Katayama, S. and Yokoyama, K.
Osaka J. Math.
43 (2006), 283–326

GLOBAL SMALL AMPLITUDE SOLUTIONS TO SYSTEMS OF NONLINEAR WAVE EQUATIONS WITH MULTIPLE SPEEDS

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(Received February 4, 2005)

Abstract

We give a global existence theorem to systems of quasilinear wave equations in three space dimensions, especially for the multiple-speed cases. It covers a wide class of quadratic nonlinearities which may depend on unknowns as well as their first and second derivatives. Our proof is achieved through total use of pointwise and L^2-estimates concerning unknowns and their first and second derivatives.

1. Introduction

Let $u = u(t,x) = (u_i(t,x))_{i=1}^m$ be an $\mathbb{R}^m$-valued unknown function, and set $\square_i = \partial_t^2 - c_i^2\Delta_x$ with some positive constants c_i $(i = 1,\dots,m)$. We consider the following system of nonlinear wave equations

$$\text{(1.1)} \qquad \square_i u_i(t,x) = F_i(u, \partial u, \nabla_x \partial u) \quad \text{for} \quad t > 0 \quad \text{and} \quad x \in \mathbb{R}^3 \quad (1 \le i \le m)$$

with initial data

$$\text{(1.2)} \qquad u_i(0,x) = \varphi_i(x),\ \ \partial_t u_i(0,x) = \psi_i(x) \quad \text{for} \quad x \in \mathbb{R}^3 \quad (1 \le i \le m).$$

We use the notation $\partial_0 = \partial_t = \partial/\partial t$ and $\partial_j = \partial/\partial x_j$ for $1 \le j \le 3$ throughout this paper. ∂u and $\nabla_x \partial u$ are $\mathbb{R}^{4m}$-valued and $\mathbb{R}^{12m}$-valued functions, whose components are $\partial_\alpha u_i$ $(1 \le i \le m,\ 0 \le \alpha \le 3)$ and $\partial_j\partial_\alpha u_i$ $(1 \le i \le m,\ 1 \le j \le 3,\ 0 \le \alpha \le 3)$, respectively. $F(u,v,w) = (F_i(u,v,w))_{1\le i\le m}$ is a given function of $(u,v,w) \in \mathbb{R}^m \times \mathbb{R}^{4m} \times \mathbb{R}^{12m}$. The components of u, v and w are denoted by u_i, $v_{i,\alpha}$ and $w_{i,j\alpha}$, respectively, where $1 \le i \le m$, $1 \le j \le 3$ and $0 \le \alpha \le 3$. Here $v_{i,\alpha}$ corresponds to $\partial_\alpha u_i$, and $w_{i,j\alpha}$ to $\partial_j\partial_\alpha u_i$. We suppose that $\varphi = (\varphi_i)_{i=1}^m$ and $\psi = (\psi_i)_{i=1}^m$ in (1.2) are rapidly decreasing functions.

We assume that $F(u,v,w)$ is linear with respect to w and satisfies

$$\text{(1.3)} \qquad F(u,v,w) = O\left(|u|^2 + |v|^2 + |w|^2\right) \quad \text{near} \quad (u,v,w) = (0,0,0).$$

2000 Mathematics Subject Classification. Primary 35L70; Secondary 35L05, 35L15, 35L55.

[1]This research was partially supported by Grant-in-Aid for Young Scientists (B) (No. 16740094 (S. Katayama), and No. 16740097 (K. Yokoyama)) of Ministry of Education, Culture, Sports, Science and Technology.

Since $F(u, v, w)$ is linear with respect to w, each equation in (1.1) takes the form

$$\Box_i u_i + \sum_{j=1}^{m} \sum_{0\le\alpha,\beta\le 3} \gamma_{ij}^{\alpha\beta}(u, \partial u)\, \partial_\alpha \partial_\beta u_j = f_i(u, \partial u) \tag{1.4}$$

for $i = 1, \dots, m$. To assure the hyperbolicity of the system, we also assume

$$\gamma_{ij}^{\alpha\beta}(u, v) = \gamma_{ij}^{\beta\alpha}(u, v) = \gamma_{ji}^{\alpha\beta}(u, v) \tag{1.5}$$

for any $1 \le i, j \le m$, $0 \le \alpha, \beta \le 3$ and $(u, v) \in \mathbb{R}^m \times \mathbb{R}^{4m}$.

The purpose of this paper is to give a condition and a proof of global existence for the Cauchy problem (1.1)–(1.2) with small data. The null condition emerged as a condition for the existence of global small amplitude solutions in [9] and [3] for the single-speed case. Its generalization to the multiple-speeds case has been studied by several researchers, see [11], [1], [17], [15], [4] and [16] for the case where F depends on ∂u, $\partial^2 u$ but not on u. The case $F = O(|u|^3 + |\partial u|^2 + |\nabla_x \partial u|^2)$ with multiple speeds was studied first in [12], whose result was generalized later in [7].

Let us review the null condition for the case $F = O(|u|^3 + |\partial u|^2 + |\nabla_x \partial u|^2)$ with multiple speeds. For simplicity, we assume that the wave propagation speeds are distinct. That is to say,

$$c_i \ne c_j \quad \text{if} \quad i \ne j. \tag{1.6}$$

Assume $F = F^{(2)} + H$, where $F^{(2)} = \left(F_i^{(2)}\right)_{i=1}^m$ is a quadratic function with respect to (v, w) and $H = (H_i)_{i=1}^m = O(|u|^3 + |v|^3 + |w|^3)$ near the origin. We introduce

$$\mathcal{N}_i = \left\{ X = (X_0, X_1, X_2, X_3) \in \mathbb{R}^4\, ;\, X_0^2 - c_i^2 \sum_{j=1}^{3} X_j^2 = 0 \right\}$$

for $i = 1, \dots, m$. For $y = (y_i)_{i=1}^m \in \mathbb{R}^m$ and $X = (X_\alpha)_{\alpha=0}^3 \in \mathbb{R}^4$, we define $V(y, X) \in \mathbb{R}^{4m}$ and $W(y, X) \in \mathbb{R}^{12m}$ by

$$V(y, X) = (V_{i,\alpha}(y, X))_{1\le i\le m, 0\le\alpha\le 3} = (y_i X_\alpha)_{1\le i\le m, 0\le\alpha\le 3},$$
$$W(y, X) = (W_{i,j\alpha}(y, X))_{1\le i\le m, 1\le j\le 3, 0\le\alpha\le 3} = (y_i X_j X_\alpha)_{1\le i\le m, 1\le j\le 3, 0\le\alpha\le 3}.$$

We say that F satisfies the null condition if $F_i^{(2)}(V(\mu, X), W(\nu, X)) = 0$ holds for any $\mu, \nu \in \mathbb{R}^m$, $X \in \mathcal{N}_i$ and $i = 1, \dots, m$. Then it was shown in [7] that there exists a global smooth solution for (1.1)–(1.2), provided that the initial data are sufficiently small.

The nonlinear terms which satisfy the null condition are explicitly described by the null forms. For arbitrary smooth functions ϕ and ψ on $\mathbb{R} \times \mathbb{R}^3$, we define new functions $Q_0(\phi, \psi)$ and $Q_{\alpha\beta}(\phi, \psi)$, as bilinear forms of $\partial\phi$ and $\partial\psi$:

$$Q_0(\phi, \psi; c_i) = \partial_t\phi\, \partial_t\psi - c_i^2 \sum_{j=1}^{3} \partial_j\phi\, \partial_j\psi, \tag{1.7}$$

$$Q_{\alpha\beta}(\phi, \psi) = \partial_\alpha\phi\, \partial_\beta\psi - \partial_\beta\phi\, \partial_\alpha\psi. \tag{1.8}$$

We call them the null forms. If the null condition is satisfied, then we can rewrite the nonlinear terms explicitly, as

$$\begin{aligned} F_i(u, \partial u, \nabla_x\, \partial u) = {}& \sum_{|a|=0,1}{}' \left\{ Q_0(u_i, \partial^a u_i; c_i) + \sum_{0\le\alpha,\beta\le3}{}' Q_{\alpha\beta}(u_i, \partial^a u_i) \right\} \\ & + \sum_{(j,k)\ne(i,i)} \sum_{\substack{0\le\alpha,\beta\le3 \\ |a|=0,1}}{}' \partial_\alpha u_j\, \partial^a \partial_\beta u_k + H_i(u, \partial u, \nabla_x\, \partial u), \end{aligned} \tag{1.9}$$

where $\partial^a = \partial_0^{a_0}\partial_1^{a_1}\partial_2^{a_2}\partial_3^{a_3}$ for a multi-index $a = (a_0, a_1, a_2, a_3)$. Here and in what follows, the expression $f = \sum'_{\lambda\in\Lambda} g_\lambda$ means that there exists a family $\{C_\lambda\}_{\lambda\in\Lambda}$ of constants such that $f = \sum_{\lambda\in\Lambda} C_\lambda g_\lambda$. We note that only the products of $\partial^a u_i$ and $\partial^b u_i$ in F_i are involved with the null forms. So we understand that the null forms weaken the effects of self-interactions and that is enough for the global existence.

Our aim in this paper is to consider the case where the quadratic parts of the nonlinear terms contain u. This case was studied by the first author in [6] and [8]. More precisely, he gave a global existence theorem for small initial data, assuming

$$\begin{aligned} F_i(u, \partial u, \nabla_x\, \partial u) = {}& \sum_{\gamma=0}^{3}{}' \partial_\gamma \left\{ Q_0(u_i, u_i; c_i) + \sum_{\alpha,\beta=0}^{3}{}' Q_{\alpha\beta}(u_i, u_i) \right\} \\ & + \sum_{(j,k)\ne(i,i)} \sum_{\substack{0\le\alpha\le3 \\ |a|,|b|=0,1}}{}' \partial_\alpha \left(\partial^a u_j\, \partial^b u_k\right) + H_i(u, \partial u, \nabla_x\, \partial u) \end{aligned} \tag{1.10}$$

in [6], while another global existence theorem for small data was proved for nonlinearity satisfying

$$\begin{aligned} F_i(u, \partial u, \nabla_x\, \partial u) = {}& \sum_{j=1}^{m} \sum_{|a|=0,1}{}' \left\{ Q_0(u_j, \partial^a u_j; c_j) + \sum_{\alpha,\beta=0}^{3}{}' Q_{\alpha\beta}(u_j, \partial^a u_j) \right\} \\ & + \sum_{k\ne l} \sum_{\substack{0\le\alpha\le3 \\ |a|,|b|=0,1}}{}' \partial^a u_k\, \partial^b \partial_\alpha u_l + H_i(u, \partial u, \nabla_x\, \partial u) \end{aligned} \tag{1.11}$$

in [8]. Note that in both cases quadratic terms depend on u itself as well as its derivatives. In this sense, he considered generalized situations. However, instead of allowing such terms, additional restrictions are imposed on quadratic terms depending only on derivatives. Remember that special forms were required only for the self-interactions $\partial^a u_i \cdot \partial^b u_i$ ($|a|, |b| = 1, 2$) of F_i in the previous case (1.9). In contrast to this, we see that some special forms are assumed also for terms like $\partial^a u_j \cdot \partial^b u_j$ ($|a|, |b| = 1, 2$) with $j \neq i$ in (1.10) and (1.11). Hence the readers may have thought that we should aim to remove these additional restrictions. But this attempt for (1.11) will not be achieved, on account of Ohta's counterexample [14]. In fact, he showed that a solution of the Cauchy problem for the systems of two wave equations

$$\Box_1 u_1(t, x) = u_2 \partial_t u_1, \quad \Box_2 u_2(t, x) = (\partial_t u_1)^2$$

can blow up in finite time if $c_1 < c_2$, however small the initial data are. Note that there is no self-interaction in this system. So we cannot always combine nonlinear terms freely, even if they are favorable in different situations (observe that the above nonlinear terms $u_2 \partial_t u_1$ and $(\partial_t u_1)^2$ are included in (1.11) and (1.9), respectively).

Though we should give up a global existence theorem unifying (1.9) and (1.11), we can prove global existence for the following nonlinearity, which means that (1.10) and (1.11) can be unified:

$$\begin{aligned} F_i(u, \partial u, \nabla_x \partial u) = &\sum_{\alpha=0}^{3} \partial_\alpha G_{i,\alpha}(u, \partial u) + N_i(\partial u, \nabla_x \partial u) \\ &+ R_i(u, \partial u, \nabla_x \partial u) + H_i(u, \partial u, \nabla_x \partial u) \\ &\qquad (1 \le i \le m), \end{aligned} \tag{1.12}$$

$$G_{i,\alpha}(u, \partial u) = \sum_{j \neq i} \sideset{}{'}\sum_{|a|,|b|=0,1} \partial^a u_j \, \partial^b u_j, \tag{1.13}$$

$$N_i(\partial u, \nabla_x \partial u) = \sum_{0 \le j \le m} \sideset{}{'}\sum_{|a|=0,1} \left\{ Q_0(u_j, \partial^a u_j; c_j) + \sideset{}{'}\sum_{0 \le \alpha, \beta \le 3} Q_{\alpha\beta}(u_j, \partial^a u_j) \right\}, \tag{1.14}$$

$$R_i(u, \partial u, \nabla_x \partial u) = \sum_{k \neq l} \sideset{}{'}\sum_{\substack{0 \le \alpha \le 3 \\ |a|,|b|=0,1}} \partial^a u_k \, \partial^b \partial_\alpha u_l, \tag{1.15}$$

$$H_i(u, v, w) = O\left(|u|^3 + |v|^3 + |w|^3\right) \quad \text{near the origin.} \tag{1.16}$$

As we have observed, we need some assumptions not only for self-interactions but also for the terms like $\partial^a u_j \cdot \partial^b u_j$ for $j = 1, \dots, m$. So we require that they should take either the null forms or the divergence-type forms.

In order to describe the main result, we introduce some notation briefly. $\Gamma = (\Gamma_0, \dots, \Gamma_7)$ denotes the collection of vector fields S, Ω, and ∂, where $S = t\partial_t + x \cdot \nabla_x$ and $\Omega = (x_2\partial_3 - x_3\partial_2, x_3\partial_1 - x_1\partial_3, x_1\partial_2 - x_2\partial_1)$. We write

$$|v(t, x)|_s = \sum_{|a| \le s} |\Gamma^a v(t, x)|,$$

where $\Gamma^a = \Gamma_0^{a_0} \cdots \Gamma_7^{a_7}$. Moreover, we set

$$E_s(t) = E_s[u](t) = \| |u(t,\cdot)|_s \|_{L^2} + \| |\partial u(t,\cdot)|_s \|_{L^2} + \sum_{i=1}^m \| \langle c_i t - |\cdot| \rangle |\partial u_i(t,\cdot)|_{s-1} \|_{L^2},$$

where $\langle \rho \rangle = \sqrt{1+\rho^2}$ for $\rho \in \mathbb{R}$. We use this notation $\langle \rho \rangle$ throughout this paper.

Theorem 1.1. *Assume that* (1.5) *and* (1.6) *hold. Suppose that the nonlinear term* $F = (F_i)_{i=1}^m$ *is given by* (1.12)–(1.16). *Let* $\nu \in (0, 1/2]$. *Then there exists a positive constant* ε, *such that if*

$$\sup_{x\in\mathbb{R}^3} \left\{ \langle |x| \rangle^2 |u(0,x)|_{13} + \langle |x| \rangle^3 |\partial u(0,x)|_{14} + \langle |x| \rangle^{2+\nu} |\partial_t u(0,x)|_{17} \right\} + E_{22}(0) \le \varepsilon,$$

then the Cauchy problem (1.1)–(1.2) *has a unique global solution* $u \in C^\infty\left([0,\infty) \times \mathbb{R}^3 ; \mathbb{R}^m\right)$.

It should be emphasized that we cannot prove the theorem only by combining the estimates in [6] and [8]. The method in [6] depends on the peculiarity of nonlinear terms (1.10), while the estimates in [8] rely on fairly good decay of solutions with nonlinear terms (1.11), which cannot be expected for the solutions of [6]. Since the estimates which we require for N_i and R_i have been established already in former works, the difficulty of considering the unified nonlinearity lies on the treatment of the divergence-type terms. The missing tools for the estimates of the divergence-type terms are pointwise estimates of the second derivatives. See Corollary 3.4 and the proof of Lemma 6.6 below.

REMARK. (i) We can generalize the theorem above to the case where (1.6) is not satisfied. We define

$$I(i) = \{ j \in \{1, \ldots, m\} ; c_j = c_i \} \quad \text{for} \quad 1 \le i \le m$$

and assume that

$$G_{i,\alpha}(u, \partial u) = \sum_{j \notin I(i)} \sum_{k,l \in I(j)} \sum_{|a|,|b|=0,1}' \partial^a u_k \, \partial^b u_l,$$

$$N_i(\partial u, \nabla_x \partial u) = \sum_{\substack{k,l\in I(j) \\ 0 \le j \le m}} \sum_{|a|=0,1}' \left\{ Q_0(u_k, \partial^a u_l; c_j) + \sum_{0\le\alpha,\beta\le 3}' Q_{\alpha\beta}(u_k, \partial^a u_l) \right\},$$

$$R_i(u, \partial u, \nabla_x \partial u) = \sum_{I(k) \ne I(l)} \sum_{\substack{0\le\alpha\le3 \\ |a|,|b|=0,1}}' \partial^a u_k \, \partial^b \partial_\alpha u_l,$$

instead of (1.13)–(1.15). The global existence is proved without essential modifications to our proof below.
(ii) There are some nonlinearities to which we can apply our method, though they do not explicitly satisfy the conditions of Theorem 1.1. For example, consider a system of two wave equations

$$\Box_1 u_1 = u_2^2, \quad \Box_2 u_2 = (\partial_\alpha u_1)(\partial_\beta u_2), \tag{1.17}$$

where $c_1 \neq c_2$, and $0 \leq \alpha, \beta \leq 3$. Note that this system does not satisfy the conditions of Theorem 1.1, because there exists a term which do not contain any derivative. However, by introducing new unknowns $v_1 = \partial_\alpha u_1$ and $v_2 = u_2$, we can rewrite the above system as

$$\Box_1 v_1 = \partial_\alpha \left(v_2^2\right), \quad \Box_2 v_2 = v_1(\partial_\beta v_2), \tag{1.18}$$

to which Theorem 1.1 is applicable. Thus the reduced system (1.18) possesses a global solution for small data. Now it is easy to obtain a global solution for the original system (1.17).

