ON THE CHARACTERS OF THE PROLONGED DIFFERENTIAL SYSTEM

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§ 1. Introduction

Let $M = M(D, \pi)$ be a (real analytic) fibered manifold of dimension n over a manifold D of dimension p, with projection π. We denote by $M'(D, \alpha)$ the prolonged fibered manifold of $M(D, \pi)$. Every point of M' is a p-dimensional contact element of M, and a p-dimensional contact element X of M belongs to M' if and only if $\pi_* X = T_{\pi(x)}(D)$, where x is the origin of X. We write $x = \beta(X)$ and $\alpha = \pi \circ \beta$. We also denote by $M''(D, \alpha')$ the prolonged fibered manifold of $M'(D, \alpha)$, where $\alpha' = \alpha \circ \beta'$ and β' is the projection of each $X' \in M''$ to its origin in M'.

Let $\mathfrak{A}$ be a (homogeneous and d-closed real analytic) differential system on the fibered manifold $M(D, \pi)$. Then, on the fibered manifold $M'(D, \alpha)$, the prolonged differential system of $\mathfrak{A}$ is well defined. We denote it by $\mathfrak{A}'$. Let $X' \in M''$ and $X \in M'$ be integral elements of $\mathfrak{A}'$ and $\mathfrak{A}$ respectively. We denote their characters by $s'_k(X')$ and $s_k(X)$ $k = 0, 1, \ldots, p$. The following theorem was first proved by E. Cartan [1] in the special case and later by Matsushima [3] in the general case.

THEOREM. *If X' is an integral element of $\mathfrak{A}'$ such that $X = \beta'(X')$ is an ordinary integral element of $\mathfrak{A}$, then X' is also an ordinary integral element of $\mathfrak{A}'$ and the characters satisfy the following relations*

$$(1) \qquad s'_k(X') = \sum_{j=k}^{p} s_j(X), \qquad k = 1, 2, \ldots, p-1.$$

In this note we investigate the case where it is not assumed that X is ordinary. In this case we have the inequalities (2) instead of (1). In fact, we can prove the following theorem.

THEOREM. *Let X' be an arbitrary integral element of $\mathfrak{A}'$ and let X be its origin.*

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Then the following inequaities hold:

$$s'_k(X') \leqq \sum_{j=k}^{p} s_j(X), \qquad k=0, 1, \ldots, p. \tag{2}$$

Further, if the sets of 0*-forms of* $\mathfrak{A}'$ *and* $\mathfrak{A}$ *are regular at* X *and* $x=\beta(X)$ *respectively, then the element* X *is an ordinary integral element of* $\mathfrak{A}$ *if and only if* (1) *hold for all* $k=0, 1, \ldots, p$.

In § 2, we state some definitions and lemmas and in § 3 give a proof of our theorem.

§ 2. Definitions and Lemmas

Let $X \in M'$ be an integral element of $\mathfrak{A}$, and E^k be a k-dimensional element in X. We denote by $t(E^k)$ the rank of the polar system for E^k, and by $t_k(X)$ the maximum value of $t(E^k)$ for arbitrary $E^k \subset X$. Then, as is well known, there is a flag $\beta(X)=E^0 \subset E^1 \subset \cdots \subset E^{p-1} \subset X$, such that $t_k(X)=t(E^k)$ for $k=0, 1, \ldots, p-1$. Such a flag $\{E^k\}$ is called a non-singular flag in X. Let $x=\beta(X)$, $\bar{x}=\pi(x)$ and $\pi_* E^k=\bar{E}^k$. Then we have a flag $\{\bar{E}^k\}$ in $T_{\bar{x}}(D)$. Conversely, for any flag $\{\bar{E}^k\}$ in $T_{\bar{x}}(D)$, there is a unique flag $\{E^k\}$ in the given X, which is over the flag $\{\bar{E}^k\}$. An ordered coordinate system $(x^1, \ldots, x^p)$ at $\bar{x}$ in D is called non-singular for X, if and only if the flag in X, which is over the flag $\{(\bar{\partial}_1, \ldots, \bar{\partial}_k)\}$, is non-singular (We denote by $(\bar{\partial}_1, \ldots, \bar{\partial}_k)$ the element spanned by vectors $\bar{\partial}_j=(\partial/\partial x^j)_{\bar{x}}$, $j=1, \ldots, k$).

