ON q-GALOIS EXTENSIONS OF SIMPLE RINGS

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To the memory of Professor TADASI NAKAYAMA

In 1952, the late Professor T. Nakayama succeeded in constructing the Galois theory for finite dimensional simple ring extensions [7]. And, we believe, the theory was essentially due to the following proposition: If a simple ring A is Galois and finite over a simple subring B then A is B'-A-completely reducible for any simple intermediate ring B' of A/B [7, Lemmas 1.1 and 1.2[1)]]. Moreover, as was established in [5], Nakayama's idea was still efficient in considering the infinite dimensional Galois theory of simple rings.

In this paper, we shall present first such a generalization of the proposition stated above that contains [5, Lemma 2] as well. And then, by the aid of this generalization, several facts obtained in [6] and [8] for division rings will be extended to simple rings. In fact, under the assumption that a simple ring extension in question is h-q-Galois and left locally finite, many important results previously obtained in [2]-[10] can be unified.

Throughout the present paper, $A=\sum_1^n De_{ij}$ will represent a simple ring where $E=\{e_{ij}\text{'s}\}$ is a system of matrix units and $D=V_A(E)$ a division ring, and B a simple subring of A containing the identity 1 of A. And we use the following conventions: V and H mean $V_A(B)$ and $V_A^2(B)=V_A(V_A(B))$, respectively. If H is a simple ring, we set $H=\sum Kd_{hk}$ where $\Delta=\{d_{hk}\text{'s}\}$ is a system of matrix units and $K=V_H(\Delta)$ a division ring. If T is a regular subring of A containing B, $\mathfrak{G}(T, A/B)$ will mean the set of all the B-(ring) isomorphisms of T onto regular subrings of A. And finally, A/B is said to be *h-Galois*[2)] if B is regular and $\mathfrak{G}A_r$ is dense in $\mathrm{Hom}_{B_l}(A, A)$, where $\mathfrak{G}$ is the

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1) These lemmas were stated under the weaker assumption that A/B is (finite and) weakly normal.

2) In [4], A/B was defined to be h-Galois if (i) B is regular, (ii) A is Galois over B' and $V_A^2(B')$ is simple for any regular subring B' of A left finite over B, and (iii) $A'=V_A^2(A')$ and $[A':H]_l=[V:V_A(A')]_r$ for every regular subring A' of A left finite over H, and it

(*Continued on next page*)

group of all the B-automorphisms of A. As to other notations and terminologies used here, we follow [4] and [5].

The following propositions previously known will play important roles in the present study.

PROPOSITION 1. *Let B' be a subring of A containing* 1 *of A, $V' = V_A(B')$ and $H' = V_A^2(B')$.*

(a) *If A is $B' \cdot V'$-A-irreducible, then A is homogeneously completely reducible as B'-A-module and as V'-A-module, both V' and H' are simple rings, $[A \mid B_l' \cdot A_r] = [V' \mid V']$ and $[A \mid V_l' \cdot A_r] = [H' \mid H']$*[3].

(b) *If B' is an intermediate ring of A/B left (resp. right) finite over B and A is $B' \cdot V'$-A-irreducible (resp. A-$B' \cdot V'$-irreducible), then $[V : V']_r \leq [B' : H^*]_l$ (resp. $[V : V']_l \leq [B' : H^*]_r$) for any simple intermediate ring H^* of $H \cap B'/B$.*

(c) *If B' is an intermediate ring of $A/B[E]$ left (resp. right) finite over B and A is left (resp. right) locally finite over B, then $[V : V']_l \leq [B' : B]_l$ (resp. $[V : V']_r \leq [B' : B]_r$).* ([2, Lemma 1 and Cor. 2].)

PROPOSITION 2. *Let A be outer Galois and left locally finite over B, and A' an intermediate ring of A/B.*

(a) *A' is simple, A/A' is (two-sided) locally finite, and each B-(ring) isomorphism of A' into A can be extended to an element of $\mathfrak{G}$.*

(b) *A/B is h-Galois, and there exists a* 1-1 *dual correspondence between closed subgroups of $\mathfrak{G}$ and intermediate rings of A/B, in the usual sense of Galois theory.*

(c) *If $[A' : B]_l < \infty$ then $[A' : B]_l = [A' : B]_r = \# (\mathfrak{H} \mid A')$ for any Galois group $\mathfrak{H}$ of A/B.* ([3, Th. 1.1], [3, Cor. 1.4], [4, Lemma 1.8] and [9].)

PROPOSITION 3. *Let A be Galois over B with a regular Galois group $\mathfrak{H}$, and H a simple ring left locally finite over B. And let T be an intermediate ring of A/B such that $[T : B]_l < \infty$ and A is T-A-irreducible.*

(a) *If $[T : H \cap T]_l = [V : V_A(T)]_r$ then* $\mathrm{Hom}_{B_l}(T, A) = (\mathfrak{H} \mid T) A_r$ *and $\mathfrak{G} \mid T = \mathfrak{H} \mid T$.*

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was shown that if A/B is h-Galois and left locally finite then $\mathfrak{G} A_r$ is dense in $\mathrm{Hom}_B(A, A)$. And more recently, in [2], T. Nagahara has shown the converse implication. However, one will see later the converse implication to be true even under a somewhat weakened assumption. (Cf. Ths. 2 and 8.)

[3] $[A \mid B_l' \cdot A_r]$ and $[V' \mid V']$ denote the length of the composition series of the B'-A-module A and the length of the composition series of the V'-module V' (the capacity of the simple ring V'), respectively.

(b) *If* $T' = J(\mathfrak{G}(T), A)$ *then* $[H \cap T' : B] < \infty$. ([4, Lemma 3.1] and [5, Lemma 5].)

1. Preliminaries. The present section starts with the following brief lemma.

LEMMA 1. *Let* B' *be a simple intermediate ring of* A/B *with* $[B' | B'] = n$ (= *capacity of* A). *If* a *is an arbitrary element of* A *and* T *an arbitrary simple intermediate ring of* A/B' *then* $[aB' | B'] \geq [aT | T]$. *And, if* A/B *is left locally finite and* $[B' : B]_l < \infty$ *then there exists an intermediate ring* B'' *of* A/B' *such that* $[B'' : B]_l < \infty$ *and* $[aB'' | B''] = [aA | A]$.

Proof. Without loss of generality, we may assume that B' contains E and $aB' = \sum_1^n ae_{ii}B' = \oplus_1^m ae_{ii}B'$ ($m = [aB' | B']$). As each $e_{ii}T = e_{ii}B'T$ is a minimal right ideal of T, $aT = aB'T = \sum_1^m ae_{ii}T$ implies then $[aT | T] \leq m$. Now, the rest of the proof will be obvious.

The proof of the next lemma proceeds in the usual way (cf. [4]), and may be omitted.

LEMMA 2. *Let* B' *be a simple intermediate ring of* A/B *with* $[B' | B'] = n$, α *and* β *elements of* $\mathfrak{G}(B', A/B)$, *and* $\mathfrak{H}$ *a subset of* $\mathfrak{G}(B', A/B)$.

(a) αA_r *is* B'_r-A_r*-irreducible and* α *is linearly independent over* A_r.

(b) *Let* $\mathfrak{m}$ *be a* B'_r-A_r*-submodule of* $\mathfrak{H}A_r$. $\mathfrak{m}$ *is* B'_r-A_r*-irreducible if and only if* $\mathfrak{m} = \sigma u_l A_r$ *with some* $\sigma \in \mathfrak{H}$ *and some non-zero* $u \in V$.

(c) αA_r *is* B'_r-A_r*-isomorphic to* βA_r *if and only if* $\alpha = \beta \tilde{u}$ *with some* $u \in V^{\cdot}$ (*the multiplicative group of the regular elements of* V), *and so if* α *is contained in* $\mathfrak{H}A_r$ *then* $\alpha = \sigma \tilde{v}$ *with some* $\sigma \in \mathfrak{H}$ *and* $v \in V^{\cdot}$.

We consider here the following conditions:

(1) $\mathrm{Hom}_{B_l}(B', A) = \mathfrak{G}(B', A/B)A_r$ for any regular intermediate ring B' of A/B with $[B' : B]_l < \infty$.

(1′) $\mathrm{Hom}_{B_r}(B', A) = \mathfrak{G}(B', A/B)A_l$ for any regular intermediate ring B' of A/B with $[B' : B]_r < \infty$.

(2) $\mathfrak{G}(B_1, A/B) | B_2 \subseteq \mathfrak{G}(B_2, A/B)$ for any regular subrings $B_1 \supseteq B_2$ of A containing B with $[B_1 : B]_l < \infty$.

(2′) $\mathfrak{G}(B_1, A/B) | B_2 \subseteq \mathfrak{G}(B_2, A/B)$ for any regular subrings $B_1 \supseteq B_2$ of A containing B with $[B_1 : B]_r < \infty$.