The plan of this paper is as follows. In Section 2 we introduce the notation used throughout this paper. In Sections 3 and 4 we collect some basic pointwise and energy estimates which we require. Then we obtain energy and pointwise estimates for smooth and small solutions in Sections 5 and 6. Finally, the proof of Theorem 1.1 will be given in Section 7.

2. Notation

We define the scaling operator S and the angular-momentum operators Ω_{jk} by

$$S = t\partial_t + \sum_{j=1}^{3} x_j \partial_j \quad \text{and} \quad \Omega_{jk} = x_j \partial_k - x_k \partial_j \quad \text{for} \quad 1 \leq j < k \leq 3.$$

We also set

$$\Gamma_0 = S, \quad \Gamma_1 = \Omega_{12}, \quad \Gamma_2 = \Omega_{13}, \quad \Gamma_3 = \Omega_{23}, \quad \Gamma_k = \partial_{k-4} \quad (4 \leq k \leq 7)$$

and $\Gamma = (\Gamma_0, \dots, \Gamma_7)$, so that we can use multi-index notation Γ^a for a product $\Gamma_0^{a_0}\Gamma_1^{a_1}\cdots\Gamma_7^{a_7}$, where $a = (a_0, \dots, a_7) \in (\mathbb{Z}_+)^8$. In order to deal with products of the differential operators above, we frequently use the commutation relations

$$\begin{aligned}
&[S, \partial_\alpha] = -\partial_\alpha, \quad [S, \Omega_{jk}] = 0,\\
&[\Omega_{jk}, \partial_\alpha] = -\delta_{\alpha j}\partial_k + \delta_{\alpha k}\partial_j,\\
&[\Omega_{jk}, \Omega_{pq}] = \delta_{jp}\Omega_{qk} + \delta_{jq}\Omega_{kp} - \delta_{kp}\Omega_{qj} - \delta_{kq}\Omega_{jp}
\end{aligned}$$

for $0 \le \alpha \le 3$, $1 \le j < k \le 3$ and $1 \le p < q \le 3$, where δ_{ab} is the Kronecker delta, and Ω_{jk} for $j > k$ is given by $\Omega_{jk} = -\Omega_{kj}$. From these identities we obtain

$$\Gamma^a \Gamma^b v = \Gamma^{a+b} v + \sum_{|c| \le |a|+|b|-1}{}' \Gamma^c v,$$
$$\partial_\alpha \Gamma^a v = \Gamma^a \partial_\alpha v + \sum_{\substack{0 \le \beta \le 3 \\ |b| \le |a|-1}}{}' \Gamma^b \partial_\beta v, \quad \Gamma^a \partial_\alpha v = \partial_\alpha \Gamma^a v + \sum_{\substack{0 \le \beta \le 3 \\ |b| \le |a|-1}}{}' \partial_\beta \Gamma^b v$$

for any smooth function v. We have also $[\Box_i, \Gamma_0] = 2\Box_i$ and $[\Box_i, \Gamma_j] = 0$ for $1 \le j \le 7$, which yield

$$\Box_i(\Gamma^a v) = \Gamma^a(\Box_i v) + \sum_{|b| \le |a|-1}{}' \Gamma^b(\Box_i v). \tag{2.1}$$

The followings are used in the subsequent sections, to evaluate several quantities by using pointwise and L^2-estimates. Let s be a non-negative integer. Then for a smooth function $v(t, x)$, we define

$$|v(t,x)|_s = \sum_{|a| \le s} |\Gamma^a v(t,x)|$$

and

$$\|v(t, \cdot)\|_s = \| |v(t, \cdot)|_s \|_{L^2(\mathbb{R}^3)}.$$

Finally, we introduce two linear operators. For each $i \in \{1, \dots, m\}$, we write $U_i^*[f, g]$ for the solution to the Cauchy problem

$$\begin{cases} \Box_i U_i^*[f,g](t,x) = 0 & \text{in} \quad (0,\infty) \times \mathbb{R}^3, \\ U_i^*[f,g](0,x) = f(x), \quad \partial_t U_i^*[f,g](0,x) = g(x) & \text{for} \quad x \in \mathbb{R}^3. \end{cases}$$

Similarly, $U_i[\Phi]$ stands for the solution to the Cauchy problem

$$\begin{cases} \Box_i U_i[\Phi](t,x) = \Phi(t,x) & \text{in} \quad (0,\infty) \times \mathbb{R}^3, \\ U_i[\Phi](0,x) = \partial_t U_i[\Phi](0,x) = 0 & \text{for} \quad x \in \mathbb{R}^3. \end{cases}$$

Since the commutation relations $[\Box_i, \Gamma_\alpha] = 2\delta_{0\alpha}\Box_i$ imply $\Box_i \Gamma_\alpha U_i[\Phi] = \Gamma_\alpha \Phi + 2\delta_{0\alpha}\Phi$, we easily get

$$\Gamma_\alpha U_i[\Phi] = U_i[\Gamma_\alpha \Phi] + 2\delta_{0\alpha} U_i[\Phi] + \delta_{4\alpha} U_i^*[0, \Phi(0, \cdot)]. \tag{2.2}$$

Here we use the representation

$$v(t,x) = U_i^*[v(0, \cdot), \partial_t v(0, \cdot)](t,x) + U_i[\Box_i v](t,x). \tag{2.3}$$

As an immediate consequence of (2.3) and (2.1), we also have

$$\Gamma^a v(t,x) = U_i^*[\Gamma^a v(0,\,\cdot\,), \partial_t \Gamma^a v(0,\,\cdot\,)](t,x) + {\sum_{|b|\le|a|}}' U_i\left[\Gamma^b \Box_i v\right](t,x). \tag{2.4}$$

3. Pointwise estimates

The aim of this section is to give some pointwise estimates for solutions of wave equations. We start with estimates of $U_i^*[f,g]$ and $U_i[\Phi]$ together with estimates of their first derivatives.

Lemma 3.1. *For $\nu > 0$, and $i = 1, \dots, m$, we have*

$$\begin{aligned}&\langle t+|x|\rangle\langle c_i t - |x|\rangle^\nu |U_i^*[f,g](t,x)| \\ &\le C \sup_{|y|\le c_i t+|x|} \left\{ \sum_{|a|\le 1} |y|^{|a|}\langle |y|\rangle^{1+\nu}|\nabla^a f(y)| + |y|\langle|y|\rangle^{1+\nu}|g(y)| \right\},\end{aligned} \tag{3.1}$$

$$\begin{aligned}&\langle t+|x|\rangle\langle c_i t - |x|\rangle^\nu |\partial U_i^*[f,g](t,x)| \\ &\le C \sup_{|y|\le c_i t+|x|} \sum_{|a|\le 1} |y|^{|a|}\langle |y|\rangle^{1+\nu}\{|\nabla^a \nabla f(y)| + |\nabla^a g(y)|\}.\end{aligned} \tag{3.2}$$

The above constant C depends only on c_i and ν.

Proof. See Proposition 3.3 and the subsequent remark in Kubota–Yokoyama [12]. In [12], it was actually shown that

$$\langle t+|x|\rangle\langle c_i t - |x|\rangle^\nu |U_i^*[f,g](t,x)| \le C \sup_{y\in\mathbb{R}^3} \langle|y|\rangle^{2+\nu}\left\{ \sum_{|a|\le 1}|\nabla^a f(y)| + |g(y)| \right\}.$$

But (3.1) is obtained by making slight modification to the proof of [12]. (3.2) is an immediate consequence of (3.1), since $\partial_{x_j} U_i^*[f,g] = U_i^*[\partial_{x_j} f, \partial_{x_j} g]$ and $\partial_t U_i^*[f,g] = U_i^*\left[g, c_i^2 \Delta f\right]$. See also Asakura [2]. $\square$

To describe the estimates for $U_i[\Phi]$ which we require, we introduce two kinds of weights. We set

$$w(t,r) = \left\{ \langle r\rangle^{-1} + \sum_{j=1}^m \langle c_j t - r\rangle^{-1} \right\}^{-1}, \tag{3.3}$$

$$w_i(t,r) = \left\{ \langle r\rangle^{-1} + \sum_{j\ne i} \langle c_j t - r\rangle^{-1} \right\}^{-1} \qquad (i = 1, \dots, m). \tag{3.4}$$

We also set

$$D_i(t,r)=\left\{(\tau,y)\in\mathbb{R}\times\mathbb{R}^3\,;\,0\le\tau\le t,\ \ c_i\tau+|y|\le c_it+r\right\}. \tag{3.5}$$

Lemma 3.2. *For $\mu>0$, $\nu>0$, $i=1,\dots,m$, and $\alpha=0,\dots,3$, we have*

$$\begin{aligned}&\langle t+|x|\rangle\langle c_it-|x|\rangle^{\nu}|U_i[\Phi](t,x)|\\&\le C\sup_{(\tau,y)\in D_i(t,|x|)}|y|\langle\tau+|y|\rangle^{1+\nu}w(\tau,|y|)^{1+\mu}|\Phi(\tau,y)|,\end{aligned} \tag{3.6}$$

$$\begin{aligned}&\langle|x|\rangle\langle c_it-|x|\rangle^{1+\nu}|U_i[\partial_\alpha\Phi](t,x)|\\&\le C\sup_{(\tau,y)\in D_i(t,|x|)}|y|\langle\tau+|y|\rangle^{1+\nu}w(\tau,|y|)^{1+\mu}|\Phi(\tau,y)|_1,\end{aligned} \tag{3.7}$$

$$\begin{aligned}&\langle|x|\rangle\langle c_it-|x|\rangle^{\nu}|U_i[\partial_\alpha\Phi](t,x)|\\&\le C\sup_{(\tau,y)\in D_i(t,|x|)}|y|\langle\tau+|y|\rangle^{\nu}w_i(\tau,|y|)^{1+\mu}|\Phi(\tau,y)|_1,\end{aligned} \tag{3.8}$$

where the constant C depends on c_i,μ,ν.

Note that the weight $w_i(t,r)$ is stronger than $w(t,r)$ along the cone $c_it=r$. Hence (3.8) for $\nu>1$ is a weaker result than (3.7). However, the inequality (3.8) is no longer true for $0<\nu\le 1$, if we replace $w_i(t,r)$ by $w(t,r)$.

Proof. Although we can get (3.6)–(3.8) by making slight modifications to the proofs of Yokoyama [17] or Kubota–Yokoyama [12], we give a proof in Section 8 for completeness. □

In addition, we need pointwise estimates of the second derivatives. As it was shown by Klainerman–Sideris [10], we can draw out the decaying factor $\langle c_it-r\rangle$ like (3.9)–(3.11) simply by manipulating differential operators Γ_α and $\Box_i$, as far as the temporal differentiations or the Laplacian are involved. We can play a similar game for the spatial second derivatives, but unfortunately only a factor $\langle r\rangle$ can be obtained instead of $\langle c_it-r\rangle$ (see (3.12) below). We will observe in Section 6 that it is sufficient for our present purpose.

Lemma 3.3. *Let $v\in C^2\left((0,\infty)\times\mathbb{R}^3\right)$. Then we have*

$$\langle c_it-|x|\rangle|\Delta v(t,x)|\le C\left(\sum_{|a|\le1}\left|\partial\Gamma^a v(t,x)\right|+t|\Box_i v(t,x)|\right), \tag{3.9}$$

$$\langle c_it-|x|\rangle\left|\partial_t^2 v(t,x)\right|\le C\left(\sum_{|a|\le1}\left|\partial\Gamma^a v(t,x)\right|+|x|\,|\Box_i v(t,x)|\right), \tag{3.10}$$

$$\langle c_i t - |x|\rangle |\nabla_x \partial_t v(t,x)| \le C \left(\sum_{|a|\le 1} \left|\partial \Gamma^a v(t,x)\right| + t|\Box_i v(t,x)| \right), \tag{3.11}$$

$$\langle |x| \rangle \left|\nabla_x^2 v(t,x)\right| \le C \left(|\Omega \nabla_x v(t,x)| + \langle |x| \rangle |\Delta v(t,x)|\right). \tag{3.12}$$

Proof. See Lemma 2.3 of [10] for the proof of (3.9)–(3.11). In order to prove (3.12) we note that

$$x \wedge \Omega = x(x\cdot\nabla) - |x|^2\nabla, \quad \Omega \wedge \nabla = -x\Delta + (x\cdot\nabla)\nabla, \tag{3.13}$$

where $\Omega = (\Omega_1, \Omega_2, \Omega_3) = (\Omega_{23}, \Omega_{31}, \Omega_{12})$. Hence we obtain

$$-(x\wedge\Omega)_i \,\partial_j v + x_i (\Omega\wedge\nabla)_j v = |x|^2 \partial_i \partial_j v - x_i x_j \Delta v \tag{3.14}$$

for $i, j = 1, 2, 3$, which imply (3.12). ☐

Corollary 3.4. *Let $v \in C^2((0,\infty)\times\mathbb{R}^3)$. Then we have*

$$\left|\partial^2 v(t,x)\right| \le C w(t,|x|)^{-1} \sum_{|a|\le 1} |\partial \Gamma^a v(t,x)| + C \frac{\langle t+|x|\rangle}{\langle c_i t - |x|\rangle} |\Box_i v(t,x)| \tag{3.15}$$

for $i = 1, \dots, m$, where $\partial^2 v = (\partial_\alpha \partial_\beta v)_{0\le\alpha,\beta\le 3}$, and $w(t,r)$ is defined by (3.3).

Proof. It follows from Lemma 3.3 that

$$\begin{aligned} &|\Delta v(t,x)| + |\partial_t \partial v(t,x)| \\ &\le C \langle c_i t - |x|\rangle^{-1} \left(\sum_{|a|\le 1} |\partial\Gamma^a v(t,x)| + \langle t+|x|\rangle |\Box_i v(t,x)| \right), \end{aligned} \tag{3.16}$$

$$\left|\nabla_x^2 v(t,x)\right| \le C \langle |x|\rangle^{-1} \sum_{|a|\le 1} |\partial\Gamma^a v(t,x)| + C|\Delta v(t,x)|. \tag{3.17}$$

Noting that Δv on the right-hand side of (3.17) can be controlled by (3.16), we obtain (3.15). ☐

Lastly, we present the following well-known Sobolev type inequalities.

Lemma 3.5. *Let v be a smooth function. Then we have*

$$|x|^{1/2} |v(x)| \le C \sum_{|a|\le 1} \|\partial_x \Omega^a v\|_{L^2}, \tag{3.18}$$

$$|x|\,|v(x)| \le C \sum_{|a|\le 2} \|\Omega^a v\|_{L^2} + C \sum_{|a|\le 1} \|\Omega^a \partial_x v\|_{L^2}. \tag{3.19}$$

Proof. See Lemma 4.2 of Klainerman–Sideris [10] and Lemma 6.1 of Sideris–Tu [15]. □

REMARK. By combining the standard Sobolev inequality and Lemma 3.5, we can replace $|x|$ with $\langle |x| \rangle$ in the above inequality, and we get

$$\langle |x|\rangle^{1/2}|v(x)| \le C \sum_{|a|+|b|\le 1} \left\| \partial_x \partial_x^a \Omega^b v \right\|_{L^2}, \tag{3.20}$$

$$\langle |x|\rangle |v(x)| \le C \sum_{|a|+|b|\le 2} \left\| \Omega^a\, \partial_x^b v \right\|_{L^2}. \tag{3.21}$$

4. Energy estimates

In this section, we collect several L^2-estimates concerning the operators U_i and U_i^*. We start with the standard energy inequalities.

Lemma 4.1. *Let $f \in H^1(\mathbb{R}^3)$, $g \in L^2(\mathbb{R}^3)$ and $\Phi \in L^1\left([0,T); L^2(\mathbb{R}^3)\right)$. Then we have*

$$\left\| \partial U_i^*[f,g](t,\ \cdot\ ) \right\|_{L^2(\mathbb{R}^3)} \le C \left(\|\nabla_x f\|_{L^2(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)} \right), \tag{4.1}$$

$$\|\partial U_i[\Phi](t,\ \cdot\ )\|_{L^2(\mathbb{R}^3)} \le C \int_0^t \|\Phi(\tau,\ \cdot\ )\|_{L^2(\mathbb{R}^3)}\, d\tau \tag{4.2}$$

for any $t \in [0,T)$, where C is a constant independent of T.

The following conformal energy was used in Klainerman [9] (see also [8]). It plays an important role in our proof, since it is useful not only for estimating the L^2 norms of u but also for the weighted estimates of the first derivatives (see Lemma 4.3 below).

Lemma 4.2. *Let v be a smooth solution of*

$$\left(\partial_t^2 - c_i^2 \Delta_x\right) v(t,x) = \Phi(t,x) \quad \text{in} \quad (0,T)\times\mathbb{R}^3. \tag{4.3}$$

Then we have

$$\begin{aligned} &\sum_{|a|\le 1} \|\Gamma^a v(t,\ \cdot\ )\|_{L^2} + \sum_{j=1}^{3} \|L_{ij} v(t,\ \cdot\ )\|_{L^2} \\ &\le C\left(\|\langle|\cdot|\rangle \partial v(0,\ \cdot\ )\|_{L^2} + \|v(0,\ \cdot\ )\|_{L^2}\right) + C \int_0^t \|\langle \tau + |\cdot|\rangle \Phi(\tau,\ \cdot\ )\|_{L^2}\, d\tau, \end{aligned} \tag{4.4}$$

where $L_{ij} = (x_j/c_i)\partial_t + c_i t \partial_j$ $(i = 1,\dots,m;\ j = 1,2,3)$.