Let (x^i) be a non-singular coordinate system at $\bar{x}=\alpha(X)$ for an integral element X, such that $x^i(\bar{x})=0$. Then it is easily seen that there is an open and dense subset, say G, of $GL(p)$, such that any coordinate system (y^i) at $\bar{x}$, defined by $y^i=\sum_j \alpha^i_j x^i$, is non-singular for X if $(\alpha^i_j) \in G$. By this remark, we can easily verify the following lemma.

LEMMA 1. *Let* $M_1(D, \pi_1)$ *and* $M_2(D, \pi_2)$ *be fibered manifolds over the same manifold* D, *and let* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *be differential systems on* $M_1(D, \pi_1)$ *and* M_2 (D, π_2) *respectively. If* X_1 *and* X_2 *are integral elements of* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *respectively such that their origins are over the same point* $\bar{x}$ *of* D, *then there exists a coordinate system* (x^i) *at* $\bar{x}$, *which is non-singular for both* X_1 *and* X_2.

For any integral element X of $\mathfrak{A}$, we denote by $t_{-1}(X)$ the rank of $d\mathfrak{A}^{(0)}_{\beta(X)}$, where $\mathfrak{A}^{(0)}$ is the set of 0-forms of $\mathfrak{A}$. We define the characters s_k $(k=0,$

$1, \ldots, p)$ by $s_k(X) = t_k(X) - t_{k-1}(X)$, for $k = 0, \ldots, p-1$, and by $s_p(X) = n - p - t_{p-1}(X)$. Our s_k are the same as the ones in [3] except s_0.

Let $\{f\}$ be a set of (real analytic) functions defined on a neighborhood of a point a in a manifold and $f(a) = 0$ for every $f \in \{f\}$. We say that $\{f\}$ is regular at the point a, if there exist a subset $\{f_1, \ldots, f_m\}$ of $\{f\}$ and a neighborhood V of a, such that (i) $(df_1)_a, \ldots, (df_m)_a$ are linearly independent, and (ii) if $p \in V$ and $f_i(p) = 0$, $i = 1, \ldots, m$, then $f(p) = 0$ for any $f \in \{f\}$.

An integral element X of the differential system $\mathfrak{A}$ on the fibered manifold $M(D, \pi)$ is called ordinary if and only if the following conditions (i) and (ii) are satisfied:

(i) $\mathfrak{A}^{(0)}$ and $\mathfrak{A}'^{(0)}$ are regular at $\beta(X)$ and X respectively,

(ii) the dimension of the submanifold of M', defined by the set of equations $\mathfrak{A}'^{(0)} = 0$ in a neighborhood of X, is $n' - t_{-1}(X) + \sum_{k=1}^{p} k s_k(X)$, where $n' = \dim M' = n + p(n-p)$.

For later use, we state the following

LEMMA 2. *Let V be a finite dimensional vector space and $V = V_0 + V_1 + \cdots + V_{p+1}$ be a direct sum decomposition of V. Denote by $i_{k_1, \ldots, k_q}$ the identity map of $V_{k_1} + \cdots + V_{k_q}$ into V and suppose that a family of sets of linear functionals $\{\Psi^{(k)} \mid k = 0, 1, \ldots, p\}$ on V is given and each set of linear functionals $\Psi^{(k)}$ satisfies the condition $i^*_{k+1, k+2, \ldots, p+1}\Psi^{(k)} = \{0\}$. Then there exists a set of non-negative integers $\{\delta_0, \delta_1, \ldots, \delta_{p-1}\}$, such that*

$$\operatorname{rank}\Big(i^*_{k, k+1, \ldots, p+1}\Big(\bigcup_{l=0}^{p} \Psi^{(l)}\Big)\Big) = \sum_{j=k-1}^{p-1} t_j + \sum_{j=k}^{p-1} \delta_j, \quad k = 0, 1, \ldots, p,$$

where $t_{k-1} = \operatorname{rank}(i^*_k \Psi^{(k)})$.