Remark 1. If the condition (1) is satisfied, then $J(\mathfrak{G}(B', A/B), B') = B$

for any regular intermediate ring B' of A/B with $[B' : B]_l < \infty$. In fact, if b' is an arbitrary element of $J(\mathfrak{G}(B', A/B), B')$ not contained in B then $T = B[b']$ is a subring of B' properly containing B. Since $\mathrm{Hom}_{B_l}(B', A) = \mathfrak{G}(B', A/B)A_r$, we have $\mathrm{Hom}_{B_l}(T, A) = \mathrm{Hom}_{B_l}(B', A) | T = (\mathfrak{G}(B', A/B) | T)A_r = 1 | T)A_r$, whence it follows a contradiction $[T : B]_l = 1$.

Now, we shall prove our first theorem which contains evidently the proposition cited at the opening as well as [5, Lemma 2].

Theorem 1. *Let A/B be left locally finite, and the condition* (1) *satisfied. If T is a simple intermediate ring of A/B with $[T : B]_l < \infty$ then A is T-A-completely reducible. In particular, if T is a regular subring of A with $[T : B]_l < \infty$ then A is homogeneously T-A-completely reducible with $[A | T_l \cdot A_r] = [V_A(T) | V_A(T)]$ and T is f-regular.*

Proof. Let M be an arbitrary minimal T-A-submodule of A such that thh composition series of M as right A-module is of the shortest length among minimal T-A-submodules of A. Then, $M = eA$ with a non-zero idempotent e. In virtue of Lemma 1, we can find an intermediate ring T^* of $A/T[E, e]$ with $[T^* : B]_l < \infty$ and $[eT^* | T^*] = [eA | A]$. One may remark here that $TeT^* = eT^*$. In fact, for each $t \in T$ there exists some $a \in A$ with $ea = te \in T^*$, so that $te = e \cdot ea \in eT^*$. By Lemma 2 (a), $\mathrm{Hom}_{B_l}(T^*, A) = \mathfrak{G}(T^*, A/B)A_r$ is T_r^*-A_r-completely reducible. Accordingly, the T_r^*-A_r-module $\mathrm{Hom}_{T_l}(T^*, A) = \oplus_1^t \mathfrak{M}_j$ with T_r^*-A_r-irreducible $\mathfrak{M}_j$. By Lemma 2 (b), $\mathfrak{M}_j = \sigma_j u_{jl} A_r$ with some $\sigma_j \in \mathfrak{G}(T^*, A/B)$ and non-zero $u_j \in V$. Since $\mathfrak{M}_j \subseteq \mathrm{Hom}_{T_l}(T^*, A)$ and $TeT^* = eT^* (\subseteq T^*)$, each $M_j = (Te)\mathfrak{M}_j$ is a T-A-submodule of A. Further, there holds $M_j = u_j \cdot (Te)\sigma_j \cdot A = u_j \cdot (TeT^*)\sigma_j \cdot A = u_j \cdot (eT^*)\sigma_j \cdot A = u_j \cdot e\sigma_j \cdot A$, whence it follows $[M_j | A] = [u_j \cdot e\sigma_j \cdot A | A] \leq [e\sigma_j \cdot A | A] \leq [e\sigma_j \cdot T^* \sigma_j | T^* \sigma_j] = [eT^* | T^*] = [M | A]$ by Lemma 1. Recalling here that $[M | A]$ is the least, we see that each M_j is either 0 or T-A-irreducible. Finally, noting that A is $T_l \cdot \mathrm{Hom}_{T_l}(A, A)$-irreducible, there holds $A = e(T_l \cdot \mathrm{Hom}_{T_l}(A, A)) = (Te)\mathrm{Hom}_{T_l}(T^*, A) = (Te)\sum \mathfrak{M}_j = \sum M_j$, which proves evidently the complete reducibility of A as T-A-module. Now, the latter assertion will be evident by Prop. 1 (b).

The next has been proved in [2] and [5]. Nevertheless, according to the idea in [7], we shall present here another proof that needs only Lemma 2 and Th. 1.

COROLLARY 1. *Let A be left locally finite over a regular subring B, and $\mathfrak{H}A_r$ is dense in* $\mathrm{Hom}_{B_l}(A, A)$ *for an automorphism group $\mathfrak{H}$ containing $\tilde{V}$. If B' is a regular intermediate ring of A/B with $[B' : B]_l < \infty$ then $\mathfrak{H}(B')A_r$ is dense in* $\mathrm{Hom}_{B'_l}(A, A)$ *and $J(\mathfrak{H}(B'), A) = B'$.*

Proof. Let T be an arbitrary intermediate ring of $A/B'[E]$ with $[T : B]_l < \infty$. Evidently, $\mathrm{Hom}_{B'_l}(T, A)$ is a T_r-A_r-submodule of $\mathrm{Hom}_{B_l}(T, A) = (\mathfrak{H}|T)A_r$. And then, by Lemma 2(b), $\mathrm{Hom}_{B'_l}(T, A) = \oplus(\sigma_i u_{il}|T)A_r$ with some $\sigma_i \in \mathfrak{H}$ and non-zero $u_i \in V$. In general, if $\tau w_l | T(\tau \in \mathfrak{H}, w \in V)$ is contained in $\mathrm{Hom}_{B'_l}(T, A)$, one will easily see that τw_l is contained in $V_{\mathfrak{A}}(B'_l)$ $(\mathfrak{A} = \mathrm{Hom}(A, A))$. Now, let σu_l be an arbitrary $\sigma_i u_{il}$. Since A is homogeneously B'-A-completely reducible by Th. 1, a standard argument enables us to find such an inversible element $\nu \in V_{\mathfrak{A}}(B'_l)$ that $a_r \nu = \nu(a\sigma)_r$ for all $a \in A$. As $\nu^{-1}\sigma u_l$ is then contained in $V_{\mathfrak{A}}(B'_l \cdot A_r) = V_A(B')_l$, $\sigma u_l = \nu v_{1l} + \cdots + \nu v_{ml}$ with some $v_j \in V_A(B')^{\cdot}$. Noting that T contains E, one will easily see that every $(\nu v_{jl}|T)A_r$ is a T_r-A_r-irreducible submodule of $\mathrm{Hom}_{B'_l}(T, A)$, so that $(\nu v_{jl}|T)A_r = (\tau w_{jl}|T)A_r$ with some $\tau \in \mathfrak{H}$ and $w_j \in V$ (Lemma 2). We have then $A = v_j A = v_j \cdot A\nu = v_j \cdot (T \cdot A\sigma^{-1})\nu = v_j \cdot T\nu \cdot A = T(\nu v_{jl}|T)A_r = T(\tau w_{jl}|T)A_r = w_j \cdot T\tau \cdot A = w_j A$, whence it follows $w_j \in V^{\cdot}$. Hence, $\tau \tilde{w}_j = \tau w_{jl} w_{jr}^{-1}$ is contained in $V_{\mathfrak{A}}(B'_l) \cap \mathfrak{H} = \mathfrak{H}(B')$. It follows therefore $\mathrm{Hom}_{B'_l}(T, A) = (\mathfrak{H}(B')|T)A_r$, which forces $\mathfrak{H}(B')A_r$ to be dense in $\mathrm{Hom}_{B'_l}(A, A)$. Finally, to be easily verified, $B_l = V_{\mathfrak{A}}^2(B_l) = V_{\mathfrak{A}}(\mathfrak{H}A_r)$, which implies $J(\mathfrak{H}, A) = B$. And hence, by the fact proved above, $J(\mathfrak{H}(B'), A) = B'$.

Patterning after the proof of [2, Lemma 2], we readily obtain the next:

LEMMA 3. *Let H be simple, and T an intermediate ring of $A/B[\Delta]$. If there exists an automorphism group $\mathfrak{H}$ of $H[T]$ with $J(\mathfrak{H}, H[T]) = T$ and $H\mathfrak{H} = H$, and if $H \cap T$ is simple, then T is linearly disjoint from H.*

The following proposition is a part of [2, Th. 1]. However, for the sake of completeness, we shall give here the proof.