Proof. Using a certain change of variables, we may assume $c_i = 1$. For simplicity of exposition, we write L_j for L_{ij} with $c_i = 1$, i.e., $L_j = x_j\partial_t + t\partial_j$. We introduce

$$\begin{aligned}|v(t,x)|^2_{\Gamma,L,1} &= \sum_{|a|\le 1} |\Gamma^a v(t,x)|^2 + \sum_{j=1}^{3} |L_j v(t,x)|^2 \\ &= v^2 + (\partial_t v)^2 + \sum_{j=1}^{3}(\partial_j v)^2 + (Sv)^2 + \sum_{1\le j<k\le 3}(\Omega_{jk}v)^2 + \sum_{j=1}^{3}(L_j v)^2\end{aligned}$$

and

$$E[v](t,x) = \frac{1}{2}|v(t,x)|^2_{\Gamma,L,1} + 2tv(t,x)\partial_t v(t,x) - \frac{3}{2}v(t,x)^2. \tag{4.5}$$

We can rewrite $E[v]$ as

$$\begin{aligned}E[v](t,x) = \frac{1}{2}\left(1+t^2+|x|^2\right)&\left\{(\partial_t v)^2 + \sum_{j=1}^{3}(\partial_j v)^2\right\} + \sum_{j=1}^{3} 2tx_j(\partial_j v)(\partial_t v) \\ &+ 2tv(\partial_t v) - v^2.\end{aligned}$$

Set

$$Kv := \left(1+t^2+|x|^2\right)\partial_t v + 2tx\cdot\nabla_x v + 2tv.$$

Multiplying (4.3) by Kv and integrating by parts, Klainerman showed that

$$\frac{d}{dt}\int_{\mathbb{R}^3} E[v](t,x)\,dx = \int_{\mathbb{R}^3}(Kv)(t,x)\Phi(t,x)\,dx \tag{4.6}$$

(see Klainerman [9], Section 3). He also showed that there exists a constant C such that

$$\frac{1}{C}\int_{\mathbb{R}^3}|v(t,x)|^2_{\Gamma,L,1}\,dx \le \int_{\mathbb{R}^3} E[v](t,x)\,dx \le C\int_{\mathbb{R}^3}|v(t,x)|^2_{\Gamma,L,1}\,dx \tag{4.7}$$

(see Klainerman [9], Lemma 3.1). Now, we define $\|v(t)\|_E^2 = \int E[v](t,x)\,dx$. Since $Kv = \partial_t v + t(S+2)v + |x|L_r v$ with $L_r = \sum_{j=1}^{3}(x_j/|x|)L_j$, we have

$$|Kv(t,x)| \le C(1+t+|x|)|v(t,x)|_{\Gamma,L,1}.$$

Therefore it follows from (4.6) and (4.7) that

$$\begin{aligned}\frac{d}{dt}\|v(t)\|_E^2 &\le C\int_{\mathbb{R}^3}\langle t+|x|\rangle|\Phi(t,x)|\,|v(t,x)|_{\Gamma,L,1}\,dx \\ &\le C\|\langle t+|\cdot|\rangle\Phi(t,\cdot)\|_{L^2}\|v(t)\|_E.\end{aligned} \tag{4.8}$$

Gronwall's lemma applied to (4.8) implies

$$\|v(t)\|_E \le \|v(0)\|_E + C\int_0^t \|\langle \tau + |\cdot|\rangle \Phi(\tau,\,\cdot)\|_{L^2}\,d\tau.$$

In view of (4.7), this completes the proof (see also Hörmander [5], Section 6.3, or Katayama [8], Section 3). □

Corollary 4.3. *Let $i \in \{1, \dots, m\}$. Then we have*

$$\|U_i[\Phi](t,\,\cdot)\|_1 \le C\int_0^t \|\langle \tau + |\cdot|\rangle \Phi(\tau,\,\cdot)\|_{L^2}\;d\tau, \tag{4.9}$$

$$\left\|U_i^*[f,g](t,\,\cdot)\right\|_1 \le C\left(\|f\|_{L^2} + \|\langle|\cdot|\rangle \nabla_x f\|_{L^2} + \|\langle|\cdot|\rangle g\|_{L^2}\right), \tag{4.10}$$

$$\|\langle c_i t - |\cdot|\rangle \partial U_i[\Phi](t,\,\cdot)\|_{L^2} \le C\int_0^t \|\langle \tau + |\cdot|\rangle \Phi(\tau,\,\cdot)\|_{L^2}\;d\tau, \tag{4.11}$$

$$\left\|\langle c_i t - |\cdot|\rangle \partial U_i^*[f,g](t,\,\cdot)\right\|_{L^2} \le C\left(\|f\|_{L^2} + \|\langle|\cdot|\rangle \nabla_x f\|_{L^2} + \|\langle|\cdot|\rangle g\|_{L^2}\right). \tag{4.12}$$

Proof. (4.9) and (4.10) are apparent consequences of Lemma 4.2. (4.11) and (4.12) follow immediately from Lemma 4.2 and the following inequality which is essentially due to Lindblad [13]:

$$\langle c_i t - |x|\rangle |\partial v(t,x)| \le C\left(\sum_{|a|=1} |\Gamma^a v(t,x)| + \sum_{j=1}^{3} |L_{ij} v(t,x)|\right). \tag{4.13}$$

In order to prove (4.13), we just need the following identities, which can be verified easily by direct calculations:

$$\left(c_i^2 t^2 - |x|^2\right)\partial_t v = c_i^2 t (Sv) - c_i \sum_{j=1}^{3} x_j L_{ij} v, \tag{4.14}$$

$$\left(c_i^2 t^2 - |x|^2\right)\partial_j v = c_i t (L_{ij} v) - x_j (Sv) + \sum_{k \ne j} x_k (\Omega_{jk} v) \quad (j = 1, 2, 3). \tag{4.15}$$

□

REMARK. By substituting $t = 0$ to the identity (4.15), we have

$$|x|\,|\nabla_x v(x)| \le |x \cdot \nabla_x v(x)| + |\Omega v(x)|.$$

We will use this inequality for functions on $\mathbb{R} \times \mathbb{R}^3$ in the following form:

$$|x|\,|\nabla_x v(0,x)| \le |Sv(0,x)| + |\Omega v(0,x)|, \quad \langle|x|\rangle |\nabla_x v(0,x)| \le |v(0,x)|_1. \tag{4.16}$$

To conclude this section, we state L^2-estimates for second derivatives.

Lemma 4.4. *Let v be a smooth function decaying sufficiently fast at spatial infinity. Then we have*

$$\|\langle c_i t - |\cdot|\rangle \nabla_x \partial v(t,\cdot)\|_{L^2} \le C\left(\sum_{|a|\le 1} \|\partial \Gamma^a v(t,\cdot)\|_{L^2} + t\|\Box_i v(t,\cdot)\|_{L^2}\right), \tag{4.17}$$

$$\left\|\langle c_i t - |\cdot|\rangle\, \partial_t^2 v(t,\cdot)\right\|_{L^2} \le C\left(\sum_{|a|\le 1} \|\partial \Gamma^a v(t,\cdot)\|_{L^2} + \||\cdot|\Box_i v(t,\cdot)\|_{L^2}\right). \tag{4.18}$$

Proof. Estimates of $\|\langle c_i t - |\cdot|\rangle \Delta v\|_{L^2}$, $\|\langle c_i t - |\cdot|\rangle\, \partial_t^2 v\|_{L^2}$ and $\|\langle c_i t - |\cdot|\rangle \nabla_x\, \partial_t v\|_{L^2}$ follow immediately from Lemma 3.3. Performing integration by parts in the right-hand side of

$$\sum_{j,k=1}^{3} \|\langle c_i t - |\cdot|\rangle\, \partial_j \partial_k v\|_{L^2}^2 = \sum_{j,k=1}^{3} \int \langle c_i t - |\cdot|\rangle^2 (\partial_j \partial_k v)(\partial_j \partial_k v)\, dx,$$

we obtain (4.17). See the proof of Lemma 3.1 in [10] for the details. □

Corollary 4.5. *The following estimate holds for $i = 1, \dots, m$:*

$$\begin{aligned} &\left\|\langle c_i t - |\cdot|\rangle\, \partial^2 U_i[\Psi](t,\cdot)\right\|_{L^2} \\ &\le C \int_0^t \|\Psi(\tau,\cdot)\|_1\, d\tau + C\|\langle t + |\cdot|\rangle \Psi(t,\cdot)\|_{L^2} + C\|\langle |\cdot|\rangle \Psi(0,\cdot)\|_{L^2}. \end{aligned} \tag{4.19}$$

Proof. Set $v = U_i[\Psi]$. Lemma 4.4 yields

$$\|\langle c_i t - |\cdot|\rangle\, \partial^2 v\|_{L^2} \le C \sum_{|a|\le 1} \|\partial \Gamma^a v\|_{L^2} + C\|\langle t + |\cdot|\rangle \Psi(t,\cdot)\|_{L^2}. \tag{4.20}$$

Now, using (2.2), (4.10) and (4.2) to estimate $\|\partial \Gamma^a v\|_{L^2}$ in (4.20), we obtain the result. □

5. Energy estimates for small solutions

First of all, we recall the estimates for the null forms. See [7], [15] or [17] for the proof.

Lemma 5.1. *Let $\phi_1(t,x)$ and $\phi_2(t,x)$ be smooth functions. Let $c_0 = \min\{c_1, \dots, c_m\}/2$ and $|x| \ge c_0 t$. Then we have*

$$\langle t + |x|\rangle |Q_0(\phi_1, \phi_2; c_i)| \le C\left(\langle c_i t - |x|\rangle |\partial \phi_1|\, |\partial \phi_2| + |\partial \phi_1|\, |\phi_2|_1 + |\phi_1|_1 |\partial \phi_2|\right), \tag{5.1}$$

$$\langle t + |x|\rangle |Q_{\alpha\beta}(\phi_1, \phi_2)| \le C\left(|\partial \phi_1|\, |\phi_2|_1 + |\phi_1|_1 |\partial \phi_2|\right) \tag{5.2}$$

for $i = 1, 2, \dots, m$ *and* $0 \le \alpha, \beta \le 3$. *Here* $Q_0(\phi, \psi\,; c_i)$ *and* $Q_{\alpha\beta}(\phi, \psi)$ *are defined by* (1.7) *and* (1.8), *respectively*.

The aim in this section is to derive an L^2-estimate for a small solution of (1.1). We define $E_{2K}(t) = E_{2K}[u](t)$ by

$$E_{2K}(t) = \|u(t, \cdot)\|_{2K} + \|\partial u(t, \cdot)\|_{2K} + \sum_{i=1}^{m} \|\langle c_i t - |\cdot|\rangle |\partial u_i(t, \cdot)|_{2K-1}\|_{L^2}\,. \tag{5.3}$$

A bound of $E_{2K}(t)$ is given in the following proposition.

Proposition 5.2. *Let* $u \in C^\infty([0, T] \times \mathbb{R}^3)$ *be a solution of the Cauchy problem* (1.1)–(1.2) *for some* $T > 0$. *Assume* (1.5), (1.6) *and* (1.12)–(1.16). *Then there are positive constants* $A_1 \ll 1$ *and* C_1, *both independent of* u *and* T, *such that the following holds: If the solution* u *satisfies*

$$\sum_{i=1}^{m} \langle |x| \rangle \langle c_i t - |x| \rangle |u_i(t, x)|_{K+2} \le A \quad \textit{for} \quad 0 \le t \le T \quad \textit{and} \quad x \in \mathbb{R}^3 \tag{5.4}$$

with some $A \in (0, A_1]$, *then we have*

$$E_{2K}(t) \le C_1 E_{2K}(0) \langle t \rangle^{C_1 A}. \tag{5.5}$$

REMARK. An estimate essentially similar to (5.5) can be found in [8], but its condition was $\sum_{i=1}^{m} \langle t + |x| \rangle \langle c_i t - |x| \rangle |u_i(t, x)|_{K+2} \le A$, which is stronger than (5.4). This is one of our modified points.

Proposition 5.2 follows from Lemmas 5.4 and 5.5 below. The main tools required in this section are the results in Section 4 and the following standard energy estimate for hyperbolic systems.

Lemma 5.3. *Let* $v = (v_1, \dots, v_m)$ *be a smooth solution to*

$$\Box_i v_i(t, x) + \sum_{\substack{0 \le \alpha, \beta \le 3 \\ 1 \le j \le m}} \gamma_{ij}^{\alpha\beta}(t, x)\, \partial_\alpha \partial_\beta v_j(t, x) = f_i(t, x) \quad (1 \le i \le m) \tag{5.6}$$

for $(t, x) \in [0, T] \times \mathbb{R}^3$, *where* $\gamma_{ij}^{\alpha\beta} = \gamma_{ij}^{\beta\alpha} = \gamma_{ji}^{\alpha\beta}$. *If* $v(t, x)$ *vanishes sufficiently fast at spatial infinity and*

$$\|\gamma(t, \cdot)\|_{L^\infty} = \sum_{\substack{0 \le \alpha, \beta \le 3 \\ 1 \le i, j \le m}} \left\| \gamma_{ij}^{\alpha\beta}(t, \cdot) \right\|_{L^\infty} < \frac{1}{2} \quad \textit{for} \quad 0 \le t \le T,$$

then

$$\|\partial v(t,\,\cdot\,)\|_{L^2} \le C\|\partial v(0,\,\cdot\,)\|_{L^2} + C\int_0^t \|f(\tau,\,\cdot\,)\|_{L^2} + C\int_0^t \|\partial\gamma(\tau,\,\cdot\,)\|_{L^\infty}\|\partial v(\tau,\,\cdot\,)\|_{L^2}\,d\tau \tag{5.7}$$

for $0 \le t \le T$.

We begin with this standard energy inequality to get the following Lemma 5.4. Here we do not take advantage of the special structures of the nonlinearities. If we take them into consideration for the lower energy, then we can show that $\|\partial u(t,\,\cdot\,)\|_{2K-2}$ remains small as t gets large. But this estimate will not be discussed here, because it will not be used later in our proof.

Lemma 5.4. *Let* $u \in C^\infty([0,T]\times\mathbb{R}^3)$ *be a solution of the Cauchy problem* (1.1)–(1.2) *for some* $T > 0$. *Assume* (1.3), (1.5) *and* (5.4). *Then we have*

$$\|\partial u(t,\,\cdot\,)\|_{2K} \le C\|\partial u(0,\,\cdot\,)\|_{2K} + CA\int_0^t \langle\tau\rangle^{-1}\{\|u(\tau,\,\cdot\,)\|_{2K} + \|\partial u(\tau,\,\cdot\,)\|_{2K}\}\,d\tau \tag{5.8}$$

for $0 \le t \le T$.

Proof. We apply Γ^a to

$$\Box_i u_i + \sum_{\substack{0\le\alpha,\beta\le3\\1\le j\le m}} \gamma_{ij}^{\alpha\beta}(u,\partial u)\,\partial_\alpha\partial_\beta u_j = f_i(u,\partial u)$$

for all a with $|a| \le 2K$. Then we have

$$\Box_i\Gamma^a u_i + \sum_{\substack{0\le\alpha,\beta\le3\\1\le j\le m}} \gamma_{ij}^{\alpha\beta}(u,\partial u)\,\partial_\alpha\partial_\beta\Gamma^a u_j = \tilde f_{i,a},$$

where

$$\begin{aligned}\tilde f_{i,a} &= \Gamma^a f_i(u,\partial u) - [\Gamma^a,\Box_i]u_i - \sum_{\substack{0\le\alpha,\beta\le3\\1\le j\le m}}\left[\Gamma^a,\gamma_{ij}^{\alpha\beta}\partial_\alpha\partial_\beta\right]u_j\\ &= {\sum_{|b|\le|a|}}' \Gamma^b f_i + \sum_{j,\alpha,\beta}\ {\sum_{\substack{|b|+|c|\le|a|\\|c|\le|a|-1}}}' \left(\Gamma^b\gamma_{ij}^{\alpha\beta}\right)\Gamma^c\,\partial_\alpha\partial_\beta u_j - \sum_{j,\alpha,\beta}\gamma_{ij}^{\alpha\beta}\left[\Gamma^a,\partial_\alpha\partial_\beta\right]u_j.\end{aligned}$$

Therefore it follows that

$$\begin{aligned}\left|\tilde f_{i,a}(t,x)\right| &\le C|u(t,x)|_{K+2}\,(|u(t,x)|_{2K} + |\partial u(t,x)|_{2K})\\ &\le CA\langle t\rangle^{-1}\,(|u(t,x)|_{2K} + |\partial u(t,x)|_{2K}),\end{aligned}$$

which implies

$$\left\| \tilde{f}_{i,a}(t,\ \cdot\ )\right\|_{L^2} \le CA\langle t\rangle^{-1}\{\|u(t,\ \cdot\ )\|_{2K} + \|\partial u(t,\ \cdot\ )\|_{2K}\}\,.$$

Besides this we also have

$$\sum_{\substack{0\le\alpha,\beta\le 3\\ 1\le i,j\le m}} \left|\gamma_{ij}^{\alpha\beta}(u,\partial u)(t,x)\right| \le C|u(t,x)|_1 < \frac{1}{2},$$
$$\left|\partial\gamma_{ij}^{\alpha\beta}(u,\partial u)(t,x)\right| \le C|u(t,x)|_2 \le CA\langle t\rangle^{-1},$$

if we take A sufficiently small. So the energy inequality (5.7) leads us to

$$\|\partial\Gamma^a u(t,\ \cdot\ )\|_{L^2} \le C\|\partial\Gamma^a u(0,\ \cdot\ )\|_{L^2} + CA\int_0^t \langle\tau\rangle^{-1}\{\|u(\tau,\ \cdot\ )\|_{2K} + \|\partial u(\tau,\ \cdot\ )\|_{2K}\}\,d\tau.$$

This completes the proof. □

We next derive estimates of $\|u_i(t,\ \cdot\ )\|_{2K}$ and $\|\langle c_i t - |\cdot|\rangle|\partial u_i(t,\ \cdot\ )|_{2K-1}\|_{L^2}$.