This lemma is easily verified by applying the following Lemma 3 repeatedly.

LEMMA 3. *Let $V = V_0 + V_1$ be a direct sum decomposition of a finite dimensional vector space V, and Ψ_0 and Ψ_1 be sets of linear functionals on V such that $i_1^*\Psi_0 = \{0\}$,* $\operatorname{rank}(i_0^*\Psi_0) = t_0$ *and* $\operatorname{rank}(i_1^*\Psi_1) = t_1$. *Then, for some $\delta \geqq 0$,* $\operatorname{rank}(\Psi_0 \cup \Psi_1) = t_0 + t_1 + \delta$.

This lemma is obvious.

§ 3. Proof of Theorem

Let X' be an integral element of $\mathfrak{A}'$, $\beta'(X') = X$ and $\beta(X) = x$. By Lemma

1, we can find a coordinate system (x^i) at $\pi(x)$ in D, which is non-singular for both X' and $\dot{X}$. Since $M(D, \pi)$ is a fibered manifold, there is a coordinate system $(x^i \circ \pi, x^\lambda)$, $i=1, \ldots, p$; $\lambda = p+1, \ldots, n$, defined on a neighborhood U of $x \in M$. We shall use the notation x^i instead of $x^i \circ \pi$, and adopt such conventions in the sequel. Let $U' = \beta^{-1}(U)$. The coordinate system (x^i, x^λ) on U introduces the canonical coordinate system $(x^i, x^\lambda, l_i^\lambda)$ on U'.

We write $\partial_i = \partial/\partial x^i$, $\partial_\lambda = \partial/\partial x^\lambda$, $\partial_\lambda^i = \partial/\partial l_i^\lambda$ and put $e_i = \partial_i + \sum_\lambda l_i^\lambda \partial_\lambda$, $e_\lambda = \partial_\lambda$. These are vector fields on U', and every $X \in U'$ is spanned by $\beta_*(e_i)_X$, $i=1, \ldots, p$.

We suppose that $\{\theta_\alpha^{(k)} \mid \alpha \in A_k,\ k=0, 1, \ldots, p\}$ is a local expression of $\mathfrak{A}$ on U where $\{\theta_\alpha^{(k)} \mid \alpha \in A_k\}$ is the set of k-forms and $\{d\theta_\alpha^{(k)} \mid \alpha \in A_k\} \subset \{\theta_\alpha^{(k+1)} \mid \alpha \in A_{k+1}\}$. ($A_k$ are sets of indices). For vector fields $e_{j_1}, \ldots, e_{j_k}$ $(1 \leqq j_1, \ldots, j_k \leqq n)$ defined above, $\langle \beta^* \theta_\alpha^{(k)}, e_{j_1} \wedge \cdots \wedge e_{j_k} \rangle$ is a function on U'. We denote it by $H_{\alpha j_1 \cdots j_k}^{(k)}$. (In particular, $H_\alpha^{(0)} = \beta^* \theta_\alpha^{(0)}$). We consider a set

$$\Theta = \{H_{\alpha i_1 \cdots i_k}^{(k)} \mid 1 \leqq i_1 < \cdots < i_k \leqq p,\ \alpha \in A_k,\ k=0, 1, \ldots, p\},$$

of functions on U' and a set of 1-forms $\varphi^\lambda = dx^\lambda - \sum_i l_i^\lambda dx^i$, $(\lambda = p+1, \ldots, n)$ on U'. Then

$$\Theta \cup d\Theta \cup \{\varphi^\lambda\} \cup \{d\varphi^\lambda\} \tag{3}$$

is a local expression of $\mathfrak{A}'$ on U'.