PROPOSITION 4. *If B is a regular subring of A, the following conditions are equivalent to each other:*

(A) *A is h-Galois and left locally finite over B.*

(A′) *$\mathfrak{H}A_l$ is dense in* $\mathrm{Hom}_{B_r}(A, A)$ *and A/B is right locally finite.*

(B) *A is Galois and left locally finite over B, and $B \cdot V$-A-irreducible.*

(B′) *A is Galois and right locally finite over B, and A-B·V-irreducible.*

(C) *A is Galois and left locally finite over B, and A-B·V-irreducible.*

(C′) *A is Galois and right locally finite over B, and B·V A-irreducible.*

Proof. (A) $\Rightarrow$ (B) is obvious by Th. 1 and Cor. 1. Next, we shall prove (B) $\Rightarrow$ (C′) $\Rightarrow$ (A′). As A is $B\cdot V\text{-}A$-irreducible, H is simple by Prop. 1 (a). For an arbitrary intermediate ring T of $A/B[E, \Delta]$ with $[T : B]_r < \infty$, we set $T' = J(\mathfrak{G}(T), A)$ and $H' = H \cap T'$. Then, $[V : V_A(T')]_l = [V : V_A(T)]_l \leq [T : B]_r$ by Prop. 1 (b), and so Lemma 3 and Prop. 1 (b) imply $[T' : H']_r = [T'\cdot H : H]_r \leq [V_A^2(T') : H]_r \leq [V : V_A(T')]_l < \infty$. On the other hand, noting that A is $A\text{-}T'$-irreducible, Prop. 1 (b) yields also $[V : V_A(T')]_l \leq [T' : H']_r < \infty$. Combining those above, we obtain $[T' : H']_r = [V : V_A(T')]_l$. Since $[T' : B]_r = [T' : H']_r \cdot [H' : B] < \infty$ by Prop. 3 (b), the proposition symmetric to Prop. 3 (a) yields $\mathrm{Hom}_{F_r}(T', A) = (\mathfrak{G} | T')A_l$, which proves (C′) $\Rightarrow$ (A′). In case the condition (B) is satisfied, for an arbitrary intermediate ring T of $A/B[E, \Delta]$ with $[T : B]_l < \infty$ there holds $[V : V_A(T)]_l \leq [T : B]_l < \infty$ (Prop. 1 (c)). And so, repeating the above argument, we obtain $[T : B]_r \leq [T' : B]_r < \infty$, which means A/B is right locally finite. We have proved thus (A) $\Rightarrow$ (B) $\Rightarrow$ (C′) $\Rightarrow$ (A′), and symmetrically (A′) $\Rightarrow$ (B′) $\Rightarrow$ (C) $\Rightarrow$ (A).

COROLLARY 2. *Let A be left locally finite over a regular subring B. If the condition* (1) *is satisfied, then* (*H is simple and*) *A is h-Galois and locally finite over H. And, if A/B is Galois and the condition* (1) *is satisfied then A/B is h-Galois, and conversely.*

Proof. Let B' be an arbitrary intermediate ring of $A/B[E]$ with $[B' : B]_l < \infty$. Then, by Prop. 1 (c), we have $[V : V_A(B')]_l \leq [B' : B]_l < \infty$. Since A is $B\cdot V\text{-}A$-irreducible (Th. 1), A is $V\cdot H\text{-}A$-irreducible much more and H is simple by Prop. 1 (a). And then, by Prop. 1 (b), it follows $[V_A^2(B') : H]_r \leq [V : V_A(B')]_l < \infty$, which proves evidently the right local finiteness of A/H. Hence, Prop. 4 asserts that A/H is locally finite and h-Galois. The latter assertion is a direct consequence of Th. 1 and Prop. 4.

The following theorem coincides essentially with [10, Th. 3].

THEOREM 2. *Let A be left locally finite over a regular subring B, and the condition* (1) *satisfied. If A′ is a simple intermediate ring of A/H with* $[A' : H]_l < \infty$, *then A′ is f-regular and* $V_A^2(A') = A'$.

Proof. By Cor. 2, A/H is h-Galois and locally finite. If A_0 is an arbitrary intermediate ring of $A/A'[E]$ with $[A_0 : H]_l < \infty$ then A is A_0-A-irreducible and A-V·H-irreducible (Prop. 4). Hence, $[A_0 : H]_l \geq [V : V_A(A_0)]_r \geq [V_A^2(A_0) : H]_l \geq [A_0 : H]_l$ by Prop. 1 (b), whence it follows $[A_0 : H]_l = [V : V_A(A_0)]_r$. And then, Prop. 3 (a) asserts that $\mathrm{Hom}_{H_l}(A_0, A) = (\tilde{V}\,|\,A_0)A_r$, which means that $\tilde{V}A_r$ is dense in $\mathrm{Hom}_{H_l}(A, A)$. And then, the proof of [10, Th. 3] asserts that A' is regular. Accordingly, $[V : V_A(A')]_r \leq [A' : H]_l < \infty$ by Th. 1 and Prop. 1 (b), and $V_A^2(A') = J(\tilde{V}(A'), A) = A'$ by Cor. 1.

LEMMA 4. *Let A/B be left locally finite, and the condition* (1) *satisfied. If ρ is a B-ring homomorphism of an intermediate ring A_1 of A/B with $[A_1 : B]_l < \infty$ onto a simple intermediate ring A_2 of A/B such that $V_A(A_2)$ is a division ring, then ρ is contained in $\mathfrak{G}(A_0, A/B)|A_1$ for any regular intermediate ring A_0 of A/A_1 with $[A_0 : B]_l < \infty$.*

Proof. Let $\mathfrak{H} = \mathfrak{G}(A_0, A/B)$. Since $[A_2 : B]_l \leq [A_1 : B]_l < \infty$ and $V_A(A_2)$ is a division ring, A is A_2-A-irreducible (Th. 1). And, we have $\mathrm{Hom}_{B_l}(A_1, A) = (\mathfrak{H}\,|\,A_1)A_r = \sum_1^s(\sigma_i\,|\,A_1)A_r$ with some $\sigma_i \in \mathfrak{H}$, for $[\mathrm{Hom}_{B_l}(A_1, A) : A_r]_r = [A_1 : B]_l < \infty$. Now, the rest of the proof proceeds in the same way as in the proof of [4, Lemma 3.11].

THEOREM 3. *Let A/B be left locally finite, and the conditions* (1), (2) *satisfied. If $B_1 \supseteq B_2$ are regular intermediate rings of A/B with $[B_1 : B]_l < \infty$ then $\mathfrak{G}(B_2, A/B) = \mathfrak{G}(B_1, A/B)\,|\,B_2$.*

Proof. Let σ be an arbitrary element of $\mathfrak{G}(B_2, A/B)$, and $B_3 = B_2\sigma$. We set $V_i = V_A(B_i) = \sum_1^{m_i} U_i g_{pq}^{(i)} (i = 2, 3)$, where $\{g_{pq}^{(i)}\text{'s}\}$ is a system of matrix units and $U_i = V_{V_i}(\{g_{pg}^{(i)}\text{'s}\})$ is a division ring. If $m_2 \geq m_3$ then we can consider the subrings A_2, A_3 of A defined as follows:

$$A_2 = \textstyle\sum_1^{m_3} B_2 g_{pq}^{(2)} + B_2 g, \text{ where } g = \sum_{m_3+1}^{m_2} g_{pp}^{(2)}, \text{ and}$$
$$A_3 = \textstyle\sum_1^{m_3} B_3 g_{pq}^{(3)}.$$

Evidently, A_2 is an intermediate ring of A/B_2 with $[A_2 : B]_l < \infty$, A_3 a simple intermediate ring of A/B_3, and $V_A(A_3) = U_3$ a division ring. As $\{g_{pq}^{(i)}\text{'s}\}$ is linearly independent over B_i, we can define a B-linear map ρ of A_2 onto A_3 by the following rule:

$$\begin{cases} (B_2 g)\rho = 0, \\ (\sum_1^{m_3} b_{pq}^{(2)} g_{pq}^{(2)})\rho = \sum_1^{m_3} (b_{pq}^{(2)}\sigma) g_{pq}^{(3)} \qquad (b_{pq}^{(2)} \in B_2). \end{cases}$$

Then, one will easily see that ρ is a ring homomorphism and $\sigma = \rho | B_2$. If A_0 is an arbitrary regular intermediate ring of $A/A_2[B_1]$ with $[A_0 : B]_l < \infty$ then ρ is contained in $\mathfrak{G}(A_0, A/B) | A_2$ (Lemma 4), so that $\sigma = \rho | B_2 \in \mathfrak{G}(A_0, A/B) | B_2 = (\mathfrak{G}(A_0, A/B) | B_1) | B_2 \subseteq \mathfrak{G}(B_1, A/B) | B_2$ by (2). On the other hand, if $m_2 \leq m_3$ then the same argument applied to σ^{-1} (instead of σ) enables us to find a simple intermediate ring A_0 of A/B_3 with $[A_0 : B]_l < \infty$ such that $V_A(A_0)$ is a division ring and $\sigma^{-1} = \rho | B_3$ for some $\rho \in \mathfrak{G}(A_0, A/B)$. Applying again the above argument to ρ^{-1}, we can find a simple intermediate ring A^* of $A/(A_0\rho)[B_1]$ with $[A^* : B]_l < \infty$ such that $V_A(A^*)$ is a division ring and $\rho^{-1} = \tau | A_0\rho$ for some $\tau \in \mathfrak{G}(A^*, A/B)$. Then, $\sigma = \rho^{-1} | B_2 = \tau | B_2 \in \mathfrak{G}(A^*, A/B) | B_2 \subseteq \mathfrak{G}(B_1, A/B) | B_2$. Hence, in either cases, we have seen $\mathfrak{G}(B_2, A/B) \subseteq \mathfrak{G}(B_1, A/B) | B_2$, whence it follows eventually $\mathfrak{G}(B_2, A/B) = \mathfrak{G}(B_1, A/B) | B_2$.

COROLLARY 3. *Let A be left locally finite over a regular subring B, and $\mathfrak{H}$ an automorphism group of A containing $\tilde{V}$. If $\mathfrak{H}A_r$ is dense in* $\mathrm{Hom}_{B_l}(A, A)$ *then $\mathfrak{G}(B', A/B) = \mathfrak{H} | B'$ for each regular intermediate ring B' of A/B with $[B' : B]_l < \infty$. In particular, if A/B is h-Galois and left locally finite, then the condition* (2) *is fulfilled.* (Cf. [4, Cor. 3.7].)