Lemma 5.5. *Let* $u \in C^\infty([0,T]\times\mathbb{R}^3)$ *be a solution of the Cauchy problem* (1.1)–(1.2) *for some* $T > 0$. *Suppose that the assumptions in* Proposition 5.2 *are fulfilled*. *Then we have*

$$\begin{aligned}&\|u_i(t,\ \cdot\ )\|_{2K} + \|\langle c_i t - |\cdot|\rangle|\partial u_i(t,\ \cdot\ )|_{2K-1}\|_{L^2}\\ &\le CE_{2K}(0) + CA\int_0^t\langle\tau\rangle^{-1}E_{2K}(\tau)\,d\tau + CA\,(\|u(t,\ \cdot\ )\|_{2K} + \|\partial u(t,\ \cdot\ )\|_{2K})\,.\end{aligned}\tag{5.9}$$

Proof. We first represent $\Gamma^a u_i$ by using the formula (2.4). Then we have

$$\Gamma^a u_i = U_i^*[\Gamma^a u_i(0,\ \cdot\ ), \partial_t\Gamma^a u_i(0,\ \cdot\ )] + {\sum_{|b|\le|a|}}' U_i\left[\Gamma^b F_i\right].\tag{5.10}$$

We proved L^2-estimates concerning $U_i^*[f,g]$ and $U_i[\Phi]$ in Corollary 4.3, so the estimate proceeds as follows.

$$\begin{aligned}&\|u_i(t,\ \cdot\ )\|_{2K} + \|\langle c_i t - |\cdot|\rangle|\partial u_i(t,\ \cdot\ )|_{2K-1}\|_{L^2}\\ &\le C\sum_{|a|\le 2K-1}\left\{\|\Gamma^a u_i(t,\ \cdot\ )\|_1 + \|\langle c_i t - |\cdot|\rangle\partial\Gamma^a u_i(t,\ \cdot\ )\|_{L^2}\right\}\\ &\le C\sum_{|a|\le 2K-1}\left\{\|\langle|\cdot|\rangle\partial\Gamma^a u_i(0,\ \cdot\ )\|_{L^2} + \|\Gamma^a u_i(0,\ \cdot\ )\|_{L^2}\right\}\\ &\quad + C\sum_{|a|\le 2K-1}\left\{\|U_i[\Gamma^a F_i](t,\ \cdot\ )\|_1 + \|\langle c_i t - |\cdot|\rangle\partial U_i[\Gamma^a F_i](t,\ \cdot\ )\|_{L^2}\right\}.\end{aligned}\tag{5.11}$$

Note that $\|\langle|\cdot|\rangle\partial\Gamma^a u_i(0,\cdot)\|_{L^2}+\|\Gamma^a u_i(0,\cdot)\|_{L^2}\le CE_{2K}(0)$ for $|a|\le 2K-1$. So it remains to estimate the second term on the right-hand side of (5.11). We should not use Corollary 4.3 for them right now, because it cannot deal with the divergence-type terms.

Let $|a|\le 2K-1$. We rewrite the terms $u_j(\partial_\alpha\partial_\beta\Gamma^a u_k)$ and $(\partial_\gamma u_j)(\partial_\alpha\partial_\beta\Gamma^a u_k)$ as

$$\left(\partial^b u_j\right)\left(\partial_\alpha\partial_\beta\Gamma^a u_k\right)=\partial_\alpha\left\{\left(\partial^b u_j\right)\left(\partial_\beta\Gamma^a u_k\right)\right\}-\left(\partial_\alpha\partial^b u_j\right)\left(\partial_\beta\Gamma^a u_k\right)\quad(|b|\le 1),$$

so that we can avoid loss of derivatives. We also use a similar trick to handle $(\Gamma^b u_j)(\Gamma^c\partial_\alpha u_k)$ $(j\neq k)$, which may appear in $\Gamma^a R_i$, when $|b|>|c|$ (see Lemma 5.4 in [8] for the details). Then we obtain decompositions of the following type:

$$\Gamma^a F_i(u,\partial u,\nabla_x\partial u)=\sum_{\alpha=0}^{3}\partial_\alpha g_\alpha+q+r+h, \tag{5.12}$$

where

$$g_\alpha=\sum_{j,k=1}^{m}\sum_{|d|\le 1}\sideset{}{'}\sum_{\substack{|b|\le K+1\\|c|\le 2K-1}}\Gamma^b u_j\Gamma^c\partial^d u_k, \tag{5.13}$$

$$q=\sum_{j=1}^{m}\sideset{}{'}\sum_{\substack{|b|\le K\\|c|\le 2K-1}}Q_0\left(\Gamma^b u_j,\Gamma^c u_j;c_j\right)+\sum_{\substack{1\le j\le m\\0\le\alpha,\beta\le 3}}\sideset{}{'}\sum_{\substack{|b|\le K\\|c|\le 2K-1}}Q_{\alpha\beta}\left(\Gamma^b u_j,\Gamma^c u_j\right), \tag{5.14}$$

$$r=\sum_{\substack{1\le j,k\le m\\j\neq k}}\sideset{}{'}\sum_{\substack{|b|\le K+1\\|c|\le 2K-1}}\Gamma^b u_j\Gamma^c\partial u_k, \tag{5.15}$$

$$h=\Gamma^a H_i(u,\partial u,\nabla_x\partial u). \tag{5.16}$$

We continue the estimate to obtain

$$\begin{aligned}&\|U_i[\Gamma^a F_i](t,\cdot)\|_1+\|\langle c_i t-|\cdot|\rangle\partial U_i[\Gamma^a F_i](t,\cdot)\|_{L^2}\\&\le\sum_{\alpha=0}^{3}\{\|U_i[\partial_\alpha g_\alpha](t,\cdot)\|_1+\|\langle c_i t-|\cdot|\rangle\partial U_i[\partial_\alpha g_\alpha](t,\cdot)\|_{L^2}\}\\&\quad+\|U_i[q+r+h](t,\cdot)\|_1+\|\langle c_i t-|\cdot|\rangle\partial U_i[q+r+h](t,\cdot)\|_{L^2}.\end{aligned} \tag{5.17}$$

We begin with the estimates concerning g_α. In order to apply Lemma 4.1 and Corollary 4.5, we interchange the order of the operators U_i and ∂_α by the commutation relations (2.2), as follows:

$$\sum_{\alpha=0}^{3}\{\|U_i[\partial_\alpha g_\alpha](t,\cdot)\|_1+\|\langle c_i t-|\cdot|\rangle\,\partial U_i[\partial_\alpha g_\alpha](t,\cdot)\|_{L^2}\}$$

$$\leq \sum_{\alpha=0}^{3} \{\|\partial_\alpha U_i[g_\alpha](t,\ \cdot)\|_1 + \|\langle c_i t - |\cdot|\rangle\, \partial\partial_\alpha U_i[g_\alpha](t,\ \cdot)\|_{L^2}\}$$
$$+ \|U_i^*[0, g_0(0,\ \cdot)]\|_1 + \|\langle c_i t - |\cdot|\rangle\, \partial U_i^*[0, g_0(0,\ \cdot)]\|_{L^2}$$
$$\leq \sum_{\substack{0\leq\alpha,\beta\leq 3\\ |b|\leq 1}} \left\{\|\partial_\beta U_i[\Gamma^b g_\alpha](t,\ \cdot)\|_{L^2} + \|\langle c_i t - |\cdot|\rangle\, \partial\partial_\alpha U_i[g_\alpha](t,\ \cdot)\|_{L^2}\right\}$$
$$+ C\|U_i^*[0, g_0(0,\ \cdot)]\|_1 + \|\langle c_i t - |\cdot|\rangle\, \partial U_i^*[0, g_0(0,\ \cdot)]\|_{L^2}.$$

Hence Lemma 4.1 and Corollary 4.5 yield

$$\sum_{\alpha=0}^{3} (\|U_i[\partial_\alpha g_\alpha](t,\ \cdot)\|_1 + \|\langle c_i t - |\cdot|\rangle\, \partial U_i[\partial_\alpha g_\alpha](t,\ \cdot)\|_{L^2})$$
$$\leq C \sum_{\alpha=0}^{3} \left(\int_0^t \|g_\alpha(\tau,\ \cdot)\|_1\, d\tau + \|\langle t + |\cdot|\rangle g_\alpha(t,\ \cdot)\|_{L^2} + \|\langle|\cdot|\rangle g_\alpha(0,\ \cdot)\|_{L^2}\right).$$

Recalling (5.13), we have

$$|g_\alpha(t,x)|_1 \leq CA\langle t + |x|\rangle^{-1} (|u(t,x)|_{2K} + |\partial u(t,x)|_{2K}).$$

Thus we conclude

$$\sum_{\alpha=0}^{3} (\|U_i[\partial_\alpha g_\alpha]\|_1 + \|\langle c_i t - |\cdot|\rangle \partial U_i[\partial_\alpha g_\alpha]\|_{L^2})$$
$$\leq CA\left\{\int_0^t \langle\tau\rangle^{-1} (\|u(\tau,\ \cdot)\|_{2K} + \|\partial u(\tau,\ \cdot)\|_{2K})\, d\tau \right.$$
$$\left. + \|u(t,\ \cdot)\|_{2K} + \|\partial u(t,\ \cdot)\|_{2K} + \|u(0,\ \cdot)\|_{2K} + \|\partial u(0,\ \cdot)\|_{2K}\right\}. \tag{5.18}$$

The rest of the proof is aimed at the estimates concerning q, r and h. By Corollary 4.3, we get

$$\|U_i[q+r+h](t,\ \cdot)\|_1 + \|\langle c_i t - |\cdot|\rangle \partial U_i[q+r+h](t,\ \cdot)\|_{L^2}$$
$$\leq C\int_0^t \|\langle\tau + |\cdot|\rangle(q+r+h)(\tau,\ \cdot)\|_{L^2}\, d\tau. \tag{5.19}$$

In view of Lemma 5.1, we divide the region $[0,T]\times\mathbb{R}^3$ into $\{|x|\leq c_0 t\}$ and $\{|x|\geq c_0 t\}$ for the estimate of $q+r$, where $c_0 = \min\{c_1,\ldots,c_m\}/2$. So we decompose the integrand $\|\langle t + |\cdot|\rangle(q+r+h)(t,\ \cdot)\|_{L^2}$ and obtain

$$\|\langle t + |\cdot|\rangle(q+r+h)(t,\ \cdot)\|_{L^2} \leq I + II + III, \tag{5.20}$$

where

$$I = \|\langle t + |\cdot|\rangle (q + r)(t, \cdot)\|_{L^2(|x| \le c_0 t)}, \tag{5.21}$$

$$II = \|\langle t + |\cdot|\rangle (q + r)(t, \cdot)\|_{L^2(|x| \ge c_0 t)}, \tag{5.22}$$

$$III = \|\langle t + |\cdot|\rangle h(t, \cdot)\|_{L^2}. \tag{5.23}$$

Since $\langle t + |x|\rangle \le C\langle c_k t - |x|\rangle$ for $|x| \le c_0 t$, we have

$$I \le CA\langle t\rangle^{-1} \sum_{k=1}^{m} \|\langle c_k t - |\cdot|\rangle |\partial u_k(t, \cdot)|_{2K-1}\|_{L^2}, \tag{5.24}$$

recalling (5.14) and (5.15). On the other hand, Lemma 5.1 yields

$$\begin{aligned}
\langle t + |x|\rangle |q(t, x)| &\le C \sum_{j=1}^{m} \big(\langle c_j t - |x|\rangle |u_j(t, x)|_{K+2} |\partial u_j(t, x)|_{2K-1} \\
&\qquad + |u_j(t, x)|_{K+2} |u_j(t, x)|_{2K}\big) \\
&\le CA\langle t\rangle^{-1} \sum_{j=1}^{m} \big(|\partial u_j(t, x)|_{2K-1} + |u_j(t, x)|_{2K}\big)
\end{aligned}$$

for $|x| \ge c_0 t$. Moreover, since $\langle t + |x|\rangle \le C\langle c_j t - |x|\rangle \langle c_k t - |x|\rangle$ if $j \ne k$, we have

$$\begin{aligned}
\langle t + |x|\rangle |r(t, x)| &\le C \sum_{j \ne k} \langle t + |x|\rangle |u_j(t, x)|_{K+2} |\partial u_k(t, x)|_{2K-1} \\
&\le CA\langle t\rangle^{-1} \sum_{k=1}^{m} \langle c_k t - |x|\rangle |\partial u_k|_{2K-1}
\end{aligned}$$

for $|x| \ge c_0 t$. Therefore it follows that

$$II \le CA\langle t\rangle^{-1} \left\{ \|u(t, \cdot)\|_{2K} + \sum_{k=1}^{m} \|\langle c_k t - |\cdot|\rangle |\partial u_k(t, \cdot)|_{2K-1}\|_{L^2} \right\}. \tag{5.25}$$

Finally, since

$$\begin{aligned}
\langle t + |x|\rangle |h(t, x)| &\le C\langle t + |x|\rangle |u(t, x)|_{K+2}^2 \left(|u(t, x)|_{2K} + |\partial u(t, x)|_{2K}\right) \\
&\le CA^2\langle t\rangle^{-1} \left(|u(t, x)|_{2K} + |\partial u(t, x)|_{2K}\right),
\end{aligned}$$

we have

$$III \le CA^2\langle t\rangle^{-1} \left\{\|u(t, \cdot)\|_{2K} + \|\partial u(t, \cdot)\|_{2K}\right\}. \tag{5.26}$$

Therefore it follows from (5.19), (5.20) and (5.24)–(5.26) that

$$\begin{aligned}&\|U_i[q+r+h](t,\ \cdot\ )\|_1 + \|\langle c_i t - |\cdot|\rangle \partial U_i[q+r+h](t,\ \cdot\ )\|_{L^2}\\ &\le CA\int_0^t \langle\tau\rangle^{-1} E_{2K}(\tau)\,d\tau.\end{aligned} \tag{5.27}$$

Now (5.11), (5.17), (5.18) and (5.27) imply (5.9). □

Proof of Proposition 5.2. By Lemmas 5.4 and 5.5, we have

$$E_{2K}(t) \le CE_{2K}(0) + CA\int_0^t \langle\tau\rangle^{-1} E_{2K}(\tau)\,d\tau$$

for sufficiently small A. Hence Gronwall's inequality yields

$$\begin{aligned}E_{2K}(t) &\le CE_{2K}(0)\exp\left[CA\int_0^t \langle\tau\rangle^{-1}\,d\tau\right]\\ &\le CE_{2K}(0)\langle t\rangle^{CA}.\end{aligned}$$

□

6. Pointwise estimates for small solutions

We first show a refinement of Lemma 3.1, to weaken the weight imposed on the initial data.

Lemma 6.1. *Let* $u \in C^\infty([0,T]\times\mathbb{R}^3)$ *be a solution of the Cauchy problem* (1.1)–(1.2) *for some* $T > 0$. *Assume* $0 \le \lambda \le 1$, $\mu > 0$, *and* $|a| \le \kappa$. *Then we have*

$$\begin{aligned}&\langle t+|x|\rangle^\lambda \langle c_i t - |x|\rangle^\mu \left|U_i^*[\Gamma^a u_i(0,\ \cdot\ ), \partial_t\Gamma^a u_i(0,\ \cdot\ )](t,x)\right|\\ &\le C\sup_{|y|\le c_i t+|x|}\left\{\langle|y|\rangle^{\lambda+\mu}|u_i(0,y)|_\kappa + |y|\langle|y|\rangle^{\lambda+\mu}|\partial u_i(0,y)|_\kappa\right\},\end{aligned} \tag{6.1}$$

$$\begin{aligned}&\langle t+|x|\rangle^\lambda \langle c_i t - |x|\rangle^\mu \left|\partial U_i^*[\Gamma^a u_i(0,\ \cdot\ ), \partial_t\Gamma^a u_i(0,\ \cdot\ )](t,x)\right|\\ &\le C\sup_{|y|\le c_i t+|x|}\langle|y|\rangle^{\lambda+\mu}|\partial u_i(0,y)|_{\kappa+1}\end{aligned} \tag{6.2}$$

for $0 \le t \le T$ *and* $x \in \mathbb{R}^3$.

Proof. We set $u_{i,a}^* = U_i^*[\Gamma^a u_i(0,\ \cdot\ ), \partial_t\Gamma^a u_i(0,\ \cdot\ )]$ for simplicity. By Lemma 3.1, it follows that

$$\begin{aligned}&\langle t+|x|\rangle\langle c_i t-|x|\rangle^\mu |u_{i,a}^*(t,x)|\\ &\le C\sup_{|y|\le c_i t+|x|}\left\{\langle|y|\rangle^{1+\mu}|\Gamma^a u_i(0,y)| + |y|\langle|y|\rangle^{1+\mu}|\partial\Gamma^a u_i(0,y)|\right\}\\ &\le C\langle t+|x|\rangle^{1-\lambda}\sup_{|y|\le c_i t+|x|}\left\{\langle|y|\rangle^{\lambda+\mu}|\Gamma^a u_i(0,y)| + |y|\langle|y|\rangle^{\lambda+\mu}|\partial\Gamma^a u_i(0,y)|\right\}.\end{aligned}$$

Thus we obtain (6.1). To estimate the first derivative (6.2), we begin with the estimate of Lemma 3.1, and use (4.16). Then we have

$$
\begin{aligned}
&\langle t+|x|\rangle\langle c_i t-|x|\rangle^{\mu}|\partial u^*_{i,a}(t,x)| \\
&\le C \sup_{|y|\le c_i t+|x|}\left\{\langle|y|\rangle^{1+\mu}|\partial\Gamma^a u_i(0,y)|+|y|\langle|y|\rangle^{1+\mu}|\nabla\partial\Gamma^a u_i(0,y)|\right\} \\
&\le C \sup_{|y|\le c_i t+|x|}\langle|y|\rangle^{1+\mu}|\partial\Gamma^a u_i(0,y)|_1.
\end{aligned}
$$

Thus we obtain (6.2) by a similar argument as above. □

As a first step, we derive a pointwise decay estimate of the small amplitude solution from Sobolev's inequality and the L^2-estimate (5.5).