Now we write, for the given points X and $x = \beta(X)$, $e_j(x) = \beta_*(e_j)_X$ and $\varphi_x^\lambda = (dx^\lambda)_x - \sum_i l_i^\lambda(X)(dx^i)_x$. Then $(e_i(x), e_\lambda(x))$ is a basis of the vector space $T_x(M)$, and $((dx^i)_x, \varphi_x^\lambda)$ is the dual basis. Hence we have

$$\begin{aligned}(\theta_\alpha^{(l)})_x \llcorner (e_{i_1}(x) \wedge \cdots \wedge e_{i_{l-1}}(x)) &= \sum_i < (\theta_\alpha^{(l)})_x,\ e_{i_1}(x) \wedge \cdots \wedge e_{i_{l-1}}(x) \\ &\wedge e_i(x) > (dx^i)_x + \sum_\lambda < (\theta_\alpha^{(l)})_x,\ e_{i_1}(x) \wedge \cdots \wedge e_{i_{l-1}}(x) \wedge e_\lambda(x) > \varphi_x^\lambda \\ &= \sum_\lambda H_{\alpha i_1 \cdots i_{l-1} \lambda}^{(l)}(X)\, \varphi_x^\lambda.\end{aligned}$$

Therefore

$$\Phi^{(k-1)} = \{\sum H_{\alpha i_1 \cdots i_{l-1} \lambda}^{(l)}(X)\, \varphi_x^\lambda \mid 1 \leqq i_1 < \cdots < i_{l-1} \leqq k-1,\ \alpha \in A_l,\ l=1, \ldots, k\} \tag{4}$$

is the polar system for the $(k-1)$-dimensional element $(e_1(x), \ldots, e_{k-1}(x))$, for $k=1, \ldots, p$. In addition, putting

$$\Phi^{(-1)} = \{(d\theta_\alpha^{(0)})_x \mid \alpha \in A_0\}, \tag{5}$$

we have $\operatorname{rank}(\Phi^{(k-1)}) = t_{k-1}(X)$ for $k = 0, 1, \ldots, p$.

We now proceed to examine the characters $s'_k(X')$. We suppose that X' is spanned by $e'_1, \ldots, e'_p$, where

$$e'_i = (\partial_i)_X + \sum_\lambda t^\lambda_i (\partial_\lambda)_X + \sum_{\lambda j} t^\lambda_{ij} (\partial^i_\lambda)_X, \quad i = 1, \ldots, p.$$

Since X' is the integral element of $\mathfrak{A}'$ with the origin X, we have $t^\lambda_i = l^\lambda_i(X)$, $t^\lambda_{ij} = t^\lambda_{ji}$ and $e'_i(H) = 0$ for any $H \in \Theta$. If we set $\psi^\lambda = (dx^\lambda)_X - \sum_i t^\lambda_i (dx^i)_X$ and $\psi^\lambda_i = (dl^\lambda_i)_X - \sum_j t^\lambda_{ij}(dx^j)_X$, we can see that the polar system for the $(k-1)$-dimensional element $(e'_1, \ldots, e'_{k-1})$ is given as follows:

$$\Phi'^{(k-1)} = (d\Theta)_X \cup \{\psi^\lambda \mid \lambda = p+1, \ldots, n\} \cup \{\psi^\lambda_i \mid i = 1, \ldots, k-1; \quad \lambda = p+1, \ldots, n\}. \tag{6}$$

We have $(d\Theta)_X = \bigcup_{k=0}^{p} \Psi^{(k)}$ by setting

$$\Psi^{(0)} = \{(dH^{(0)}_\alpha)_X \mid \alpha \in A_0\} \tag{7}$$

and

$$\Psi^{(k)} = \{(dH^{(l)}_{\alpha i_1 \cdots i_{l-1} k})_X \mid 1 \leqq i_1 < \cdots < i_{l-1} \leqq k-1, \ \alpha \in A_l, \ l = 1, \ldots, k\}. \tag{8}$$

Since $((dx^i)_X, \psi^\lambda, \psi^\lambda_i)$ is the dual basis of $(e'_i, (\partial_\lambda)_X, (\partial^i_\lambda)_X)$ in $T_X(M')$, we have $(dH)_X = \sum_\lambda (\partial_\lambda)_X(H)\psi^\lambda + \sum_{\lambda i} (\partial^i_\lambda)_X(H)\psi^\lambda_i$ for any $H \in \Theta$.