Proof. If $B_0 = B'[E]$, then $\mathfrak{G}(B_0, A/B) \subseteq \mathrm{Hom}_{B_l}(B_0, A) = (\mathfrak{H} | B_0) A_r$, whence it follows $\mathfrak{G}(B_0, A/B) = \mathfrak{H} | B_0$ (Lemma 2 (c)). Now, the same argument as in the proof of Th. 3 enables us to see that $\mathfrak{G}(B', A/B) \subseteq \mathfrak{G}(B_0, A/B) | B' = \mathfrak{H} | B'$, whence it follows $\mathfrak{G}(B', A/B) = \mathfrak{H} | B'$.

2. q-Galois Extensions. A/B is said to be *q-Galois* (resp. *right q-Galois*) if B is regular and the conditions (1), (2) (resp. (1′), (2′)) are satisfied. To be easily verified, if A is a division ring, the notion of q-Galois coincides with that of quasi-Galois defined in [8] provided A/B is left locally finite (cf. [8] and Remark 2). And, A/B is said to be *locally h-Galois* if for each finite subset F of A there exists such an intermediate ring A' of $A/B[F]$ that A'/B is h-Galois. Needless to say, if A/B is h-Galois or locally Galois then it is locally h-Galois.

PROPOSITION 5. *If A/B is locally h-Galois and left locally finite then it is q-Galois.*

Proof. Let $B_1 \supseteq B_2$ be regular intermediate rings of A/B with $[B_1 : B]_l < \infty$, and σ an arbitrary element of $\mathfrak{G}(B_1, A/B)$. Then, the simple rings $V_A(B_1)$, $V_A(B_2)$ and $V_A(B_1\sigma)$ are represented as the complete matrix rings over division rings with the systems of matrix units Γ_1, Γ_2 and Γ_3, respectively. Now, for an arbitrary finite subset F of A, choose an intermediate ring A^* of $A/B_1[B_1\sigma, F, E, \Gamma_1, \Gamma_2, \Gamma_3]$ such that A^*/B is h-Galois. Then, by Cor. 3, σ can be extended to an automorphism σ^* of A^*. Since $V_{A^*}(B)$ and $V_{A^*}(B_2\sigma) = V_{A^*}(B_2)\sigma^*$ are simple rings, they are the complete matrix rings over division rings with the systems of matrix units Γ^* and Γ_2^*, respectively. If we set $B^* = B_2[B_2\sigma, F, E, \Gamma^*, \Gamma_2^*]$ $(\subseteq A^*)$, B^* is a regular subring of A left finite over B such that $V_{B^*}(B)$ and $V_{B^*}(B_2\sigma)$ are simple. Hence, we have seen that there exists a directed set $\{B_\lambda^*\}$ of regular intermediate rings B_λ^* of $A/B_2[B_2\sigma]$ such that $[B_\lambda^* : B]_l < \infty$, $A = \cup B_\lambda^*$ and that $V_{B_\lambda^*}(B)$ and $V_{B_\lambda^*}(B_2\sigma)$ are simple. It follows therefore $V = \cup V_{B_\lambda^*}(B)$ and $V_A(B_2\sigma) = \cup V_{B_\lambda^*}(B_2\sigma)$ are simple by [4, Lemma 1.1], which proves (2). Moreover, noting that B^* contains E, we see that $\mathrm{Hom}_{B_l}(B^*, A) = \mathrm{Hom}_{B_l}(B^*, A^*)A_r = ((\mathfrak{G}(A^*/B)|B^*)A_r^*)A_r \subseteq \mathfrak{G}(B^*, A/B)A_r$. And so, by (2), it follows eventually $\mathrm{Hom}_{B_l}(B_2, A) = \mathrm{Hom}_{B_l}(B^*, A)|B_2 = (\mathfrak{G}(B^*, A/B)|B_2)A_r = \mathfrak{G}(B_2, A/B)A_r$.

We insert here [4, Th. 2.3] as an easy consequence of Cors. 2 and 3.

PROPOSITION 6. *If A/B is Galois and locally Galois then A/B is $\mathfrak{G}$-locally Galois, and conversely.*

Proof. A/B is h-Galois by Cor. 2, so that for each shade B' we have $\mathfrak{G}(B'/B) \subseteq \mathfrak{G}(B', A/B) = \mathfrak{G}|B'$ (Cor. 3). And the converse part is obvious.

By the validity of Th. 1, the proof of the next lemma proceeds just like that of [5, Lemma 8] did.

LEMMA 5. *Let A/B be left locally finite, the condition* (1) *satisfied, and A^* a regular subring of A containing B. If F is an arbitrary finite subset of A^*, then A^* contains a regular subring B' of A such that $B' \supseteq B[F]$ and $[B' : B]_l < \infty$.*

LEMMA 6. *Let A/B be q-Galois and left locally finite. If A' is an f-regular intermediate ring of A/B then $(H \cap A')\mathfrak{G}(A', A/B) \subseteq H$.*

Proof. Let σ be an arbitrary element of $\mathfrak{G}(A', A/B)$, and h an arbitrary one of $H \cap A'$. And, choose a simple intermediate ring B' of $A'/B[h]$ such that $V_A(B') = V_A(A')$ and $[B' : B]_l < \infty$. Then, by Lemma 5 the regular subring $A'\sigma$ contains a simple subring B^* containing $B'\sigma$ such that $V_A(B^*)$ is simple and $[B^* : B]_l < \infty$. Here, needless to say, $B'' = B^*\sigma^{-1}$ is a regular subring of A as an intermediate ring of A'/B'. And so, $\tau'' = \sigma^{-1} | B^*$ is contained in $\mathfrak{G}(B^*, A/B)$. If v is an arbitrary element of V, $\tau'' = \tau | B^*$ with some $\tau \in \mathfrak{G}(B^*[E, v], A/B)$ (Th. 3). As $v\tau$ is contained in V, we have $h\cdot v\tau = v\tau\cdot h$, whence it follows $h\sigma\cdot v = v\cdot h\sigma$. We see therefore $h\sigma \in H$.

Now, we can prove the following theorem that corresponds to [8, Cor. 1].

THEOREM 4. *If A is q-Galois and left locally finite over B, then H/B is outer Galois and $\mathfrak{G}(H, A/B) = \mathfrak{G}(H/B)$.*

Proof. Let B' be an arbitrary intermediate ring of $H/B[\varDelta]$ with $[B' : B]_l < \infty$ (Cor. 2). Since $B'\mathfrak{G}(B', A/B) \subseteq H$ (Lemma 6), Lemma 2 (a) yields $[\mathfrak{G}(B', A/B)H_r : H_r]_r \leq [B' : B]_l < \infty$. Hence, $\mathfrak{G}(B', A/B)H_r = \oplus_1^t \sigma_i H_r$ with some $\sigma_i \in \mathfrak{G}(B', A/B)$ and so $\mathfrak{G}(B', A/B) = \mathfrak{G}(B', H/B) = \{\sigma_1, \ldots, \sigma_t\}$ by Lemma 2 (c). Now, we set $H = \cup B_\alpha$, where B_α ranges over all the intermediate rings of $H/B[\varDelta]$ with $[B_\alpha : B]_l < \infty$. We can consider then the inverse limit $\mathfrak{H} = \varprojlim \mathfrak{G}(B_\alpha, A/B)$, that may be regarded as a set of B-(ring) isomorphisms of H into H. Since every $\mathfrak{G}(B_\alpha, A/B)$ is finite and $\mathfrak{G}(B_\alpha, A/B) | B_\beta = \mathfrak{G}(B_\beta, A/B)$ for each $B_\alpha \supseteq B_\beta$ (Th. 3), we obtain $\mathfrak{H} | B_\alpha = \mathfrak{G}(B_\alpha, A/B)$ ([1, Cor. 3.9]). If T is an arbitrary subring of H properly containing B with $[T : B]_l < \infty$ then there exists some B_α containing T and then $J(\mathfrak{G}(B_\alpha, A/B), B_\alpha) = B$ by Remark 1. Combining this with $\mathfrak{H} | B_\alpha = \mathfrak{G}(B_\alpha, A/B)$, we readily see that $J(\mathfrak{H}, H) = B$. Further, if σ is in $\mathfrak{H}$ then for each B_α we can find a positive integer n_α such that $\sigma^{n_\alpha} | B_\alpha = 1$, which proves $H\sigma = H$, that is, σ is an automorphism of H. Finally, if τ is an arbitrary element of $\mathfrak{G}(H, A/B)$ then $H\tau \subseteq H$ (Lemma 6), and so we obtain $\mathfrak{G}(H, A/B) = \mathfrak{G}(H/B)$ by Prop. 2 (a).

