M)/H$ in the quotient. Let N be a relatively compact, connected open submanifold of M with smooth boundary ∂N such that N contains the loops $\gamma_1, \ldots, \gamma_n$. Then the open submanifold $Y := p^{-1}(N)$ of X is connected, and its boundary $\partial Y = p^{-1}(\partial N)$ is smooth. Consider the manifold dbl Y. It is easy to see that the action of T on Y passes to dbl Y. Thus, dbl Y is a Galois covering of the double dbl N of N, with T as the deck transformation group. The manifold dbl N being compact, dbl Y is of class P_G, and Theorem 1.2 implies that X is of class P_G.

Now assume that T is an extension of a finitely generated transient group, i.e., assume that there is an exact sequence $0 \to H \to T \to T' \to 0$ where T' is finitely generated and transient. Consider the intermediate covering $X/H \to M$. Since this covering has T' as the deck transformation group, X/H has Green functions by the first part of the proof. Since $X \to X/H$ is a covering, X is of class P_G, too. This completes the proof. $\square$

REMARK. To illustrate the "doubling" procedure described above, let $\exp : X := \mathbb{C} \to M := \mathbb{C}\backslash\{0\}$ be the usual covering. One can choose N to be the annulus $\{z : 1 < |z| < e\}$, so that Y is the strip $\{z : 0 < \operatorname{Re} z < 1\}$. After the doubling, one obtains a covering of the torus dbl N by the cylinder dbl Y.

Some Corollaries of Theorem 1.1

1. On commutator subgroups of Fuchsian groups. According to a theorem of Myrberg [11], if a Riemann surface X is covered by the unit disc Δ, and G is the corresponding Fuchsian group acting on Δ, then $X \in P_G$ if and only if G is of convergence type; that is,

$$\sum_{g\in G}(1 - |g(z)|) < \infty$$

for one, and hence for all $z \in \Delta$ (see also [16], X.13). The following statement is an immediate corollary of Myrberg's characterization and Theorem 1.1.

Corollary 1.3. *Let X be a Riemann surface covered by the unit disc Δ, and let $\pi_1(X) = G \subset \operatorname{Aut}(\Delta)$ be its covering group. A normal subgroup $H \triangleleft G$ is of convergence type if the quotient group G/H is an extension of a finitely generated transitive group.*

Now we shall consider the particular case of abelian coverings; that is Galois coverings whose deck transformation groups are abelian. Rank of an abelian covering is defined to be the rank of its deck transformation group. Theorem 1.1 and the Varopoulos' characterization of finitely generated transitive groups imply that for $3 \leq r \leq \infty$, an abelian covering of rank r of a Riemannian manifold is of class P_G. This latter assertion is a generalization of a theorem proved in 1953 by Mori [10] in the case where the base is a compact Riemann surface.

It has been shown by McKean-Sullivan and Lyons-McKean [9], [7] that the maximal abelian (hence, $\mathbb{Z}^2$-) covering of $\mathbb{C}\backslash\{0,1\}$ is of class P_G. Hence, we have a complete list of Riemann surfaces which do not have an abelian cover of class P_G: Since the deck transformation group of the maximal abelian cover is the abelianization of the fundamental group, the genus of such a surface should be ≤ 1 and it cannot have too many punctures. Namely, these are the sphere $S^2 = \mathbb{P}^1_{\mathbb{C}}$, the complex plane $\mathbb{C}$, the punctured plane $\mathbb{C}\backslash\{0\}$, the tori $\mathbb{T}$, and the punctured tori $\mathbb{T}\backslash\{q\}$. The only non-trivial case is that of a punctured torus, so we describe its maximal abelian cover. If $p : \mathbb{C} \to \mathbb{T}$ is the universal covering of $\mathbb{T}$, then $\mathbb{C}\backslash p^{-1}(q)$ is the maximal abelian cover of $\mathbb{T}\backslash\{q\}$, which is easily seen to be not of class P_G. Also, note that $\mathbb{T}\backslash\{q\}$ is the only surface in the above list which is covered by the unit disc Δ. So, we have the following consequence of Theorem 1.1:

Corollary 1.4. *Let a Riemann surface X be covered by the unit disc Δ, and let $\pi_1(X) = G \subset \mathrm{Aut}(\Delta)$ be its covering group. Then*
(i) *If $X \neq \mathbb{T}\backslash\{q\}$, then the commutator subgroup $[G,G]$ is of convergence type.*
(ii) *If H is a subgroup of G such that $[G,G] \subset H \subset G$, and the rank of the abelian group G/H is ≥ 3, then H is of convergence type.*

2. Carathéodory hyperbolicity of metabelian covers. A Riemann surface X is called *Carathéodory hyperbolic* if bounded holomorphic functions separate the points of X. It is interesting to know when Carathéodory hyperbolic surfaces appear as "small" covers of Riemann surfaces. In [6] Lin and Zaidenberg shows that if a Riemann surface R has an abelian cover Y of class P_G, then Y has a Carathéodory hyperbolic, abelian cover X, such that X is a metabelian (i.e. two-step solvable) Galois covering of R. Hence, Theorem 1.1 implies the following corollary.

Corollary 1.5. *If R is not one of the surfaces S^2, $\mathbb{C}$, $\mathbb{C}\backslash\{0\}$, $\mathbb{T}$, $\mathbb{T}\backslash\{q\}$, then it admits a metabelian, Carathéodory hyperbolic Galois covering $X \to R$.*

The converse of this corollary is also true, for it is obvious that the surfaces S^2, $\mathbb{C}$, $\mathbb{C}\backslash\{0\}$, $\mathbb{T}$ do not possess *any* cover carrying bounded analytic functions. For the surface $\mathbb{T}\backslash\{q\}$, recall that its maximal abelian cover is of class O_G. One of the results in [8] asserts that an abelian cover of an O_G-manifold is of class O_{HB}, that is it has no bounded non-constant harmonic functions (sometimes such a surface is also called a Liouville surface). This shows that a metabelian cover of $\mathbb{T}\backslash\{q\}$ is of class O_{HB}, so in particular it has no bounded analytic functions, and it cannot be Carathéodory hyperbolic.