Therefore we have

$$(dH^{(l)}_{\alpha i_1 \cdots i_{l-1} k})_X \equiv \sum_\lambda H^{(l)}_{\alpha i_1 \cdots i_{l-1} \lambda}(X)\, \psi^\lambda_k \qquad (\operatorname{mod} \psi^\lambda, \psi^\lambda_{i_1}, \ldots, \varphi^\lambda_{i_{l-1}}), \ l \geqq 1. \tag{9}$$

We consider the direct sum decomposition $T_X(M') = V_0 + V_1 + \cdots + V_{p+1}$, where $V_0 = ((\partial_{p+1})_X, \ldots, (\partial_n)_X)$, $V_k = ((\partial^k_{p+1})_X, \ldots, (\partial^k_n)_X)$, $k = 1, \ldots, p$, and $V_{p+1} = X'$. We can apply Lemma 2 to our vector space $T_X(M')$ and the family of sets of linear functionals $\{\Psi^{(k)}\}$ defined by (7) and (8). In fact, we can see $i^*_{k+1, \ldots, p+1}\Psi^{(k)} = \{0\}$ for $k = 0, 1, \ldots, p$, by (7), (8) and (9). Further we have, $\operatorname{rank}(i^*_k \Psi^{(k)}) = t_{k-1}(X)$ for $k = 1, \ldots, p$ by (4), (8), (9) and for $k = 0$ by (5), (7). Therefore, by Lemma 2, there exists a set of non-negative integers $\{\delta_0, \ldots, \delta_{p-1}\}$ such that

$$t'_{-1}(X') = \operatorname{rank}((d\Theta)_X) = \sum_{j=-1}^{p-1} t_j(X) + \sum_{j=0}^{p-1} \delta_j,$$

$$t_{k-1}(X') = \operatorname{rank}(\Phi'^{(k-1)}) = k(n-p) + \operatorname{rank}(i^*_{k, \ldots, p+1}(d\Theta)_X)$$

$$= k(n-p) + \sum_{j=k-1}^{p-1} t_j(X) + \sum_{j=k}^{p-1}\delta_j \text{ for } k=1, \ldots, p-1$$

and $t'_{p-1}(X') = p(n-p) + t_{p-1}(X)$.

Since by definition, $s'_k(X') = t'_k(X') - t'_{k-1}(X')$ for $k=0, 1, \ldots, p-1$ and $s'_p(X') = n' - p - t'_{p-1}(X')$, we obtain

$$\text{(10)} \qquad \begin{cases} s'_k(X') = \sum_{j=k}^{p} s_j(X) - \delta_k, \; k=0, 1, \ldots, p-1, \\ s'_p(X') = s_p(X). \end{cases}$$

Since $\delta_k \geqq 0$, the inequalities (2) in the theorem are verified.

The latter half of the theorem also follows from the above computations. Since the set of functions Θ is regular at X by the assumption, the submanifold of M', defined by $\Theta = 0$ in a neighborhood of the point X, is of dimension $n' - t'_{-1}(X') = n - t_{-1}(X) + \sum_{k=1}^{p} ks_k(X) - \sum_{k=0}^{p-1}\delta_k$. Therefore X is ordinary if and only if $\sum_{k=0}^{p-1}\delta_k = 0$. We can see by (10) that this condition is equivalent to the equalities (1) for all $k = 0, 1, \ldots, p$. Thus the proof is complete.

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