COROLLARY 4. *Let A/B be q-Galois and left locally finite. If A' is a simple intermediate ring of A/H with $[A' : H]_l < \infty$ then A' is f-regular and $\mathfrak{G}(A', A/B) | H \subseteq \mathfrak{G}(H/B)$.*

Proof. The first assertion is contained in Th. 2, and then $H\mathfrak{G}(A', A/B) \subseteq H$ (Lemma 6). Recalling now that H/B is outer Galois (Th. 4), the latter is obvious by Prop. 2 (a).

3. h-q-Galois Extensions. A/B is said to be *h-q-Galois* (resp. *right h-q-Galois*) if B is regular and A/B' is q-Galois (resp. right q-Galois) for each regular intermediate ring B' of A/B with $[B' : B]_l < \infty$ (resp. $[B' : B]_r < \infty$). If A/B is left locally finite and locally h-Galois then it is h-q-Galois by Prop. 5 and Cor. 1. Moreover, in case A is a division ring, the notion of q-Galois coincides with that of h-q-Galois (Lemma 2).

Now, assume that A/B is h-q-Galois and left locally finite. If B' is a regular intermediate ring of A/B with $[B' : B]_l < \infty$, then A/B' is q-Galois and $V_A^2(B')/B'$ is outer Galois (Th. 4), and so $H[B']$ is a simple ring (Prop. 2). Recalling that A/H is locally finite (Cor. 2), Th. 2 yields $H[B'] = V_A^2(B')$. (This fact will be used often without mention in the sequel.) Since $\mathfrak{G}(V_A^2(B')/B')|H \subseteq \mathfrak{G}(H/B)$ (Cor. 4), $\sigma \to \sigma|H$ is a continuous monomorphism of compact $\mathfrak{G}(V_A^2(B')/B')$ into $\mathfrak{G}(H/H \cap B')$ and its image is a Galois group of $H/H \cap B'$. Hence, we see that $\sigma \to \sigma|H$ is an isomorphism onto $\mathfrak{G}(H/H \cap B')$. (Cf. [4] or [9]). By the aid of this fact, the same argument as in the proof of [5, Lemma 9] enables us to see that if A is h-q-Galois and left locally finite over B and A' is a regular intermediate ring of A/B with $[H[A'] : H]_l < \infty$ then $H[A']$ is outer Galois and locally finite over A' and $\mathfrak{G}(H[A']/A') \approx \mathfrak{G}(H/H \cap A')$ by contraction. Accordingly, by the validity of Lemma 5, we can apply the same argument as in the proof of [5, Th. 6] to obtain the next theorem that is stated without proof.

THEOREM 5. *Let A be h-q-Galois and left locally finite over B. If A' is a regular intermediate ring of A/B, and H' an intermediate ring of H/B that is Galois over B, then $H'[A']$ is outer Galois and locally finite over A' and $\mathfrak{G}(H'[A']/A') \approx \mathfrak{G}(H'/H' \cap A')$ (algebraically and topologically) by contraction.*

As the first corollary to Th. 5, we shall remark that if A/B is h-q-Galois and left locally finite then the condition (2) can be sharpened as follows:

(2^*) $\mathfrak{G}(A_1, A/B)|A_2 \subseteq \mathfrak{G}(A_2, A/B)$ for each f-regular intermediate rings $A_1 \supseteq A_2$ of A/B.

To prove (2^*), let σ be an arbitrary element of $\mathfrak{G}(A_1, A/B)$, and B_1 a simple intermediate ring of A_1/B with $[B_1 : B]_l < \infty$ and $V_A(B_1) = V_A(A_1)$. If B_2 is an arbitrary regular subring of A between A_1 and B with $[B_2 : B]_l < \infty$, then we can find a regular subring B^* of A between $A_1\sigma$ and $(B_1[B_2])\sigma$ with $[B^* : B]_l < \infty$ (Lemma 5). Evidently $B' = B^*\sigma^{-1}$ is regular as an intermediate ring of A_1/B_1. Hence, $\sigma' = \sigma | B'$ is in $\mathfrak{G}(B', A/B)$, and so $B_2\sigma = B_2\sigma'$ is regular by the condition (2). Now, let B_2 be specialized as a simple intermediate ring of A_2/B with $[B_2 : B]_l < \infty$ and $V_A(B_2) = V_A(A_2)$. Since $A_2 = (H \cap A_2)[B_2]$ by Th. 5 and Prop. 2, Lemma 6 yields $V_A(A_2\sigma) = V_A(((H \cap A_2)\sigma)[B_2\sigma]) = V_A(B_2\sigma)$. Hence, $V_A(B_2\sigma)$ being simple by the above remark, it follows that $\sigma | A_2$ is contained in $\mathfrak{G}(A_2, A/B)$.

Corollary 5. *Let A/B be h-q-Galois and left locally finite. If B' is a regular intermediate ring of A/B with $[B' : B]_l < \infty$ then $\mathfrak{G}(B', A/B) = \mathfrak{G}(V_A^2(B'), A/B) | B'$.*

Proof. By Th. 5, $H^* = V_A^2(B') = H[B']$ is outer Galois over B'. We set here $H^* = \cup B'_\alpha$, where B'_α ranges over all the $\mathfrak{G}(H^*/B')$-invariant shades. Now, let ρ be an arbitrary element of $\mathfrak{G}(B', A/B)$. Then, the set $\mathfrak{E}_\alpha = \{\rho' \in \mathfrak{G}(B'_\alpha, A/B)\ ;\ \rho' | B' = \rho\}$ is non-empty (Th. 3). If ρ' and ρ'' are in $\mathfrak{E}_\alpha$ then $\rho'' = \rho'\varepsilon$ with some $B'\rho$-(ring) isomorphism ε between regular subrings $B'_\alpha\rho'$ and $B'_\alpha\rho''$. As $B'_\alpha = (H \cap B'_\alpha)[B']$ (Th. 5 and Prop. 2), $B'_\alpha\rho' \subseteq H[B'\rho] = V_A^2(B'\rho)$ by Lemma 6. And so, recalling that A is q-Galois and left locally finite over $B'\rho$ and $B'_\alpha\rho'/B'\rho$ is Galois, by [4, Cor. 3.9], Lemma 6 and Prop. 2 (a), we see that $\mathfrak{G}(B'_\alpha\rho'/B'\rho) = \mathfrak{G}(V_A^2(B'\rho)/B'\rho) | B'_\alpha\rho' = \mathfrak{G}(B'_\alpha\rho', A/B'\rho)$. Consequently, $\mathfrak{G}(B'_\alpha\rho', A/B'\rho) = \mathfrak{G}(B'_\alpha\rho'/B'\rho) \approx \mathfrak{G}(B'_\alpha/B')$ is finite, and so $\mathfrak{E}_\alpha$ is finite, too. Thus, by [1, Th. 3.6], the inverse limit $\mathfrak{E} = \varprojlim \mathfrak{E}_\alpha$ is non-empty, which means that $\rho \in \mathfrak{G}(B', A/B)$ can be extended to an isomorphism ρ^* of H^* into A. Since $(H \cap B'_\alpha)\rho' \subseteq H$ for each $\rho' \in \mathfrak{E}_\alpha$ (Lemma 6), $H^*\rho^* = (\cup (H \cap B'_\alpha)[B'])\rho^*$ is to be regular. Hence, we have seen $\mathfrak{G}(B', A/B) \subseteq \mathfrak{G}(H^*, A/B) | B'$. The converse inclusion is secured by (2^*).

Corollary 6. *Let A/B be h-q-Galois and left locally finite. If B' is a regular intermediate ring of $A/B[\Delta]$ with $[B' : B]_l < \infty$ then $H^*[B'] = H^* \cdot B$ and $[H^*[B'] : H^*]_l = [A^* : H \cap A^*]_l = [B' : H \cap B']_l$ for each intermediate ring H^* of $H/H \cap B'$ and each intermediate ring A^* of $H[B']/B'$.*

Proof. We set $H' = H \cap B'$ and $\mathfrak{G}' = \mathfrak{G}(H[B']/B')$. Then, H' is simple by Th. 4 and Prop. 2. If M is an arbitrary $\mathfrak{G}(H/H')$-invariant shade then $\mathfrak{G}(M[B']/B') = \mathfrak{G}' | M[B'] \approx \mathfrak{G}' | M = \mathfrak{G}(M/H')$ (Th. 5), which implies $[M[B'] : B'] = [M : H']$. Accordingly, we obtain $[M[B'] : M]_l = [B' : H']_l$. On the other hand, by the validity of Th. 5, Lemma 3 applies to obtain $[M \cdot B' : M]_l$

$=[B':H']_l$. It follows therefore $M[B']=M\cdot B'$. Now, it will be easy to see that $H[B']=H\cdot B'=\oplus_1^t Hb_i'$, where $\{b_i'\text{'s}\}$ is an arbitrary linearly independent left H'-basis of B'. And so, we have $H^*[B']=J(\mathfrak{G}'(H^*[B']), \oplus_1^t Hb_i') = \oplus_1^t H^*b_i'$ (Prop. 2 (b)), whence $H^*[B']=H^*\cdot B'$. And, at the same time, the latter assertion is also obvious by Th. 5 and Prop. 2 (b).