Lemma 6.2. *Let* $u \in C^\infty([0,T]\times\mathbb{R}^3)$ *be a solution of the Cauchy problem* (1.1)–(1.2) *for some* $T>0$. *Assume* (1.5), (1.6) *and* (1.12)–(1.16). *Let* $0<\delta<1/2$ *and* $0<\nu<1-2\delta$. *Then there exist two positive constants* A_2 *and* C *such that*

$$
\sum_{i=1}^{m}\sup_{\substack{0\le t\le T\\ x\in\mathbb{R}^3}}\langle|x|\rangle\langle c_i t-|x|\rangle|u_i(t,x)|_{K+2}\le A \tag{6.3}
$$

implies

$$
\begin{aligned}
&\langle t+|x|\rangle^{\nu}\langle c_i t-|x|\rangle^{\delta}|u_i(t,x)|_{2K-3} \\
&+\langle t+|x|\rangle^{\nu-1}\langle|x|\rangle\langle c_i t-|x|\rangle^{1+\delta}|\partial u_i(t,x)|_{2K-4}\le C E_{2K}(0)
\end{aligned}\tag{6.4}
$$

for $0\le t\le T$ *and* $x\in\mathbb{R}^3$, *provided* $A\in(0,A_2]$. *Here the above constants* A_2 *and* C *may depend on* δ *and* ν, *but are independent of* T *and* u.

Proof. Using the representation (2.4) and Lemma 6.1, we have

$$
\begin{aligned}
&\langle t+|x|\rangle^{\nu}\langle c_i t-|x|\rangle^{\delta}|u_i(t,x)|_{2K-3} \\
&\le C\sum_{|a|\le 2K-3}\langle t+|x|\rangle^{\nu}\langle c_i t-|x|\rangle^{\delta} \\
&\quad\times\left\{|U_i^*[\Gamma^a u_i(0,\,\cdot\,),\partial_t\Gamma^a u_i(0,\,\cdot\,)](t,x)|+|U_i[\Gamma^a F_i](t,x)|\right\} \\
&\le C\sup_{y\in\mathbb{R}^3}\left\{\langle|y|\rangle|u_i(0,y)|_{2K-3}+\langle|y|\rangle^2|\partial u_i(0,y)|_{2K-3}\right\} \\
&\quad + C\sum_{|a|\le 2K-3}\langle t+|x|\rangle^{\nu}\langle c_i t-|x|\rangle^{\delta}|U_i[\Gamma^a F_i](t,x)|.
\end{aligned}\tag{6.5}
$$

To estimate the sup norm above, we apply (3.21). Then we immediately see that

$$
\begin{aligned}
&\sup_{y\in\mathbb{R}^3}\left\{\langle|y|\rangle|u_i(0,y)|_{2K-3}+\langle|y|\rangle^2|\partial u_i(0,y)|_{2K-3}\right\}\\
&\le C\sum_{\substack{|a|+|b|\le 2\\ |c|\le 2K-3}}\left\{\|\nabla^a\Omega^b\Gamma^c u_i(0,\,\cdot\,)\|_{L^2}+\|\nabla^a\Omega^b\langle|\cdot|\rangle\Gamma^c\partial u_i(0,\,\cdot\,)\|_{L^2}\right\}\\
&\le CE_{2K}(0).
\end{aligned}\tag{6.6}
$$

To estimate the force terms, we only have to notice that they are quadratic near the origin. Then it follows from Lemma 3.5 and the smallness assumption (6.3) of $|u(t,x)|_{K+2}$ that

$$
\begin{aligned}
|y|\,|\Gamma^a F_i(\tau,y)| &\le C|y|\,|u(\tau,y)|_{K+2}(|u(\tau,y)|_{2K-2}+|\partial u(\tau,y)|_{2K-2})\\
&\le CA\langle|y|\rangle^{-1}\left(\sum_{j=1}^{m}\langle c_j\tau-|y|\rangle^{-1}\right)(\|u(\tau,\,\cdot\,)\|_{2K}+\|\partial u(\tau,\,\cdot\,)\|_{2K})
\end{aligned}
$$

for $|a|\le 2K-3$ and $0\le\tau\le T$. Thus we obtain

$$
|y|\langle\tau+|y|\rangle w(\tau,|y|)|\Gamma^a F_i(\tau,y)|\le CAE_{2K}(\tau),
$$

where the weight $w(t,r)$ is defined by (3.3). Moreover, in view of Proposition 5.2, we get

$$
|y|\langle\tau+|y|\rangle^{1+\delta}w(\tau,|y|)^{1+\delta/2}|\Gamma^a F_i(\tau,y)|\le CAE_{2K}(0)\langle t+|x|\rangle^{2\delta}\tag{6.7}
$$

for $|a|\le 2K-3$, $0\le\tau\le t$ and $c_i\tau+|y|\le c_it+|x|$, provided that A is so small to satisfy $A\le A_1$ and $C_1A\le\delta/2$. Here A_1 and C_1 are the constants in Proposition 5.2. Thus Lemma 3.2 yields

$$
\langle t+|x|\rangle\langle c_it-|x|\rangle^{\delta}|U_i[\Gamma^a F_i](t,x)|\le CAE_{2K}(0)\langle t+|x|\rangle^{2\delta}\tag{6.8}
$$

for $|a|\le 2K-3$. Hence (6.5)–(6.8) imply

$$
\langle t+|x|\rangle^{\nu}\langle c_it-|x|\rangle^{\delta}|u_i(t,x)|_{2K-3}\le CE_{2K}(0).\tag{6.9}
$$

We next estimate $|\partial u_i(t,x)|_{2K-4}$. By (2.4) and Lemma 6.1, we have

$$
\begin{aligned}
&\langle t+|x|\rangle^{\nu-1}\langle |x|\rangle\langle c_i t-|x|\rangle^{1+\delta}|\partial u_i(t,x)|_{2K-4}\\
&\le C\sum_{|a|\le 2K-4}\langle t+|x|\rangle^{\nu-1}\langle |x|\rangle\langle c_i t-|x|\rangle^{1+\delta}\\
&\qquad\qquad \times\left\{|\partial U_i^*[\Gamma^a u_i(0,\ \cdot\ ),\partial_t\Gamma^a u_i(0,\ \cdot\ )](t,x)|+|\partial U_i[\Gamma^a F_i](t,x)|\right\}\\
&\le C\sup_{y\in\mathbb{R}^3}\langle |y|\rangle^2|\partial u_i(0,y)|_{2K-3}\\
&\quad + C\sum_{|a|\le 2K-4}\langle t+|x|\rangle^{\nu-1}\langle |x|\rangle\langle c_i t-|x|\rangle^{1+\delta}|\partial U_i[\Gamma^a F_i](t,x)|.
\end{aligned}
\tag{6.10}
$$

Observing that, by (6.7) and Lemma 3.2, we obtain

$$
\langle |x|\rangle\langle c_i t-|x|\rangle^{1+\delta}|\partial U_i[\Gamma^a F_i](t,x)|\le CAE_{2K}(0)\langle t+|x|\rangle^{2\delta}
\tag{6.11}
$$

for $|a|\le 2K-4$, we conclude from (6.10) and (6.6) that

$$
\langle t+|x|\rangle^{\nu-1}\langle |x|\rangle\langle c_i t-|x|\rangle^{1+\delta}|\partial u_i(t,x)|_{2K-4}\le CE_{2K}(0).
$$

This completes the proof. □

Now we set

$$
a_1(t)=a_1[u](t)=\sup_{x\in\mathbb{R}^3}\sum_{i=1}^m\langle |x|\rangle\langle c_i t-|x|\rangle|u_i(t,x)|_{K+2},
\tag{6.12}
$$

$$
a_2(t)=a_2[u](t)=\sup_{x\in\mathbb{R}^3}\sum_{i=1}^m\langle |x|\rangle\langle c_i t-|x|\rangle w(t,|x|)^{\nu}|\partial u_i(t,x)|_{K+3},
\tag{6.13}
$$

$$
a_3(t)=a_3[u](t)=\sup_{x\in\mathbb{R}^3}\sum_{i=1}^m\langle |x|\rangle\langle c_i t-|x|\rangle^{\nu}|u_i(t,x)|_{2K-5},
\tag{6.14}
$$

and

$$
A(T)=A[u](T)=\sup_{0\le t\le T}\{a_1(t)+a_2(t)+a_3(t)\},
\tag{6.15}
$$

where $w(t,r)$ is defined by (3.3). Our aim in this section is to give a bound of $A[u](T)$ for a small amplitude solution of (1.1).

Proposition 6.3. *Let $u\in C^\infty([0,T]\times\mathbb{R}^3)$ be a solution of the Cauchy problem* (1.1)–(1.2) *for some $T>0$. Assume* (1.5), (1.6) *and* (1.12)–(1.16). *Assume moreover $K+6\le 2K-5$ and $0<\nu\le 1/2$ in the definition of $A(T)$ above. Then there exist* *positive numbers* A_0 *and* C_0, *both independent of* u *and* T, *such that the following holds*: *If* $A(T) \le A_0$, *then we have*

$$A(T) \le C_0 \sup_{y\in\mathbb{R}^3}\Big\{\langle|y|\rangle^2|u(0,y)|_{K+2} + \langle|y|\rangle^3|\partial u(0,y)|_{K+2} + \langle|y|\rangle^{2+\nu}|\partial_t u(0,y)|_{2K-5}\Big\} + C_0 E_{2K}(0). \tag{6.16}$$

The proof of this proposition will be given at the end of this section, after we prove three lemmas below.

Lemma 6.4. *Let* $u \in C^\infty([0,T]\times\mathbb{R}^3)$ *be a solution of the Cauchy problem* (1.1)–(1.2) *for some* $T > 0$. *Assume* (1.5), (1.6) *and* (1.12)–(1.16). *If* $0 < \nu < 1$ *and* $A(T) \le A$, *then we have*

$$\begin{aligned}&\langle|x|\rangle\langle c_i t - |x|\rangle^\nu |u_i(t,x)|_{2K-5}\\ &\le C \sup_{y\in\mathbb{R}^3}\Big\{\langle|y|\rangle^{1+\nu}|u_i(0,y)|_{2K-5} + \langle|y|\rangle^{2+\nu}|\partial u_i(0,y)|_{2K-5}\Big\}\\ &\quad + CA(T)E_{2K}(0) + CA(T)^2\end{aligned} \tag{6.17}$$

for $0 \le t \le T$ *and* $x \in \mathbb{R}^3$, *provided that* A *is sufficiently small.*

Proof. By (2.4) and Lemma 6.1, we have

$$\begin{aligned}&\langle|x|\rangle\langle c_i t - |x|\rangle^\nu |u_i(t,x)|_{2K-5}\\ &\le C \sup_{y\in\mathbb{R}^3}\Big\{\langle|y|\rangle^{1+\nu}|u_i(0,y)|_{2K-5} + \langle|y|\rangle^{2+\nu}|\partial u_i(0,y)|_{2K-5}\Big\}\\ &\quad + C\sum_{|a|\le 2K-5}\langle|x|\rangle\langle c_i t - |x|\rangle^\nu |U_i[\Gamma^a F_i](t,x)|.\end{aligned} \tag{6.18}$$

In order to estimate the effects of the force terms, we use the decomposition (1.12). That is,

$$\begin{aligned}&|U_i[\Gamma^a F_i](t,x)|\\ &\le \sum_{\alpha=0}^{3}|U_i[\Gamma^a\partial_\alpha G_{i,\alpha}](t,x)| + |U_i[\Gamma^a N_i](t,x)|\\ &\quad + |U_i[\Gamma^a R_i](t,x)| + |U_i[\Gamma^a H_i](t,x)|\\ &\le C\sum_{\alpha,\beta=0}^{3}\sum_{|b|\le|a|}\Big|U_i\left[\partial_\beta\Gamma^b G_{i,\alpha}\right](t,x)\Big| + |U_i[\Gamma^a N_i](t,x)|\\ &\qquad + |U_i[\Gamma^a R_i](t,x)| + |U_i[\Gamma^a H_i](t,x)|.\end{aligned} \tag{6.19}$$

We estimate the each term above in the following. Firstly, we choose sufficiently small $\delta > 0$ so that we have $\delta \le \nu < 1 - 2\delta$. Then, by the pointwise estimate of Lemma 6.2

and the definition (6.15), we get

$$\begin{aligned}\left|\Gamma^b G_{i,\alpha}(\tau,y)\right|_1 &\le C\sum_{j\ne i}|u_j(\tau,y)|_{K+2}|u_j(\tau,y)|_{2K-3}\\ &\le CA(T)E_{2K}(0)\langle|y|\rangle^{-1}\langle\tau+|y|\rangle^{-\nu}w_i(\tau,|y|)^{-1-\delta}\end{aligned}\tag{6.20}$$

for $|b|\le 2K-5$, where $w_i(t,r)$ $(i=1,\dots,m)$ are defined by (3.4). Hence it follows from (3.8) in Lemma 3.2 that

$$\langle|x|\rangle\langle c_it-|x|\rangle^{\nu}|U_i[\partial_\beta\Gamma^b G_{i,\alpha}](t,x)|\le CA(T)E_{2K}(0)\tag{6.21}$$

for $|b|\le 2K-5$. Likewise, we compute pointwise bounds for the force terms by using Lemma 6.2 and (6.15), and apply Lemma 3.2 in the following. To estimate the null forms, we divide $[0,T]\times\mathbb{R}^3$ into $\{|y|\le c_0\tau\}$ and $\{|y|\ge c_0\tau\}$. If $|y|\le c_0\tau$, simply because N_i are quadratic, we obtain

$$\begin{aligned}|\Gamma^a N_i(\tau,y)| &\le C\sum_{j=1}^m|u_j(\tau,y)|_{K+2}|\partial u_j(\tau,y)|_{2K-4}\\ &\le CA(T)E_{2K}(0)\sum_{j=1}^m\langle|y|\rangle^{-2}\langle\tau+|y|\rangle^{1-\nu}\langle c_j\tau-|y|\rangle^{-2-\delta}\\ &\le CA(T)E_{2K}(0)\langle|y|\rangle^{-1}\langle\tau+|y|\rangle^{-1-\nu}\langle|y|\rangle^{-1-\delta},\end{aligned}$$

provided $|a|\le 2K-5$. If $|y|\ge c_0\tau$ to the contrary, we employ Lemma 5.1. Since $\langle|y|\rangle^{-1}\le C\langle\tau+|y|\rangle^{-1}$, we easily have

$$\begin{aligned}|\Gamma^a N_i(\tau,y)| &\le C\sum_{j=1}^m\langle\tau+|y|\rangle^{-1}\{\langle c_j\tau-|y|\rangle|u_j(\tau,y)|_{K+2}|\partial u_j(\tau,y)|_{2K-4}\\ &\qquad\qquad+|u_j(\tau,y)|_{K+2}|u_j(\tau,y)|_{2K-3}\}\\ &\le CA(T)E_{2K}(0)\langle|y|\rangle^{-1}\langle\tau+|y|\rangle^{-1-\nu}\sum_{j=1}^m\langle c_j\tau-|y|\rangle^{-1-\delta}.\end{aligned}$$

To sum up, we have proved

$$|\Gamma^a N_i(\tau,y)|\le CA(T)E_{2K}(0)\langle|y|\rangle^{-1}\langle\tau+|y|\rangle^{-1-\nu}w(\tau,|y|)^{-1-\delta}\tag{6.22}$$

for $|a|\le 2K-5$. Therefore, (3.7) in Lemma 3.2 implies

$$\langle t+|x|\rangle\langle c_it-|x|\rangle^{\nu}|U_i[\Gamma^a N_i](t,x)|\le CA(T)E_{2K}(0)\tag{6.23}$$

for $|a|\le 2K-5$. In the estimates of the nonresonant terms $\Gamma^a R_i$, we note that at least two of three decaying factors $\langle|y|\rangle^{-1}$, $\langle c_j\tau-|y|\rangle^{-1}$ and $\langle c_k\tau-|y|\rangle^{-1}$ are equivalent to $\langle \tau + |y| \rangle^{-1}$ everywhere, by virtue of the difference of the wave propagation speeds. Remember also that we have chosen δ satisfying $0 < \delta \le \nu < 1 - 2\delta$. Then Lemma 6.2 and (6.15) lead to

$$
\begin{aligned}
&|\Gamma^a R_i(\tau, y)| \\
&\le C \sum_{j \ne k} \left(|u_j(\tau, y)|_{K+2} |\partial u_k(\tau, y)|_{2K-4} + |u_j(\tau, y)|_{2K-5} |\partial u_k(\tau, y)|_{K+3} \right) \\
&\le C A(T) E_{2K}(0) \sum_{j \ne k} \langle |y| \rangle^{-2} \langle \tau + |y| \rangle^{1-\nu} \langle c_j \tau - |y| \rangle^{-1} \langle c_k \tau - |y| \rangle^{-1-\delta} \\
&\qquad + C A(T)^2 \sum_{j \ne k} \langle |y| \rangle^{-2} \langle c_j \tau - |y| \rangle^{-\nu} \langle c_k \tau - |y| \rangle^{-1} w(\tau, |y|)^{-\nu} \\
&\le C \left(A(T) E_{2K}(0) + A(T)^2 \right) \langle |y| \rangle^{-1} \langle \tau + |y| \rangle^{-1-\nu} w(\tau, |y|)^{-1-\delta}
\end{aligned}
\tag{6.24}
$$

for $|a| \le 2K - 5$. Therefore, it follows that

$$
\langle t + |x| \rangle \langle c_i t - |x| \rangle^{\nu} |U_i[\Gamma^a R_i](t, x)| \le C \left(A(T) E_{2K}(0) + A(T)^2 \right) \tag{6.25}
$$

for $|a| \le 2K - 5$. Lastly,

$$
\begin{aligned}
|\Gamma^a H_i(\tau, y)| &\le C |u(\tau, y)|_{K+2}^2 \left(|\partial u(\tau, y)|_{2K-4} + |u(\tau, y)|_{2K-3} \right) \\
&\le C A(T)^2 E_{2K}(0) \langle |y| \rangle^{-3} \langle \tau + |y| \rangle^{1-\nu} \sum_{j=1}^{m} \langle c_j t - |y| \rangle^{-2-\delta} \\
&\le C A(T)^2 E_{2K}(0) \langle |y| \rangle^{-1} \langle \tau + |y| \rangle^{-1-\nu} w(\tau, |y|)^{-2-\delta}
\end{aligned}
\tag{6.26}
$$

for $|a| \le 2K - 5$, so we obtain

$$
\langle t + |x| \rangle \langle c_i t - |x| \rangle^{\nu} |U_i[\Gamma^a H_i](t, x)| \le C A(T)^2 E_{2K}(0) \quad (|a| \le 2K - 5). \tag{6.27}
$$

Thus we have proved the lemma, by (6.21), (6.23), (6.25) and (6.27). $\square$

Lemma 6.5. *Let $u \in C^\infty([0, T] \times \mathbb{R}^3)$ be a solution of the Cauchy problem* (1.1)–(1.2) *for some $T > 0$. Assume* (1.5), (1.6) *and* (1.12)–(1.16). *If $K + 6 \le 2K - 5$ and $A(T) \le 1$, then we have*

$$
\begin{aligned}
&\langle |x| \rangle \langle c_i t - |x| \rangle |u_i(t, x)|_{K+2} \\
&\le C \sup_{y \in \mathbb{R}^3} \left\{ \langle |y| \rangle^2 |u(0, y)|_{K+2} + \langle |y| \rangle^3 |\partial u(0, y)|_{K+2} \right\} + C A(T)^2
\end{aligned}
\tag{6.28}
$$

for $0 \le t \le T$ and $x \in \mathbb{R}^3$.