2. Existence of bounded harmonic functions on finite covers of P_G-surfaces

An unwritten rule in the classification theory of Riemann surfaces states that "passage to covers produces more and more functions". In this section we consider the following question: While passing to covers, exactly when the bounded harmonic functions appears?

Let R be a Riemann surface of class O_G, and let X be a rank ≥ 3 abelian cover of R. Then, as we have noticed above, $X \in O_{HB}$ by a result in [8]. On the other hand, Theorem 1.1 implies that $X \in P_G$. So there are many Riemann surfaces $X \in P_G \cap O_{HB}$. Let us denote by Z the maximal abelian (hence, $\mathbb{Z}^\infty$-) cover of X. Lyons and Sullivan [8] observed that Z is of class P_{HB}, that is, it does carry a non-constant bounded harmonic function. The theorem below states that X has a *finite* Galois cover Y which is of class P_{HB}. However, by an argument due to V. Lin, Y does not carry any non-constant bounded analytic functions (see Remark 3 below).

Theorem 2.1. *Any Riemann surface X of class P_G has a finite cover Y which is of class P_{HB} and, moreover, Y carries a Dirichlet finite bounded harmonic function.*

Proof of Theorem 2.1. A P_G-surface of genus $g = 0$ already carries a Dirichlet finite non-constant bounded harmonic function (see [15], III.5G). Hence, setting $X = Y$ we are done. If $g \neq 0$ then there exists a closed analytic curve γ on X which does not divide the surface. We denote two sides of γ by γ^+, γ^-, and we cut X along γ. Let $\tilde{X}$ be a second copy of X, $\tilde{\gamma}$ be the copy of γ in $\tilde{X}$ with corresponding sides $\tilde{\gamma}^+$, $\tilde{\gamma}^-$. Gluing X to $\tilde{X}$ via natural identifications $\gamma^+ \longleftrightarrow \tilde{\gamma}^-$, $\gamma^- \longleftrightarrow \tilde{\gamma}^+$, we obtain a $\mathbb{Z}_2$-covering Y of X.

CLAIM. The surface Y is of class P_{HB}.

An immediate way to see this is to observe that the harmonic boundary of Y consists of two points, which implies the existence of a Dirichlet finite non-constant bounded harmonic function on Y (see [13], or [15], III.3F). However, we shall give a more elementary proof based on the following theorem:

Theorem 2.2 (Bader-Parreau [2], Nevanlinna [12]). *Let Y_1, Y_2 be two disjoint subsurfaces of Y with analytic boundaries ∂Y_1, ∂Y_2. Assume that there exists two non-constant bounded harmonic functions u_1 on Y_1 and u_2 on Y_2 such that $u_1 \equiv 0$ on ∂Y_1 and $u_2 \equiv 0$ on ∂Y_2. Then Y carries a non-constant bounded harmonic function. Moreover, if u_1, u_2 have finite Dirichlet integrals, then Y carries a non-constant bounded harmonic function with finite Dirichlet integral.*

Proof of Claim. The surface X being of class P_G, the harmonic measure ω of the ideal boundary of X with respect to γ is non-vanishing, that is, ω is a bounded non-constant harmonic function with finite Dirichlet integral on $X\backslash\gamma$ which vanishes on γ (See [16] X.1). Let $\tilde{\omega}$ be the same function on $\tilde{X}\backslash\tilde{\gamma}$. Setting $Y_1 := X\backslash\gamma$, $Y_2 := \tilde{X}\backslash\tilde{\gamma}$, $u_1 := \omega$, $u_2 := \tilde{\omega}$, the hypotheses of Theorem 2.1 are satisfied, hence Y carries a bounded non-constant harmonic function with finite Dirichlet integral. $\square$

REMARKS. 1. A sufficient condition for a Galois cover Y of a P_G-surface X to be of class P_{HB} is the compactness of the boundary of a fundamental region of the corresponding group action on Y; this can be proved in the same way as in the proof above. Looking at the Royden boundary shows that this latter condition is valid for Riemannian manifolds, too.

2. It should be observed that a finite (even infinite) cover of a P_G-Riemann surface can be of class O_{HB}. For example, if X is a rank-3 abelian cover of a compact Riemann surface K of genus $g = 2$, and Y is the maximal abelian (i.e. rank-4) cover of K, then $X \in P_G$ by Theorem 1.1. On the other hand, an abelian cover of a compact surface is of class O_{HB} by a theorem of Lyons-Sullivan [8], so $Y \in O_{HB}$, but Y is a $\mathbb{Z}$-cover of X.

3. In contrast with the possible existence of bounded non-constant harmonic functions as stated in Theorem 2.1, a finite cover Y of an O_{HB} surface X cannot carry non-constant bounded analytic functions. The proof goes as follows[1]: Let n be the degree of a finite covering $p : Y \to X$, and let f be a bounded analytic function on Y. Let $x \in X$ and $p^{-1}(x) = \{y_1, \ldots, y_n\}$. For $j = 1, \ldots, n$ define $a_j(x) := \sigma_j(f(y_1), \ldots, f(y_n))$, where σ_j is the elementary symmetric polynomial of degree j in n variables. Then each a_j is a bounded analytic function on X. The real parts of the a_j's, being bounded harmonic functions, are constant, hence $a_j = const$, which implies that $f = const$.

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