If A/B is h-q-Galois and left locally finite, we can prove the following sharpening of Th. 3, which is at the same time an extension of [6, Th. 5] to simple rings.

Theorem 6. *Let A/B be h-q-Galois and left locally finite. If $A_1 \supseteq A_2$ are f-regular intermediate rings of A/B then $\mathfrak{G}(A_2, A/B)=\mathfrak{G}(A_1, A/B)|A_2$.*

Proof. (I) We shall prove first our theorem for regular intermediate rings $A_1\supseteq A_2$ of A/H with $[A_1:H]_l<\infty$. By the validity of (2^*), it suffices to prove that $\mathfrak{G}(A_2, A/B)\subseteq\mathfrak{G}(A_1, A/B)|A_2$. Choose a simple intermediate ring B_2' of A_2/B with $[B_2':B]_l<\infty$ and $V_A(B_2')=V_A(A_2)$ (Th. 2). And then, between A_1 and B_2' there exists a regular subring B_1 of A with $[B_1:B]_l<\infty$ and $A_1=V_A^2(B_1)=H[B_1]$. If $B_2=A_2\cap B_1$ then $B_2'\subseteq B_2\subseteq A_2=V_A^2(B_2')$, and hence B_2 is a regular subring of A left finite over B (Th. 4 and Prop. 2 (a)) and $A_2=V_A^2(B_2)=H[B_2]$. Since $\mathfrak{G}(A_2, A/B)|B_2=\mathfrak{G}(B_2, A/B)=\mathfrak{G}(B_1, A/B)|B_2=\mathfrak{G}(A_1, A/B)|B_2$ (Cor. 5 and Th. 3), for each $\sigma\in\mathfrak{G}(A_2, A/B)$ we can find some $\rho\in\mathfrak{G}(A_1, A/B)$ with $\rho|B_2=\sigma|B_2$. As $A_2\sigma=H[B_2\sigma]=H[B_2\rho]=A_2\rho$ (Cor. 4), $\sigma\rho^{-1}$ is contained in $\mathfrak{G}(A_2/B_2)=\mathfrak{G}(A_2/A_2\cap B_1)=\mathfrak{G}(A_1/B_1)|A_2$ (Th. 5). Hence, σ is in $\mathfrak{G}(A_1, A/B)|A_2$.

(II) Now, assume that A_i be f-regular, and take simple intermediate rings B_i of A_i/B with $[B_i:B]_l<\infty$ and $V_A(B_i)=V_A(A_i)$ $(i=1, 2)$. Then, $A_i'=V_A^2(B_i)=H[B_i]$ are finite over H (Cor. 2), $A_1'\supseteq A_2'\supseteq H$ and $A_i'\supseteq A_i\supseteq B_i$. Now, let σ_i be arbitrary elements of $\mathfrak{G}(A_i, A/B)$. Then, by Cor. 5 and (2^*), $\sigma_i|B_i=\tau_i|B_i$ for some $\tau_i\in\mathfrak{G}(A_i', A/B)$. Recalling that $A_i=(H\cap A_i)[B_i]$ (Th. 5 and Prop. 2), we see that $A_i\sigma_i=((H\cap A_i)\sigma_i)[B_i\sigma_i]\subseteq H[B_i\tau_i]=A_i'\tau_i$ (Lemma 6). And so, $\sigma_i\tau_i^{-1}$ is contained in $\mathfrak{G}(A_i'/B_i)|A_i$ (Th. 4 and Prop. 2 (a)), whence it follows $\sigma_i\in\mathfrak{G}(A_i', A/B)|A_i$. Combining this with ($2^*$), we obtain $\mathfrak{G}(A_i, A/B)=\mathfrak{G}(A_i', A/B)|A_i$. On the other hand, there holds $\mathfrak{G}(A_2', A/B)=\mathfrak{G}(A_1', A/B)|A_2'$ by (I). Hence, it follows $\mathfrak{G}(A_2, A/B)=\mathfrak{G}(A_2', A/B)|A_2=(\mathfrak{G}(A_1', A/B)|A_2')|A_2=(\mathfrak{G}(A_1', A/B)|A_1)|A_2=\mathfrak{G}(A_1, A/B)|A_2$, completing the proof.

Remark 2. Let A be a division ring, and left locally finite over B. Then, $\mathfrak{G}(B', A/B)$ is nothing but the set of all B-ring isomorphisms of B' into A, and the condition (2) is superfluous. Following [6] and [8], we consider the following conditions:

(1°) $\mathfrak{G}(B', A/B) \neq 1$ for each subring B' of A properly containing B with $[B' : B]_l < \infty$, and $\mathfrak{G}(B_1, A/B) | B_2 = \mathfrak{G}(B_2, A/B)$ for each intermediate rings $B_1 \supseteq B_2$ of A/B with $[B_1 : B]_l < \infty$.

(2°) H/B is Galois, and $\mathfrak{G}(B_1, A/B) | B_2 = \mathfrak{G}(B_2, A/B)$ for each intermediate rings $B_1 \supseteq B_2$ of A/B with $[B_1 : B]_l < \infty$.

(3°) H/B is Galois, and $\mathfrak{G}(A_1, A/B) | A_2 = \mathfrak{G}(A_2, A/B)$ for each intermediate rings $A_1 \supseteq A_2$ of A/H with $[A_1 : H]_l < \infty$.

(4°) $J(\mathfrak{G}(B', A/B), B') = B$ for each intermediate ring B' of A/B with $[B' : B]_l < \infty$.

If A/B is q-Galois (and necessarily h-q-Galois by Lemma 2), then all the conditions (1°)-(4°) are fulfilled by Remark 1 and Ths. 4, 6. Conversely, if (4°) is satisfied then A/B is q-Galois. To see this, it will suffice to prove that if $\{x_1, \dots, x_n\}$ is a subset of B' that is linearly left independent over B then there exists an element $\xi \in \mathfrak{G}(B', A/B)A_r$ such that $x_i\xi = 0$ for all $i \neq n$ and $x_n\xi \neq 0$, where B' is an arbitrary intermediate ring of A/B with $[B' : B]_l < \infty$. If $n = 2$, by (4°) there exists some $\rho \in \mathfrak{G}(B', A/B)$ with $(x_1x_2^{-1})\rho \neq x_1x_2^{-1}$, and then one will easily see that $\xi = \rho - 1(x_1^{-1} \cdot x_1\rho)_r$ is an element requested. Now, assume that we can find $\xi_1, \dots, \xi_{n-1} \in \mathfrak{G}(B', A/B)A_r$ such that $x_i\xi_j = \delta_{ij}x_i$ $(i, j = 1, \dots, n-1)$. There holds then $x_i(\sum \xi_j - 1) = 0$ for $i = 1, \dots, n-1$. If $x_n(\sum \xi_j - 1) \neq 0$, our assertion is true for $\xi = \sum \xi_j - 1$. If otherwise $x_n = \sum_1^{n-1} x_n\xi_j$ then, say, $\{x_1, x_n\xi_1\}$ is linearly left independent over B. We set here $\xi_1 = \sum_1^k \rho_p a_{pr}$ with $\rho_p \in \mathfrak{G}(B', A/B)$ and $a_p \in A$. If $B'' = B'[\cup B'\rho_p, \{a_p\text{'s}\}]$, then by the case $n = 2$ there exists an element $\xi' \in \mathfrak{G}(B'', A/B)A_r$ such that $x_1\xi' = 0$ and $x_n\xi_1\xi' \neq 0$. Now, it will be easy to see that $x_i\xi_1\xi' = 0$ for $i = 1, \dots, n-1$, so that $\xi = \xi_1\xi'$ contained in $\mathfrak{G}(B', A/B)A_r$ is an element requested.

Next, we shall prove the implications (2°) $\Rightarrow$ (4°) and (3°) $\Rightarrow$ (4°). In any rate, we have $J(\mathfrak{G}(B', A/B), B') \subseteq J(\tilde{V} | B', B') = H \cap B'$. If (2°) is satisfied then $\mathfrak{G}(H/B) | H \cap B' \subseteq \mathfrak{G}(B', A/B) | H \cap B'$, whence it follows $J(\mathfrak{G}(B', A/B), B') = B$. On the other hand, if (3°) is satisfied then $\mathfrak{G}(H/B) \subseteq \mathfrak{G}(H[B'], A/B) | H$

($[H[B'] : H]_l < \infty$ by Prop. 1 (b)), whence it follows again $J(\mathfrak{G}(B', A/B), B') = B$.

Since the implication (1°) $\Rightarrow$(4°) is obvious, we have proved that A is q-Galois if and only if any of the equivalent conditions (1°)-(4°) is satisfied (cf. [6, Th. 1] and [8, Th. 3]).

In case A/B is an algebraic field extension, it is well-known that A/B is Galois (in our sense) if and only if it is normal and separable. The next theorem may be regarded as an extension of this fact to simple rings, and contains [6, Cor. 3] as well as [4, Th. 3.5].