Proof. By (2.4) and Lemma 6.1, we have

$$
\begin{aligned}
&\langle |x|\rangle\langle c_i t-|x|\rangle |u_i(t,x)|_{K+2}\\
&\le C\sup_{y\in\mathbb{R}^3}\left\{\langle |y|\rangle^2|u(0,y)|_{K+2}+\langle |y|\rangle^3|\partial u(0,y)|_{K+2}\right\}\\
&\quad + C\sum_{|a|\le K+2}\langle |x|\rangle\langle c_i t-|x|\rangle |U_i[\Gamma^a F_i](t,x)|.
\end{aligned}
\tag{6.29}
$$

Then we use the decomposition (6.19) for $|a|\le K+2$, and make the estimates by similar arguments as in the previous lemma. That is, we compute pointwise bounds for the force terms by using (6.15), and apply Lemma 3.2.

We start with an estimate of $G_{i,\alpha}$. Since $K+6\le 2K-5$, we get

$$
\begin{aligned}
\left|\Gamma^b G_{i,\alpha}(\tau,y)\right| &\le C\sum_{j\ne i}|u_j(\tau,y)|_{K+2}|u_j(\tau,y)|_{2K-5}\\
&\le CA(T)^2\langle |y|\rangle^{-2}\sum_{j\ne i}\langle c_j\tau-|y|\rangle^{-1-\nu}\\
&\le CA(T)^2\langle |y|\rangle^{-1}\langle \tau+|y|\rangle^{-1}w_i(\tau,|y|)^{-1-\nu}
\end{aligned}
\tag{6.30}
$$

for $|b|\le K+5$. Note that what we actually need here is the estimate for $|b|\le K+3$. The estimate for $|b|\le K+5$ will be used to prove the next lemma.

Now, assume $|a|\le K+2$ in what follows. We estimate the null forms for $|y|\le c_0\tau$ as

$$
\begin{aligned}
|\Gamma^a N_i(\tau,y)| &\le C\sum_{j=1}^m |u_j(\tau,y)|_{K+2}|\partial u_j(\tau,y)|_{K+3}\\
&\le C\sum_{j=1}^m A(T)^2\langle |y|\rangle^{-2}\langle c_j\tau-|y|\rangle^{-2}w(\tau,|y|)^{-\nu}\\
&\le CA(T)^2\langle |y|\rangle^{-1}\langle \tau+|y|\rangle^{-2}w(\tau,|y|)^{-1-\nu},
\end{aligned}
$$

while for $|y|\ge c_0\tau$, Lemma 5.1 implies

$$
\begin{aligned}
|\Gamma^a N_i(\tau,y)| &\le C\sum_{j=1}^m\langle \tau+|y|\rangle^{-1}\{\langle c_j\tau-|y|\rangle|u_j(\tau,y)|_{K+2}|\partial u_j(\tau,y)|_{K+3}\\
&\qquad\qquad + |u_j(\tau,y)|_{K+2}|u_j(\tau,y)|_{K+4}\}\\
&\le CA(T)^2\sum_{j=1}^m\langle \tau+|y|\rangle^{-1}\langle |y|\rangle^{-2}w(\tau,|y|)^{-\nu}\langle c_j\tau-|y|\rangle^{-1}\\
&\le CA(T)^2\langle |y|\rangle^{-1}\langle \tau+|y|\rangle^{-2}w(\tau,|y|)^{-1-\nu}.
\end{aligned}
$$

Hence it follows that

$$|\Gamma^a N_i(\tau, y)| \le CA(T)^2 \langle |y|\rangle^{-1}\langle \tau + |y|\rangle^{-2} w(\tau, |y|)^{-1-\nu}. \tag{6.31}$$

As for the nonresonant terms, noting that $\langle c_j\tau - |y|\rangle\langle c_k\tau - |y|\rangle$ is bounded from below by $C\langle \tau + |y|\rangle w(\tau, |y|)$ for $c_j \neq c_k$, we obtain

$$\begin{aligned}|\Gamma^a R_i(\tau, y)| &\le C\sum_{j\neq k} |u_j(\tau, y)|_{K+2}|\partial u_k(\tau, y)|_{K+3}\\ &\le CA(T)^2\sum_{j\neq k}\langle |y|\rangle^{-2}\langle c_j\tau - |y|\rangle^{-1}\langle c_k\tau - |y|\rangle^{-1} w(\tau, |y|)^{-\nu}\\ &\le CA(T)^2\langle |y|\rangle^{-1}\langle \tau + |y|\rangle^{-2} w(\tau, |y|)^{-1-\nu}.\end{aligned} \tag{6.32}$$

Finally, the higher order terms are handled as

$$\begin{aligned}|\Gamma^a H_i(\tau, y)| &\le C|u(\tau, y)|_{K+2}^2(|\partial u(\tau, y)|_{K+3} + |u(\tau, y)|_{K+2})\\ &\le CA(T)^3\langle |y|\rangle^{-3}\sum_{j=1}^{m}\langle c_j\tau - |y|\rangle^{-3}\\ &\le CA(T)^3\langle |y|\rangle^{-1}\langle \tau + |y|\rangle^{-2} w(\tau, |y|)^{-3}.\end{aligned} \tag{6.33}$$

Now, combining the estimates (6.31)–(6.33) for $|a| \le K + 2$ with (3.6) of Lemma 3.2, and (6.30) for $|b| \le K + 3$ with (3.8) of Lemma 3.2, we obtain (6.28). □

It remains to show the estimate of $a_2(t)$. Here we need the extra decaying factor $w(t, r)^\nu$, which has played an important role in the proof of Lemma 6.5, but it will not be difficult to obtain this factor from the terms other than the divergence-type terms. To handle the effects of the divergence terms, we notice that they are written as the second derivatives plus harmless terms.

Lemma 6.6. *Let* $u \in C^\infty([0, T] \times \mathbb{R}^3)$ *be a solution of the Cauchy problem* (1.1)–(1.2) *for some* $T > 0$. *Assume* (1.5), (1.6) *and* (1.12)–(1.16). *Suppose that* $0 < \nu < 1$ *and* $K + 6 \le 2K - 5$. *If* $A(T) \le A$, *then we have*

$$\begin{aligned}&\langle |x|\rangle\langle c_i t - |x|\rangle w(t, |x|)^\nu|\partial u_i(t, x)|_{K+3}\\ &\le C\sup_{y\in\mathbb{R}^3}\langle |y|\rangle^{2+\nu}|\partial u(0, y)|_{K+4} + CA(T)E_{2K}(0) + CA(T)^2\end{aligned} \tag{6.34}$$

for $0 \le t \le T$ *and* $x \in \mathbb{R}^3$, *provided that* A *is sufficiently small.*

Proof. We begin with (2.4) and Lemma 6.1 as before. Since $w(t,r) \le \langle c_i t - r\rangle$, we get

$$
\begin{aligned}
&\langle|x|\rangle\langle c_i t-|x|\rangle w(t,|x|)^{\nu}|\partial u_i(t,x)|_{K+3}\\
&\le C\sup_{y\in\mathbb{R}^3}\langle|y|\rangle^{2+\nu}|\partial u(0,y)|_{K+4}\\
&\quad + C\sum_{|a|\le K+3}\langle|x|\rangle\langle c_i t-|x|\rangle w(t,|x|)^{\nu}|\partial U_i[\Gamma^a F_i](t,x)|.
\end{aligned}
\tag{6.35}
$$

Let $|a| \le K+3$. We split $U_i[\Gamma^a F_i]$ by using (1.12). We first deal with the terms concerning N_i, R_i and H_i. By the commutation relations (2.2), we get

$$
\begin{aligned}
&|\partial U_i[\Gamma^a N_i](t,x)| + |\partial U_i[\Gamma^a R_i](t,x)| + |\partial U_i[\Gamma^a H_i](t,x)|\\
&\le |U_i[\partial\Gamma^a N_i](t,x)| + |U_i[\partial\Gamma^a R_i](t,x)| + |U_i[\partial\Gamma^a H_i](t,x)|\\
&\quad + |U_i^*[0,\Gamma^a N_i(0,\,\cdot\,)](t,x)| + |U_i^*[0,\Gamma^a R_i(0,\,\cdot\,)](t,x)|\\
&\quad + |U_i^*[0,\Gamma^a H_i(0,\,\cdot\,)](t,x)|.
\end{aligned}
$$

We have already computed the estimate of $\Gamma^a N_i, \Gamma^a R_i$ and $\Gamma^a H_i$ in (6.22), (6.24) and (6.26) for $|a| \le 2K-5$. Therefore, it follows from Lemma 3.1 and (3.7) of Lemma 3.2 that

$$
\begin{aligned}
&\langle|x|\rangle\langle c_i t-|x|\rangle^{1+\nu}\left\{|\partial U_i[\Gamma^a N_i](t,x)| + |\partial U_i[\Gamma^a R_i](t,x)| + |\partial U_i[\Gamma^a H_i](t,x)|\right\}\\
&\le C\langle|x|\rangle\langle c_i t-|x|\rangle^{1+\nu}\Big\{|U_i[\partial\Gamma^a N_i](t,x)| + |U_i[\partial\Gamma^a R_i](t,x)|\\
&\qquad + |U_i[\partial\Gamma^a H_i](t,x)| + |U_i^*[0,\Gamma^a N_i(0,\,\cdot\,)](t,x)|\\
&\qquad + |U_i^*[0,\Gamma^a R_i(0,\,\cdot\,)](t,x)| + |U_i^*[0,\Gamma^a H_i(0,\,\cdot\,)](t,x)|\Big\}\\
&\le C\Bigg\{A(T)E_{2K}(0) + A(T)^2\\
&\qquad + \sup_{y\in\mathbb{R}^3}\langle|y|\rangle^{3+\nu}(|\Gamma^a N_i(0,y)| + |\Gamma^a R_i(0,y)| + |\Gamma^a H_i(0,y)|)\Bigg\}\\
&\le C\left(A(T)E_{2K}(0) + A(T)^2\right).
\end{aligned}
\tag{6.36}
$$

Now it remains to estimate $\partial U_i[\partial_\beta\Gamma^a G_{i,\alpha}]$ for $|a| \le K+3$. We employ (2.2) to form second derivatives:

$$
\partial U_i[\partial_\beta\Gamma^a G_{i,\alpha}] = \partial\partial_\beta U_i[\Gamma^a G_{i,\alpha}] - \delta_{0\beta}\,\partial U_i^*[0,\Gamma^a G_{i,\alpha}(0,\,\cdot\,)].
\tag{6.37}
$$

Applying Lemma 3.1 to the second term on the right-hand side above, we have

$$
\begin{aligned}
&\langle|x|\rangle\langle c_i t-|x|\rangle w(t,|x|)^{\nu}|\partial U_i^*[0,\Gamma^a G_{i,\alpha}(0,\,\cdot\,)](t,x)|\\
&\le C\sup_{y\in\mathbb{R}^3}\langle|y|\rangle^{3+\nu}|G_{i,\alpha}(0,y)|_{K+4}\\
&\le C E_{2K}(0)A(T) + C A(T)^2,
\end{aligned}
\tag{6.38}
$$

where we used Lemma 6.2 and (6.15). As for the second derivative, we use Corollary 3.4 to obtain

$$
\begin{aligned}
&\langle |x|\rangle \langle c_i t - |x|\rangle w(t,|x|)^{\nu} |\partial \partial_\beta U_i[\Gamma^a G_{i,\alpha}](t,x)| \\
&\le C\langle |x|\rangle \langle c_i t - |x|\rangle |\partial U_i[\Gamma^a G_{i,\alpha}](t,x)|_1 \\
&\quad + C\langle |x|\rangle \langle t + |x|\rangle w(t,|x|)^{\nu} |\Gamma^a G_{i,\alpha}(t,x)|.
\end{aligned} \tag{6.39}
$$

Note that we have disposed of $w(t,|x|)^{\nu-1}$ in the first term on the right-hand side. In order to estimate it further, we utilize the commutation relations repeatedly and get

$$
\begin{aligned}
\Gamma^d \partial_\beta U_i[\Gamma^a G_{i,\alpha}] = &\sum_{\substack{|c|\le K+4 \\ 0\le\gamma\le 3}}{}' U_i[\partial_\gamma \Gamma^c G_{i,\alpha}] + \sum_{|c|\le K+4}{}' U_i^*[0, \Gamma^c G_{i,\alpha}(0,\ \cdot)] \\
&+ \sum_{0\le\gamma\le 3}{}' \partial_\gamma U_i^*[0, \Gamma^a G_{i,\alpha}(0,\ \cdot)]
\end{aligned}
$$

for $|a| \le K+3$ and $|d| \le 1$. Therefore we obtain

$$
\begin{aligned}
&\langle |x|\rangle \langle c_i t - |x|\rangle |\partial U_i[\Gamma^a G_{i,\alpha}](t,x)|_1 \\
&\le C \sum_{\substack{|c|\le K+4 \\ 0\le\beta\le 3}} \langle |x|\rangle \langle c_i t - |x|\rangle |U_i[\partial_\beta \Gamma^c G_{i,\alpha}](t,x)| \\
&\quad + C \sup_{y\in\mathbb{R}^3} \langle |y|\rangle^3 |G_{i,\alpha}(0,y)|_{K+4}.
\end{aligned}
$$

Now, in view of (6.30) for $|b| \le K+5$, from (3.8) of Lemma 3.2 and (6.38) we get

$$
\langle |x|\rangle \langle c_i t - |x|\rangle |\partial U_i[\Gamma^a G_{i,\alpha}](t,x)|_1 \le C E_{2K}(0) A(T) + C A(T)^2. \tag{6.40}
$$

As for the second term on the right-hand side of (6.39), we see easily from (6.30) for $|b| \le K+3$ that it is bounded by $CA(T)^2$, because $w(t,|x|) \le C w_i(t,|x|)$. Finally, it follows from (6.37)–(6.40) that

$$
\langle |x|\rangle \langle c_i t - |x|\rangle w(t,|x|)^{\nu} |\partial U_i[\partial_\beta \Gamma^a G_{i,\alpha}](t,x)| \le C E_{2K}(0) A(T) + C A(T)^2 \tag{6.41}
$$

for $|a| \le K+3$. This completes the proof. □

Proof of Proposition 6.3. Summing up the estimates of Lemmas 6.4, 6.5 and 6.6, we get

$$
\begin{aligned}
A(T) \le\ & C \sup_{y\in\mathbb{R}^3} \left\{ \langle |y|\rangle^2 |u(0,y)|_{K+2} + \langle |y|\rangle^3 |\partial u(0,y)|_{K+2} \right\} \\
&+ C \sup_{y\in\mathbb{R}^3} \left\{ \langle |y|\rangle^{1+\nu} |u_i(0,y)|_{2K-5} + \langle |y|\rangle^{2+\nu} |\partial u_i(0,y)|_{2K-5} \right\} \\
&+ C A(T) E_{2K}(0) + C A(T)^2.
\end{aligned}
$$

Since $\nu \le 1/2$, Lemma 3.5 and (4.16) imply

$$\langle |y| \rangle^{1+\nu} |u_i(0, y)|_{2K-5} + \langle |y| \rangle^{2+\nu} |\nabla_x u_i(0, y)|_{2K-5} \le C_1 E_{2K}(0). \tag{6.42}$$

Thus we obtain (6.16), provided that $A(T)$ is sufficiently small. □

7. Proof of the main theorem

In this section we give a proof of Theorem 1.1. Suppose that all the assumptions of Theorem 1.1 are fulfilled. Because we are only considering small solutions, changing the definition of $\gamma_{ij}^{\alpha\beta}(u, v)$ in (1.4) outside some large ball of (u, v) does not affect solutions. Hence we may assume $\sum_{\alpha,\beta,i,j} \gamma_{ij}^{\alpha\beta}(u, v) \le 1/2$ for any $(u, v) \in \mathbb{R}^m \times \mathbb{R}^{4m}$. Then, by the standard argument for classical local existence theorems, we can see that the Cauchy problem (1.1)–(1.2) admits a (unique) local solution $u \in C^\infty\left([0, T) \times \mathbb{R}^3; \mathbb{R}^m\right)$ for some $T > 0$. More precisely, we have

$$u \in C^\infty\left([0, T); H^{s,p}(\mathbb{R}^3; \mathbb{R}^m)\right) \quad \text{for any} \quad s \ge 0 \quad \text{and} \quad p \ge 0, \tag{7.1}$$

where $H^{s,p}$ is given by $H^{s,p} = \left\{ f \in L^2;\ \sum_{|a| \le s} \|\langle|\cdot|\rangle^p \partial_x^a f\|_{L^2} < \infty \right\}$ with $\partial_x = (\partial_1, \partial_2, \partial_3)$. Moreover, the above solution u can be extended beyond the above time T, unless

$$\sup_{(t,x)\in[0,T)\times\mathbb{R}^3} \sum_{|a|\le 2} |\partial^a u(t, x)| = \infty \tag{7.2}$$

holds (see Hörmander [5], Theorem 6.4.11 and its remarks; see also Proposition 4.1 in [7]). Therefore, if we can show that $\sum_{|a|\le 2} \|\partial^a u(t,\ \cdot)\|_{L^\infty(\mathbb{R}^3)}$ stays small as far as the solution exists, we can extend the solution globally in time.