THEOREM 7. *If A is h-q-Galois and left locally finite over B and $[A : H]_l \leq \aleph_0$, then A/B is h-Galois and $\mathfrak{G}(A', A/B) = \mathfrak{G}|A'$ for each f-regular intermediate ring A' of A/B. In particular, if A is locally Galois over B and $[A : H]_l \leq \aleph_0$ then A/B is $\mathfrak{G}$-locally Galois.*

Proof. Since A' is f-regular, we can find an intermediate ring A'' of $A/H[E, A']$ with $[A'' : H]_l < \infty$ (Cor. 2). Now, by the validity of Cors. 2, 4 and Th. 6, we can apply the same argument as in the proof of [4, Lemma 3.9] to see that $\mathfrak{G}(A'', A/B) = \mathfrak{G}|A''$. Then, we obtain $\mathfrak{G}|A' = \mathfrak{G}(A'', A/B)|A' = \mathfrak{G}(A', A/B)$ (Th. 6), and in particular $\mathfrak{G}|H = \mathfrak{G}(H, A/B) = \mathfrak{G}(H/B)$ (Th. 4). Hence, there holds $J(\mathfrak{G}, A) = J(\mathfrak{G}|H, H) = B$. And so, A being $B\cdot V$-A-irreducible (Th. 1), A/B is h-Galois by Prop. 4. The latter assertion is [4, Th. 4.4] itself, and is clear by the former and Prop. 6.

Next, we shall prove an extension of the latter half of [2, Th. 1], that contains completely [6, Cor. 2].

THEOREM 8. *Let A/B be h-q-Galois and left locally finite. If B' is a regular intermediate ring of A/B with $[B' : B]_l < \infty$ then $\infty > [B' : B]$*[4] *$\geq [V : V_A(B')] = [V_A^2(B') : H] = [B' : H \cap B']$, and in particular A/B* is (*two-sided*) *locally finite.*

Proof. We set $V_A^2(B') = \sum K'd'_{h'k'}$, where $\Delta' = \{d'_{h'k'}\text{'s}\}$ is a system of matrix units and $K' = V_{V^2_A(B')}(\Delta')$ is a division ring (Cor. 2), and consider $T = B'[E, \Delta, \Delta']$ and $H' = H \cap T$ (simple by Th. 4 and Prop. 2). Since $H\mathfrak{G}(V_A^2(T)/T) = H$ (Cor. 4) and A is $B\cdot V$-A-irreducible (Th. 1), Prop. 1 and Lemma

[4] In case $[B' : B]_l$ coincides with $[B' : B]_r$, the equal dimensions will be denoted as $[B' : B]$.

3 yield $\infty>[T:H']_l\geq[V:V_A(T)]_l\geq[V_A^2(T):H]_r\geq[T\cdot H:H]_r=[T:H']_r$. And then, A being A-$V\cdot H$-irreducible by Cor. 2 and Prop. 4, we obtain $[T:H']_r\geq[V:V_A(T)]_r\geq[V_A^2(T):H]_l\geq[H\cdot T:H]_l=[T:H']_l$ again by Prop. 1 and Lemma 3. Hence, it follows $[T:H']=[V:V_A(T)]=[V_A^2(T):H]$ and $[T:B]_l=[T:H']_l\cdot[H':B]_l=[T:H']_r\cdot[H':B]_r=[T:B]_r$ by Prop. 2 (c). Since A/B' is h-q-Galois, by the same reason, we have $[V_A(B'):V_A(T)]=[V_A^2(T):V_A^2(B')]$ and $[T:B']_l=[T:B']_r$. Combining those above with the fact that A is $B'\cdot V'$-A-irreducible (Th. 1), it follows at once $[B':B]_r=[B':B]_l\geq[V:V_A(B')]=[V_A^2(B'):H]$ by Prop. 1 (b). Now, we shall prove $[B':H\cap B']=[V_A^2(B'):H]$. If $H^*=(H\cap B')[\varDelta]$ and $B^*=H^*[B']$ then B^* is regular as an intermediate ring of $V_A^2(B')/B'$ (Th. 4 and Prop. 2 (a)). Hence, Cor. 6 yields $[B^*:H\cap B^*]=[V_A^2(B^*):H]=[V_A^2(B'):H]$. Recalling here that $\mathfrak{H}=\mathfrak{G}(V_A^2(B')/B')=\mathfrak{G}(H[B']/B')\approx\mathfrak{G}(H/H\cap B')$ by contraction (Th. 5), Prop. 2 (c) yields $[B^*:B']=\#(\mathfrak{H}|B^*)=\#(\mathfrak{H}|H^*)=\#(\mathfrak{H}|H\cap B^*)=[H\cap B^*:H\cap B']$, whence it follows $[B':H\cap B']=[B^*:H\cap B^*]$. We have proved therefore $[B':H\cap B']=[V_A^2(B'):H]$.

LEMMA 7. *Let A be h-q-Galois and left locally finite over B. If A' is an f-regular intermediate ring of A/B then A/A' is left locally finite and* $[A':H\cap A']_l=[V:V_A(A')]$.

Proof. Let N be an arbitrary $\mathfrak{G}(H/B)$-invariant shade of $\varDelta$. Then, by Th. 5 and Prop. 2 (b), we have $[N[A']:A']=[N:N\cap A']<\infty$ and $H\cap N[A']=H\cap(N[H\cap A'])[A']=N[H\cap A']$. Since $H\cap A'$ is also a regular intermediate ring of A/B (Prop. 2 (a)), we obtain $[H\cap N[A']:H\cap A']=[N[H\cap A']:H\cap A']=[N:N\cap A']=[N[A']:A']<\infty$ again by Th. 5 and Prop. 2 (b). We choose here a simple intermediate ring B' of A'/B with $[B':B]<\infty$ and $V_A(B')=V_A(A')$, and set $B^*=N[B']$. Then, B^* is a regular subring of A with $[B^*:B]<\infty$ as an intermediate ring of $V_A^2(B')/B'$ (Th. 4 and Prop. 2). Recalling that $H[B^*]=V_A^2(B^*)\supseteq N[A']\supseteq B^*\supseteq\varDelta$, Cor. 6 and Th. 8 imply $[N[A']:H\cap N[A']]_l=[B^*:H\cap B^*]=[V:V_A(B^*)]=[V:V_A(B')]<\infty$. Combining this with $[H\cap N[A']:H\cap A']=[N[A']:A']<\infty$, it follows at once $[A':H\cap A']_l=[N[A']:H\cap N[A']]_l=[V:V_A(B')]=[V:V_A(A')]$, which is the latter assertion. Next, we shall prove the first half. Here, without loss of generality, we may assume that $A'\subseteq H$. For an arbitrary finite subset F of A, we set $B_1=B[E,\varDelta,F]$. Then, $[A'[H\cap B_1]:A']<\infty$ by Prop. 2 and

$[A'[B_1] : A'[H\cap B_1]]_l = [B_1 : H\cap B_1]_l \leq [B_1 : B] < \infty$ by Cor. 6. It follows therefore $[A'[F] : A']_l \leq [A'[B_1] : A'[H\cap B_1]]\cdot[A'[H\cap B_1] : A'] < \infty$.

The next theorem contains evidently [6, Ths. 2 and 4].

THEOREM 9. *Let A be h-q-Galois and left locally finite over B. If A' is an f-regular intermediate ring of A/B then A is h-q-Galois, right h-q-Galois and locally finite over A' and $[A' : H\cap A'] = [V : V_A(A')] = [V_A^2(A') : H]$.*