Our task is to show the following:

Proposition 7.1. *Suppose that the assumptions in* Theorem 1.1 *are fulfilled. Assume that* $\nu \in (0, 1/2]$ *and* $K + 6 \le 2K - 5$ *in the definition* (6.15) *of* $A[u](t)$. *Set* $M = \max\{1, C_0, C_1\}$, *where* C_0 *and* C_1 *are the constants given in* (6.16) *and* (6.42), *respectively. If*

$$\begin{aligned} &M \sup_{y\in\mathbb{R}^3} \Big\{ \langle |y| \rangle^2 |u(0, y)|_{K+2} + \langle |y| \rangle^3 |\partial u(0, y)|_{K+3} \\ &\qquad + \langle |y| \rangle^{2+\nu} |\partial_t u(0, y)|_{2K-5} \Big\} + M E_{2K}(0) \le \frac{A_0}{2}, \end{aligned} \tag{7.3}$$

then, for the local solution $u \in C^\infty\left([0, T) \times \mathbb{R}^3; \mathbb{R}^m\right)$, *we have* $\sup_{0\le t<T} A[u](t) \le A_0$. *Here* A_0 *is the constant appeared in* Proposition 6.3.

Proposition 7.1 implies Theorem 1.1 immediately, because we have

$$\sum_{|a|\le 2} \|\partial^a u(t,\ \cdot\ )\|_{L^\infty(\mathbb{R}^3)} \le A[u](t) \quad \text{for any} \quad t \in [0, T).$$

Proof of Proposition 7.1. Thanks to (7.1) and the Sobolev embedding theorem, $A(t) = A[u](t)$ is continuous with respect to $t \in [0, T)$.

Set $T_0 := \sup\{0 \le t < T; A(t) \le A_0\}$. (7.3) implies $A(0) \le A_0/2$, because we have

$$A(0) \le \sup_{y\in\mathbb{R}^3} \big(\langle|y|\rangle^2 |u(0, y)|_{K+2} + \langle|y|\rangle^3 |\partial u(0, y)|_{K+3}\big) + M E_{2K}(0)$$

by the definition of $A(T)$ and (6.42). Hence, by the continuity of $A(t)$, we find that T_0 is well-defined and $T_0 > 0$. Now assume $T_0 < T$. Then (7.3) and Proposition 6.3 yield $A(T_0) \le A_0/2$, and thus we see that $A(T_0 + \delta) \le A_0$ for some $\delta > 0$. This contradicts the definition of T_0, and we conclude that $T_0 = T$. This completes the proof. □

8. Appendix

In this section, we give a proof of Lemma 3.2.

Lemma 8.1. *Let* $a \ge 0$, $\mu > 0$, *and* $\nu > 0$. *Then we have*

$$\begin{aligned}
&\langle t + |x|\rangle \langle c_i t - |x|\rangle^{\nu} |U_i[\Phi](t, x)| \\
&\le C \sup_{(\tau,y)\in D_i(t,|x|)} \langle|y|\rangle \langle \tau + |y|\rangle^{1+\nu} \langle a\tau - |y|\rangle^{1+\mu} |\Phi(\tau, y)|
\end{aligned} \tag{8.1}$$

for $i = 1, \dots, m$, *where* $D_i(t, r)$ *are defined by* (3.5).

Proof. It suffices to prove Lemma 8.1 for the case where $c_i = 1$. So in the following we always assume $c_i = 1$.

Set

$$z_0(\tau, \rho) = (1 + \tau + \rho)^{1+\nu} (1 + |a\tau - \rho|)^{1+\mu}. \tag{8.2}$$

Then we have

$$|U_i[\Phi](t, x)| \le C I[z_0](t, |x|) \sup_{(\tau,y)\in D_i(t,|x|)} |y| z_0(\tau, |y|) |\Phi(\tau, y)|, \tag{8.3}$$

where

$$I[z_0](t, r) = r^{-1} \iint_{D_i(t,|x|)} z_0(\tau, \rho)^{-1}\, d\tau\, d\rho \tag{8.4}$$

(see p.613 of Yokoyama [17]). Therefore, it suffices to prove

$$I[z_0](t,r) \leq C\langle t+r\rangle^{-1}\langle t-r\rangle^{-\nu}. \tag{8.5}$$

If we set

$$\alpha = \rho + \tau, \quad \beta = \rho - a\tau,$$

the integral (8.4) can be written as

$$I[z_0](t,r) = \frac{1}{(a+1)r}\int_{|t-r|}^{t+r}(1+\alpha)^{-1-\nu}\,d\alpha \int_{\widehat{\beta}}^{\alpha}(1+|\beta|)^{-1-\mu}\,d\beta, \tag{8.6}$$

where

$$\widehat{\beta} = \frac{1}{2}\{(1-a)\alpha + (1+a)(r-t)\}. \tag{8.7}$$

Hence, noting that $(1+|\beta|)^{-1-\mu}$ is integrable on $\mathbb{R}$ for $\mu > 0$, we get

$$\begin{aligned} I[z_0](t,r) &\leq Cr^{-1}\int_{|t-r|}^{t+r}(1+\alpha)^{-1-\nu}\,d\alpha \\ &\leq Cr^{-1}\left\{(1+|t-r|)^{-\nu} - (1+t+r)^{-\nu}\right\}. \end{aligned} \tag{8.8}$$

Thus if $t+1 \leq 2r$, we obtain (8.5) immediately. If $t+1 > 2r$ to the contrary,

$$\begin{aligned} \left|(1+|t-r|)^{-\nu} - (1+t+r)^{-\nu}\right| &\leq C(1+|t-r|)^{-\nu-1}(t+r-|t-r|) \\ &\leq C\langle t+r\rangle^{-\nu-1}\min\{t,r\}. \end{aligned}$$

Therefore (8.8) implies (8.5). This completes the proof of Lemma 8.1. □

We next consider estimates for derivatives.

Lemma 8.2. *Let $a \geq 0$, $\mu > 0$, and $\nu > 0$. We further assume $\nu > 1$ if $a = c_i$. Then we have*

$$\begin{aligned} &\langle|x|\rangle\langle c_i t - |x|\rangle^{\nu}|U_i[\partial\Phi](t,x)| \\ &\leq C \sup_{(\tau,y)\in D_i(t,|x|)} \langle|y|\rangle\langle\tau+|y|\rangle^{\nu}\langle a\tau - |y|\rangle^{1+\mu}\{|\Phi(\tau,y)| \\ &\qquad\qquad + |\partial\Phi(\tau,y)| + |\Omega\Phi(\tau,y)|\} \end{aligned} \tag{8.9}$$

for $i = 1,\dots,m$, where $D_i(t,r)$ are defined by (3.5).

As before, it suffices to prove Lemma 8.2 for the case where $c_i = 1$, which is always assumed in what follows. Set

$$z(\tau,\rho) = (1+\tau+\rho)^{\nu}(1+|a\tau-\rho|)^{1+\mu}. \tag{8.10}$$

We begin with the following estimate, which is an immediate consequence of (3.25)–(3.29) and (3.39)–(3.40) of Yokoyama [17]:

$$|U_i[\partial\Phi](t,x)| \le C J[z](t,|x|) \sup_{(\tau,y)\in D_i(t,|x|)} |y| z(\tau,|y|)\{|\Phi(\tau,y)| + |\partial\Phi(\tau,y)| + |\Omega\Phi(\tau,y)|\}, \tag{8.11}$$

where

$$J[z](t,r) = r^{-1}\left[\iint_{D_I} z(\tau,\rho)^{-1}\,d\tau\,d\rho + \int_{\partial D_{II}} z(\tau,\rho)^{-1}\,d\sigma + \iint_{D_{II}} \left\{\rho^{-1} + \xi(t,r,\tau,\rho)\right\} z(\tau,\rho)^{-1}\,d\tau\,d\rho\right], \tag{8.12}$$

$$\xi(t,r,\tau,\rho) = \begin{cases} \dfrac{1}{\sqrt{\rho^2-\rho_-^2}} + \dfrac{1}{\sqrt{(\rho_+-\rho)(\rho-\rho_-)}} & (\rho_- \ge 0), \\[2ex] \dfrac{1}{\sqrt{\rho^2-\rho_-^2}} + \dfrac{1}{\sqrt{\rho_+^2-\rho^2}} & (\rho_- < 0), \end{cases} \tag{8.13}$$

$$\rho_- = t-\tau-r, \qquad \rho_+ = t-\tau+r, \tag{8.14}$$

$$D_I = \{(\tau,\rho) \mid 0<\tau<t,\ |\rho_-|<\rho<|\rho_-|+1,\ \rho<\rho_+\} \cup \{(\tau,\rho) \mid 0<\tau<t,\ \rho_+-1<\rho<\rho_+,\ |\rho_-|<\rho\}, \tag{8.15}$$

$$D_{II} = \{(\tau,\rho) \mid 0<\tau<t,\ |\rho_-|+1<\rho<\rho_+-1\}. \tag{8.16}$$

Now we find that all we have to do is to estimate $J[z](t,r)$. For this purpose, we prove a series of lemmas. The proof of Lemma 8.2 is clear from Lemmas 8.3–8.5 below.

Lemma 8.3. *Let* $a \ge 0$, $\mu > 0$, *and* $\nu > 0$. *Suppose* $\min\{t,r\} \le 1$. *Then*

$$J[z](t,r) \le C\langle r\rangle^{-1}\langle t-r\rangle^{-\nu}. \tag{8.17}$$

Proof. The assumption $\min\{t,r\}\le 1$ implies $D_{II} = \emptyset$, because

$$(\rho_+-1) - (|\rho_-|+1) = t-\tau+r-|t-\tau-r|-2 = 2(\min\{t-\tau,r\}-1) \le 0.$$

Hence we have

$$J[z](t,r) = r^{-1}\iint_{D_I} (1+\tau+\rho)^{-\nu}(1+|a\tau-\rho|)^{-1-\mu}\,d\tau\,d\rho.$$

Therefore, going the same way as in the proof of Lemma 8.1 with $\nu+1$ replaced by ν, we reach at

$$\begin{aligned}J[z](t,r) &\le Cr^{-1}\int_{|t-r|}^{t+r}(1+\alpha)^{-\nu}\,d\alpha\\&\le Cr^{-1}(1+|t-r|)^{-\nu}\int_{|t-r|}^{t+r}d\alpha\\&\le C(1+|t-r|)^{-\nu}r^{-1}\min\{t,r\}\le C\langle r\rangle^{-1}\langle t-r\rangle^{-\nu}.\end{aligned}$$

This completes the proof. □

It remains to prove the case where $r>1$ and $t>1$. In view of (8.12), we see that it suffices to prove

$$\iint_{D_I} z(\tau,\rho)^{-1}\,d\tau\,d\rho+\int_{\partial D_{II}} z(\tau,\rho)^{-1}\,d\sigma\le C\langle t-r\rangle^{-\nu}, \tag{8.18}$$

$$\iint_{D_{II}} (\rho^{-1}+\xi)z(\tau,\rho)^{-1}\,d\tau\,d\rho\le C\langle t-r\rangle^{-\nu}. \tag{8.19}$$

Lemma 8.4. *Let $a\ge 0$, $\mu>0$, and $\nu>0$. Furthermore we assume $\nu>1$ when $a=1$. Then we have*

$$\iint_{D_I} z(\tau,\rho)^{-1}\,d\tau\,d\rho+\int_{\partial D_{II}} z(\tau,\rho)^{-1}\,d\sigma\le C\langle t-r\rangle^{-\nu}. \tag{8.20}$$

Proof. We first note that

$$\iint_{D_I} z(\tau,\rho)^{-1}\,d\tau\,d\rho+\int_{\partial D_{II}} z(\tau,\rho)^{-1}\,d\sigma\le C\int_{\partial D(t,r)} z(\tau,\rho)^{-1}\,d\sigma, \tag{8.21}$$

because if $(\tau,\rho)\in\overline{D_I}$, $z(\tau,\rho)^{-1}$ is dominated by $Cz(\tau,|\rho_-|)^{-1}$ for $\rho\le r$, and by $Cz(\tau,\rho_+)$ for $\rho\ge r$. In order to estimate the right-hand side of (8.21), we divide $\partial D(t,r)$ into $\{\rho=|\rho_-|\}$, $\{\rho=\rho_+\}$ and $\{\tau=0\}$. Here we show the estimate of the integral on $\{\rho=|\rho_-|\}$ in particular, and omit the estimates of integrals on the other two regions, since they are easy to handle. The integral on $\{\rho=|\rho_-|\}$ is split as follows:

$$\int_0^t z(\tau,|\rho_-|)^{-1}\,d\tau=\int_0^{(t-r)_+} z(\tau,|\rho_-|)^{-1}\,d\tau+\int_{(t-r)_+}^t z(\tau,|\rho_-|)^{-1}\,d\tau,$$

where $(t-r)_+=\max\{t-r,0\}$.

(i) Let $t>r$, and $0<\tau<t-r$. Since $|\rho_-|=\rho_-=t-\tau-r$, we have

$$\int_0^{(t-r)_+} z(\tau,|\rho_-|)^{-1}\,d\tau\;=\int_0^{t-r}(1+\tau+\rho_-)^{-\nu}(1+|a\tau-\rho_-|)^{-1-\mu}\,d\tau$$

$$= \int_0^{t-r} (1 + |t - r|)^{-\nu} (1 + |(a+1)\tau - t + r|)^{-1-\mu}\, d\tau$$
$$\le C\langle t - r\rangle^{-\nu}.$$

(ii) Let $(t-r)_+ < \tau < t$ next. Since $|\rho_-| = -\rho_- = \tau - t + r$, we have

$$\begin{aligned}
&\int_{(t-r)_+}^{t} z(\tau, |\rho_-|)^{-1}\, d\tau \\
&= \int_{(t-r)_+}^{t} (1 - t + r + 2\tau)^{-\nu} (1 + |(a-1)\tau + t - r|)^{-1-\mu}\, d\tau \\
&=: j_0(t, r).
\end{aligned}$$

We observe that we can calculate $j_0(t, r)$ directly for $a = 1$. Since $\nu > 1$ in this case, it holds

$$\begin{aligned}
j_0(t, r) &= (1 + |t - r|)^{-1-\mu} \int_{(t-r)_+}^{t} (1 - t + r + 2\tau)^{-\nu}\, d\tau \\
&\le C(1 + |t - r|)^{-1-\mu} (1 + |t - r|)^{1-\nu} \\
&\le C\langle t - r\rangle^{-\nu}.
\end{aligned}$$

If $a \neq 1$ to the contrary, we get

$$\begin{aligned}
j_0(t, r) &\le (1 + |t - r|)^{-\nu} \int_{(t-r)_+}^{t} (1 + |(a-1)\tau + t - r|)^{-1-\mu}\, d\tau \\
&\le C\langle t - r\rangle^{-\nu}.
\end{aligned}$$

This completes the proof. □

Now we turn our attention to (8.19) whose proof is rather complicated. Firstly, (8.13) and (8.16) yield

$$\iint_{D_{II}} (\rho^{-1} + \xi) z(\tau, \rho)^{-1}\, d\tau\, d\rho \le C\{j_1(t, r) + j_2(t, r)\}, \tag{8.22}$$

where

$$\begin{aligned}
j_1(t, r) = \iint_{D_{II}^1} & z(\tau, \rho)^{-1} (1 + \rho - \rho_-)^{-1/2} \\
&\times \left\{(1 + \rho)^{-1/2} + (1 + \rho_+ - \rho)^{-1/2}\right\} d\tau\, d\rho,
\end{aligned} \tag{8.23}$$

$$\begin{aligned}
j_2(t, r) = \iint_{D_{II}^2} & z(\tau, \rho)^{-1} (1 + \rho)^{-1/2} \\
&\times \left\{(1 + \rho + \rho_-)^{-1/2} + (1 + \rho_+ - \rho)^{-1/2}\right\} d\tau\, d\rho.
\end{aligned} \tag{8.24}$$

Here we set

$$D_{II}^1 = D_{II} \cap \{\rho_- \geq 0\}, \quad D_{II}^2 = D_{II} \cap \{\rho_- < 0\}. \tag{8.25}$$

To evaluate the integrals (8.23) and (8.24), we introduce new variables of integration (α, β) by

$$\rho + \tau = \alpha, \quad \rho - a\tau = \beta. \tag{8.26}$$

We can easily check the following relations:

$$\begin{gathered}\tau = \frac{\alpha - \beta}{1+a}, \quad \rho = \frac{a\alpha + \beta}{1+a}, \\ \rho - \rho_- = r - t + \alpha, \quad \rho_+ - \rho = t + r - \alpha, \quad \rho + \rho_- = \frac{2\left(\beta - \widehat{\beta}\right)}{1+a},\end{gathered} \tag{8.27}$$

where $\widehat{\beta} = \{(1-a)\alpha + (1+a)(r-t)\}/2$ as in (8.7). Applying the transformation (8.26) to (8.23) and (8.24), we obtain the estimates

$$j_1(t,r) \leq C\{j_{11}(t,r) + j_{12}(t,r)\}, \tag{8.28}$$

$$j_2(t,r) \leq C\{j_{21}(t,r) + j_{22}(t,r)\}, \tag{8.29}$$

where

$$\begin{aligned} j_{11}(t,r) = &\int_{|t-r|}^{t+r} (1+\alpha)^{-\nu}(1+r-t+\alpha)^{-1/2}\,d\alpha \\ &\times \int_{\widehat{\beta}}^{\alpha} (1+|\beta|)^{-1-\mu}(1+a\alpha+\beta)^{-1/2}\,d\beta, \end{aligned} \tag{8.30}$$

$$\begin{aligned} j_{12}(t,r) = &\int_{|t-r|}^{t+r} (1+\alpha)^{-\nu}(1+r-t+\alpha)^{-1/2}(1+t+r-\alpha)^{-1/2}\,d\alpha \\ &\times \int_{\widehat{\beta}}^{\alpha} (1+|\beta|)^{-1-\mu}\,d\beta, \end{aligned} \tag{8.31}$$

$$\begin{aligned} j_{21}(t,r) = &\int_{|t-r|}^{t+r} (1+\alpha)^{-\nu}\,d\alpha \\ &\times \int_{\widehat{\beta}}^{\alpha} (1+|\beta|)^{-1-\mu}(1+a\alpha+\beta)^{-1/2}\left(1+\beta-\widehat{\beta}\right)^{-1/2}d\beta, \end{aligned} \tag{8.32}$$

$$\begin{aligned} j_{22}(t,r) = &\int_{|t-r|}^{t+r} (1+\alpha)^{-\nu}(1+t+r-\alpha)^{-1/2}\,d\alpha \\ &\times \int_{\widehat{\beta}}^{\alpha} (1+|\beta|)^{-1-\mu}(1+a\alpha+\beta)^{-1/2}\,d\beta. \end{aligned} \tag{8.33}$$

Lemma 8.5. *Let $a \geq 0$, $\mu > 0$ and $\nu > 0$. Moreover we assume $\nu > 1$ if $a = 1$. Then* (8.19) *is true.*

Proof. As we have observed in the above, it suffices to prove

$$j_{kl}(t,r) \le C\langle t-r\rangle^{-\nu} \quad (k,l=1,2). \tag{8.34}$$

Before we proceed to estimates for each j_{kl}, we give two basic inequalities which will be used repeatedly. □

Lemma 8.6. (i) *Let $|p| \le q$, and $\nu_1, \nu_2 \ge 1/2$. Then we have*

$$K^{\nu_1,\nu_2}_{p,q} := \int_{|p|}^{q} (1+p+\alpha)^{-\nu_1}(1+q-\alpha)^{-\nu_2}\,d\alpha \le 4. \tag{8.35}$$

(ii) *Let $0<|\lambda|\le|\mu|$, and $\nu>0$. Then we have*

$$L^{\nu}_{p,\lambda,\mu} := \int_{|p|}^{\infty} (1+\alpha)^{-1/2-\nu}(1+|\lambda\alpha+\mu p|)^{-1/2}\,d\alpha \le C\langle p\rangle^{-\nu}, \tag{8.36}$$

where C is a constant independent of p.

Proof. (i) We have $K^{\nu_1,\nu_2}_{p,q} \le \int_{|p|}^{q}(1-|p|+\alpha)^{-1/2}(1+q-\alpha)^{-1/2}\,d\alpha$ for $\nu_1,\nu_2 \ge 1/2$. Since $1+q-\alpha \ge 1+(q-|p|)/2$ for $|p| \le \alpha \le (|p|+q)/2$, we get

$$\begin{aligned}
&\int_{|p|}^{(|p|+q)/2} (1-|p|+\alpha)^{-1/2}(1+q-\alpha)^{-1/2}\,d\alpha \\
&\le \left(1+\frac{q-|p|}{2}\right)^{-1/2} \int_{|p|}^{(|p|+q)/2} (1-|p|+\alpha)^{-1/2}\,d\alpha \le 2.
\end{aligned}$$

On the other hand, since we have $1-|p|+\alpha \ge 1+(q-|p|)/2$ for $(|p|+q)/2 \le \alpha \le q$, we can estimate the integral on $[(|p|+q)/2, q]$ similarly.

(ii) For $\alpha \ge 2|\mu p|/|\lambda|$, we have $|\lambda\alpha+\mu p| \ge |\lambda|\alpha/2$. Therefore we get

$$\int_{2|\mu p|/|\lambda|}^{\infty} (1+\alpha)^{-1/2-\nu}(1+|\lambda\alpha+\mu p|)^{-1/2}\,d\alpha \le C\int_{2|\mu p|/|\lambda|}^{\infty} (1+\alpha)^{-\nu-1}\,d\alpha \le C\langle p\rangle^{-\nu}.$$

On the other hand, we have

$$\begin{aligned}
&\int_{|p|}^{2|\mu p|/|\lambda|} (1+\alpha)^{-1/2-\nu}(1+|\lambda\alpha+\mu p|)^{-1/2}\,d\alpha \\
&\le C\langle p\rangle^{-(1/2)-\nu}\int_{|p|}^{2|\mu p|/|\lambda|} (1+|\lambda\alpha+\mu p|)^{-1/2}\,d\alpha \\
&\le C\langle p\rangle^{-\nu}.
\end{aligned}$$

This completes the proof of Lemma 8.6. □

Now we resume the proof of Lemma 8.5.

1. *Estimate of* $j_{12}(t,r)$

By Lemma 8.6 (i), we obtain

$$\begin{aligned} j_{12}(t,r) &\leq C\int_{|t-r|}^{t+r}(1+\alpha)^{-\nu}(1+r-t+\alpha)^{-1/2}(1+t+r-\alpha)^{-1/2}\,d\alpha \\ &\leq C(1+|t-r|)^{-\nu}K^{1/2,1/2}_{r-t,t+r} \leq C\langle t-r\rangle^{-\nu}. \end{aligned}$$

2. *Estimates of* $j_{11}(t,r)$ *and* $j_{22}(t,r)$

We next consider the estimates of $j_{11}(t,r)$ and $j_{22}(t,r)$. We use the following lemma.

Lemma 8.7.

$$\begin{aligned} &\int_{\widehat{\beta}}^{\alpha}(1+|\beta|)^{-1-\mu}(1+a\alpha+\beta)^{-1/2}\,d\beta \\ &\leq \begin{cases} C(1+r-t+\alpha)^{-1/2-\mu} & (a=0), \\ C(1+\alpha)^{-1/2} & (a>0). \end{cases} \end{aligned} \tag{8.37}$$

Proof. Let us first consider the case $a=0$. Since $\widehat{\beta}=(\alpha+r-t)/2\geq 0$ if $a=0$, the integrand is equal to $(1+\beta)^{-3/2-\mu}$. Therefore, we can calculate the integral directly.

Let us assume $a>0$ next. Since $\widehat{\beta}\geq -a\alpha$, the left-hand side of (8.37) is bounded by

$$\int_{-a\alpha}^{-a\alpha/2}(1+|\beta|)^{-1-\mu}(1+a\alpha+\beta)^{-1/2}\,d\beta+\int_{-a\alpha/2}^{\alpha}(1+|\beta|)^{-1-\mu}(1+a\alpha+\beta)^{-1/2}\,d\beta.$$

Since $1+a\alpha+\beta\geq C(1+\alpha)$ for $-a\alpha/2\leq\beta\leq\alpha$, we obtain

$$\begin{aligned} \int_{-a\alpha/2}^{\alpha}(1+|\beta|)^{-1-\mu}(1+a\alpha+\beta)^{-1/2}\,d\beta &\leq C(1+\alpha)^{-1/2}\int_{-a\alpha/2}^{\alpha}(1+|\beta|)^{-1-\mu}\,d\beta \\ &\leq C(1+\alpha)^{-1/2}. \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} \int_{-a\alpha}^{-a\alpha/2}(1+|\beta|)^{-1-\mu}(1+a\alpha+\beta)^{-1/2}\,d\beta &\leq C(1+\alpha)^{-1-\mu}\int_{-a\alpha}^{-a\alpha/2}(1+a\alpha+\beta)^{-1/2}\,d\beta \\ &\leq C(1+\alpha)^{-1/2-\mu}. \end{aligned}$$

This completes the proof. □

Let us begin the estimates of $j_{11}(t,r)$ and $j_{22}(t,r)$. Suppose $a = 0$ first. Then Lemmas 8.7 and 8.6 (i), together with $1+\alpha \geq 1+|t-r|$, yield

$$\langle t-r\rangle^{\nu}\,(j_{11}(t,r)+j_{22}(t,r)) \leq C\int_{|t-r|}^{t+r}(1+r-t+\alpha)^{-1-\mu}\,d\alpha + C K_{r-t,t+r}^{1/2+\mu,1/2} \leq C.$$

Thus we obtain (8.34) for $(k,l) = (1,1)$ and $(2,2)$ when $a = 0$.

Assume $a > 0$ next. Then, using Lemma 8.7, we obtain

$$j_{11}(t,r)+j_{22}(t,r) \leq C L_{r-t,1,1}^{\nu} + C\int_{|t-r|}^{t+r}(1+\alpha)^{-1/2-\nu}(1+t+r-\alpha)^{-1/2}\,d\alpha.$$

We can estimate the first term on the right-hand side by using Lemma 8.6 (ii). Since $(1+\alpha)^{-\nu} \leq C\langle t-r\rangle^{-\nu}$, the second term is bounded by $C\langle t-r\rangle^{-\nu}K_{0,t+r}^{1/2,1/2}$. Thus we get (8.34) for $(k,l) = (1,1)$ and $(2,2)$ also when $a > 0$.

3. *Estimate of* $j_{21}(t,r)$

Lemma 8.8. *For* $\alpha \geq |r-t|$, *we have*

$$\begin{aligned} &\int_{\widehat{\beta}}^{\alpha}(1+|\beta|)^{-1-\mu}(1+a\alpha+\beta)^{-1/2}\left(1+\beta-\widehat{\beta}\right)^{-1/2}\,d\beta \\ &\leq C\left\{\left(1+\left|\widehat{\beta}\right|\right)^{-1-\mu/2} + (1+\alpha)^{-1/2}\left(1+\left|\widehat{\beta}\right|\right)^{-1/2}\right\}. \end{aligned} \tag{8.38}$$

Proof. Assume $\widehat{\beta} \geq 0$ first. Then we estimate $(1+|\beta|)^{-1-\mu}$ in the integrand as $(1+|\beta|)^{-1-\mu} \leq \left(1+\left|\widehat{\beta}\right|\right)^{-1-\mu/2}(1+|\beta|)^{-\mu/2}$. We also have $1+a\alpha+\beta \geq 1+\beta-\widehat{\beta}$ from the definition of $\widehat{\beta}$ (even if $\widehat{\beta} < 0$). Hence the left-hand side of (8.38) is bounded by

$$C\left(1+\left|\widehat{\beta}\right|\right)^{-1-\mu/2}\int_{\widehat{\beta}}^{\infty}(1+|\beta|)^{-\mu/2}\left(1+\beta-\widehat{\beta}\right)^{-1}d\beta \leq C\left(1+\left|\widehat{\beta}\right|\right)^{-1-\mu/2}.$$

Suppose $\widehat{\beta} < 0$ next. Note that $a > 0$ in this case. We divide the interval $\left(\widehat{\beta},\alpha\right]$ as $\left(\widehat{\beta},\alpha\right] = I_1 \cup I_2$, where $I_1 = \left(\widehat{\beta},\widehat{\beta}/2\right]$, and $I_2 = \left(\widehat{\beta}/2,\alpha\right]$. The estimate on the interval I_1 proceeds as above, because we still have $(1+|\beta|)^{-1-\mu} \leq \left(1+\left|\widehat{\beta}\right|\right)^{-1-\mu/2}(1+|\beta|)^{-\mu/2}$ and $1+a\alpha+\beta \geq 1+\beta-\widehat{\beta}$. Thus we see that the integral on I_1 is bounded by $C\left(1+\widehat{\beta}\right)^{-1-\mu/2}$. As for the integral on I_2, we note that $-a\alpha/2 \leq \widehat{\beta}/2$. Since $(1+a\alpha+\beta)^{-1/2}\left(1+\beta-\widehat{\beta}\right)^{-1/2} \leq C(1+\alpha)^{-1/2}\left(1+\left|\widehat{\beta}\right|\right)^{-1/2}$, we get

$$\begin{aligned} &\int_{I_2}(1+|\beta|)^{-1-\mu}(1+a\alpha+\beta)^{-1/2}\left(1+\beta-\widehat{\beta}\right)^{-1/2}d\beta \\ &\leq C(1+\alpha)^{-1/2}\left(1+\left|\widehat{\beta}\right|\right)^{-1/2}. \end{aligned} \tag{8.39}$$

This completes the proof. □

Now we begin the estimate of $j_{21}(t,r)$. By Lemma 8.8, we have

$$
\begin{aligned}
j_{21}(t,r) &\le C\int_{|t-r|}^{t+r}(1+\alpha)^{-\nu}\left(1+\left|\widehat{\beta}\right|\right)^{-1-\mu/2}d\alpha \\
&\quad + C\int_{|t-r|}^{t+r}(1+\alpha)^{-1/2-\nu}\left(1+\left|\widehat{\beta}\right|\right)^{-1/2}d\alpha \qquad (8.40)\\
&=: k_1(t,r)+k_2(t,r).
\end{aligned}
$$

Suppose $a\ne 1$ first. It is easy to see

$$
k_1(t,r)\le C(1+|t-r|)^{-\nu}\int_{|t-r|}^{t+r}\left(1+\left|\widehat{\beta}\right|\right)^{-1-\mu/2}d\alpha\le C\langle t-r\rangle^{-\nu}.
$$

On the other hand, by the definition of $\widehat{\beta}$, Lemma 8.6 (ii) leads to

$$
k_2(t,r)\le L^{\nu}_{r-t,(1-a)/2,(1+a)/2}\le C\langle t-r\rangle^{-\nu}.
$$

We next assume $a=1$. Remember that $\nu>1$ by the assumption. Since $\widehat{\beta}=|t-r|$ in this case, we can calculate $k_1(t,r)$, $k_2(t,r)$ directly. Thus we obtain (8.34). This completes the proof of Lemma 8.5. □

Finally we are in a position to prove Lemma 3.2.

Proof of Lemma 3.2. We may assume $c_1<c_2<\cdots<c_m$. Set $c_0=0$ and

$$
\begin{aligned}
d_j&=c_j-\frac{c_j-c_{j-1}}{3},\quad \widetilde{d}_j=c_j-\frac{2(c_j-c_{j-1})}{3}\quad (j=1,\dots,m),\\
e_j&=c_j+\frac{c_{j+1}-c_j}{3},\quad \widetilde{e}_j=c_j+\frac{2(c_{j+1}-c_j)}{3}\quad (j=0,\dots,m-1).
\end{aligned}
$$

We put $I_j=[(e_j)^{-1},(d_j)^{-1}]$ and $\widetilde{I}_j=\left[\left(\widetilde{e}_j\right)^{-1},\left(\widetilde{d}_j\right)^{-1}\right]$ for $1\le j\le m-1$. We also set $I_0=[(e_0)^{-1},\infty)$, $\widetilde{I}_0=\left[\left(\widetilde{e}_0\right)^{-1},\infty\right)$, $I_m=[0,(d_m)^{-1}]$, and $\widetilde{I}_m=\left[0,\left(\widetilde{d}_m\right)^{-1}\right]$.

We take a smooth cut-off function χ on $[0,\infty)$ such that $\chi=0$ on $[0,1]$, and $\chi=1$ on $[2,\infty)$. We also take smooth functions χ_j $(j=0,\dots,m)$ so that each χ_j is supported on $\widetilde{I}_j$, $\chi_j=1$ on I_j, and $\sum_{j=0}^m\chi_j=1$ on $[0,\infty)$. We define

$$
\begin{aligned}
\zeta_0(t,x)&=\{1-\chi(|x|)\}+\chi(|x|)\chi_0\left(\frac{t}{|x|}\right),\\
\zeta_j(t,x)&=\chi(|x|)\chi_j\left(\frac{t}{|x|}\right)\quad (j=1,\dots,m)
\end{aligned}
$$

for $(t,x)\in[0,\infty)\times\mathbb{R}^3$. Then, noting that $\sum_{j=0}^m\zeta_j(t,x)=1$ for $[0,\infty)\times\mathbb{R}^3$, we have

$$
\langle t+|x|\rangle\langle c_it-|x|\rangle^{\nu}|U_i[\Phi](t,x)|\le\sum_{j=0}^m\langle t+|x|\rangle\langle c_it-|x|\rangle^{\nu}|U_i[\zeta_j\Phi](t,x)|.
$$

Since $\langle c_j t - |x|\rangle \le Cw(t, |x|)$ in supp ζ_j, Lemma 8.1 implies

$$\langle t + |x|\rangle\langle c_i t - |x|\rangle^{\nu}|U_i[\zeta_j\Phi]| \le C \sup_{y\in D_i(t,|x|)} \langle|y|\rangle\langle\tau + |y|\rangle^{1+\nu} w(\tau, |y|)^{1+\mu}|\Phi(\tau, y)|$$

for $\nu > 0$. This proves (3.6). The proof of (3.7) is similar. We can also prove (3.8) similarly, because $\langle t + r\rangle \le Cw_i(t, r)$ in supp ζ_i and $\langle c_j t - r\rangle \le Cw_i(t, r)$ in supp ζ_j for $j \neq i$. $\square$

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