Proof. To prove the first assertion, we may restrict our attention to the case that $A'\subseteq H$. If A'' is a regular intermediate ring of A/A' with $[A'' : A']_l < \infty$ then, to be easily verified, A'' is f-regular. Since $A_0 = A''[E, \Delta]$ is left finite over A' (Lemma 7), $\mathfrak{G}(A'', A/A')A_r = (\mathfrak{G}(A_0, A/A')|A'')A_r$ (Th. 6). And so, we see that it suffices to prove that $\mathrm{Hom}_{A_l'}(A'', A) = \mathfrak{G}(A'', A/A')A_r$ for each intermediate ring A'' of $A/A'[E, \Delta]$ with $[A'' : A']_l < \infty$. By Th. 4 and Prop. 2 (a), $H'' = A''\cap H$ is a simple subring of H. As $\mathfrak{G}(H/B)|H'' = \mathfrak{G}(H, A/B)|H'' = \mathfrak{G}(V_A^2(A''), A/B)|H''$ (Ths. 4 and 6), it follows $\mathfrak{G}(H/A')|H'' = \mathfrak{G}(V_A^2(A''), A/A')|H''$ (Prop. 2 (b)). Recalling that $\mathfrak{G}(H/A')H_r$ is dense in $\mathrm{Hom}_{A_l'}(H, H)$ (Prop. 2) and that $[H'' : A'] < \infty$ (Prop. 2 (c) or Th. 8), we have then $\mathrm{Hom}_{A_l'}(H'', H) = (\mathfrak{G}(V_A^2(A''), A/A')|H'')H_r = \oplus_1^s(\sigma_i|H'')H_r$ with some $\sigma_i\in\mathfrak{G}(V_A^2(A''), A/A')$ (Lemma 2). Since $\sigma_i|H''\neq\sigma_j|H''(i\neq j)$, irreducible $(\sigma_i|A'')A_r$ is not A_r''-A_r-isomorphic to $(\sigma_j|A'')A_r$ (Lemma 2), which implies $\sum_1^s(\sigma_i\tilde{V}|A'')A_r = \oplus_1^s(\sigma_i\tilde{V}|A'')A_r$. By [4, Lemma 1.5] and Th. 8, there holds $[(\tilde{V}|A'')A_r : A_r]_r = [V : V_A(A'')] = [V_A^2(A'') : H]$. On the other hand, the same reason together with Ths. 4 and 6 implies $\infty > [(\sigma_i\tilde{V}|A'')A_r : A_r]_r = [(\tilde{V}|A''\sigma_i)A_r : A_r]_r = [V : V_A(A''\sigma_i)] = [V_A^2(A''\sigma_i) : H] = [(H[A''])\sigma_i : H\sigma_i] = [V_A^2(A'') : H]$. It follows therefore $[(\sigma_i\tilde{V}|A'')A_r : A_r]_r = [V : V_A(A'')]$, whence we obtain $[\sum_1^s(\sigma_i\tilde{V}|A'')A_r : A_r]_r = s\cdot[V : V_A(A'')] = [\mathrm{Hom}_{A_l'}(H'', H) : H_r]_r\cdot[V : V_A(A'')] = [H'' : A']\cdot[A'' : H'']_l = [A'' : A']_l$ by Lemma 7. We have proved therefore $\mathrm{Hom}_{A_l'}(A'', A) = \sum_1^s(\sigma_i\tilde{V}|A'')A_r = \mathfrak{G}(A'', A/A')A_r$ by (2^*), and A/A' is locally finite by Lemma 7 and Th. 8. The final equalities are now direct consequences of Lemma 7 and Th. 8, for $A'\cap H$ is f-regular. In particular, noting that $[A' : H\cap A'] = [V : V_A(A')]$, we can repeat a symmetric argument to see that A/A' is right h-q-Galois.

COROLLARY 7. *The following conditions are equivalent to each other:*

(Q) *A/B is h-q-Galois and left locally finite.*

(Q$'$) *A/B is right h-q-Galois and right locally finite.*

Combining Th. 9 with Th. 7, we readily obtain the following:

COROLLARY 8. *Let A be h-q-Galois and left locally finite over B and $[A : H]_l \leq \aleph_0$. If A' is an f-regular intermediate ring of A/B then A/A' is h-Galois and locally finite.*

Now, we shall add to Prop. 4 other equivalent conditions to complete [2, Th. 1].

PROPOSITION 7. *Let B be a regular subring of A. A/B is h-Galois and left locally finite over B if and only if any of the following conditions is satisfied:*

(D) *A is Galois and left locally finite over B, H is simple, and $[V_A^2(B') : H]_l = [V : V_A(B')]_r$ for every regular intermediate ring B' of A/B with $[B' : B]_l < \infty$.*

(D$'$) *A is Galois and right locally finite over B, H is simple, and $[V_A^2(B') : H]_r = [V : V_A(B')]_l$ for every regular intermediate ring B' of A/B with $[B' : B]_r < \infty$.*

(E) *A is left locally finite over B and Galois over every regular subring left finite over B, H is simple, and $[A' : H]_l = [V : V_A(A')]_r$ for every regular intermediate ring A' of A/H with $[A' : H]_l < \infty$.*

(E$'$) *A is right locally finite over B and Galois over every regular subring right finite over B, H is simple, and $[A' : H]_r = [V : V_A(A')]_l$ for every regular intermediate ring A' of A/H with $[A' : H]_r < \infty$.*

Proof. Since (A) $\Rightarrow$ (D) and (E) is evident by Cor. 1 and Th. 9, it is left to prove the converse. Now, let T be an arbitrary intermediate ring of A/B $[E, \Delta]$ with $[T : B]_l < \infty$, and set $T' = J(\mathfrak{G}(T), A)$ and $H' = H \cap T'$. Then, $[H' : B] < \infty$ by Prop. 3 (b). Noting that A is $H'[T]$-A-irreducible, Prop. 1 (b) yields $\infty > [H'[T] : H']_l \geq [V_A(H') : V_A(H'[T])]_r = [V : V_A(T')]_r$, whence it follows $[T' : H']_l \geq [V : V_A(T')]_r$. In case (D), Lemma 3 yields then $[T' : H']_l = [H \cdot T' : H]_l \leq [V_A^2(T') : H]_l = [V_A^2(T) : H]_l = [V : V_A(T)]_r = [V : V_A(T')]_r$. Hence, we have $[T' : H']_l = [V : V_A(T')]_r < \infty$, so that it follows $\mathrm{Hom}_{B_l}(T', A) = (\mathfrak{G} | T')A_r$ by Prop. 3 (a), which proves (D) $\Rightarrow$ (A). Now, we shall prove (E) $\Rightarrow$ (A). If N is an arbitrary $\mathfrak{G}(H/B)$-invariant shade of H', then $\mathfrak{G}(T) | N[T]$ and $\mathfrak{G}(T) | N$ are (outer) Galois groups of $N[T]/T$ and N/H', respectively. There holds then $[N : H'] = \# (\mathfrak{G}(T) | N) = \# (\mathfrak{G}(T) | N[T])$

$=[N[T]:T]$ (Prop. 2 (c)), and so Lemma 3 yields $[N\cdot T:H']_l=[N\cdot T:N]_l \cdot[N:H']=[T:H']_l\cdot[N[T]:T]=[N[T]:H']_l$, whence we obtain $N\cdot T=N[T]$. We readily see then $H\cdot T$ is a regular intermediate ring of A/H with $[H\cdot T:H]_l=[T:H']_l<\infty$. It follows therefore $[T:H']_l=[H\cdot T:H]_l=[V:V_A(T)]_r$, and we have $\mathrm{Hom}_{B_l}(T, A)=(\mathfrak{G}|T)A_r$ again by Prop. 3 (a).

We shall present here a notably short proof to [4, Lemma 2.2][5].

Proposition 8. *If A is Galois and left locally finite over B and $[V:C]<\infty$, then A/B is $\mathfrak{G}$-locally Galois.*

Proof. By the validity of Prop. 6, it suffices to prove that A/B is locally Galois. To be easily seen, (H is simple and) $[V_A^2(B'):H]_l=[V:V_A(B')]_r$ for each regular intermediate ring B' of A/B with $[B':B]_l<\infty$. A/B is therefore h-Galois by Prop. 7. We set here $V=\sum Ug_{pq}$, where $\Gamma=\{g_{pq}\text{'s}\}$ is a system of matrix units and $U=V_V(\Gamma)$ a division ring. Now, let B' be an arbitrary intermediate ring of $A/B[E, \Gamma]$ with $[B':B]_l<\infty$. Since $J(\mathfrak{G}|B', B')=B$, there exists a finite subset $\mathfrak{F}$ of $\mathfrak{G}$ with $J(\mathfrak{F}|B', B')=B$. If N is an arbitrary $\mathfrak{G}(H/B)$-invariant shade of $B'[\bigcup_{\sigma\in\mathfrak{F}} B'\sigma]\cap H$ then $B'[\cup B'\sigma]$ is contained in the simple ring $M=N[B']$ (Th. 5 and Prop. 2 (b)). And so, $\mathfrak{H}=\mathfrak{G}(B')[\mathfrak{F}]$ induces an automorphism group of M. Since $J(\mathfrak{H}|M, M)=B$ and $V_M(B)$ is evidently simple, M/B is Galois, which implies that A/B is locally Galois.

We shall conclude this section with the following theorem, whose first assertion is [4, Lemma 4.2].

Theorem 10. (a) *If A/B is locally Galois then H is simple and for each finite subset F of A there exists a simple intermediate ring A' of $A/H[F]$ such that $[A':H]_l<\infty$ and A'/B is Galois, and conversely provided A/B is left locally finite.*

(b) *If A/B is locally Galois then so is A/A' for every f-regular intermediate ring A' of A/B.*

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[5] The proof of Prop. 8 given in [4] enabled us moreover to see that there exists a Galois group $\mathfrak{H}$ of A/B with the property that $(\mathfrak{H}[\mathfrak{F}], A/B)$ is l.f.d. for each finite subset $\mathfrak{F}$ of $\mathfrak{G}$, which was needed only to prove the following: If A is Galois and left locally finite over B and $[V:C]<\infty$, then every $(*)$-regular subgroup of $\mathfrak{G}$ is regular. However, in [2] and [10], we have proved directly an extension of the last proposition (cf. also Th. 11 (a)).

Proof. (a) Let $V=\sum Ug_{pq}$, where $\Gamma=\{g_{pq}\text{'s}\}$ is a system of matrix units and $U=V_V(\Gamma)$ a division ring. If B' is an arbitrary shade of $B[E, \Gamma]$, then $A'=V_A^2(B')=H[B']=\bigcup N_\alpha[B']$, where N_α ranges over all the $\mathfrak{G}(H/B)$-invariant shades. Now, let B'' be a shade of $N_\alpha[B']$, and $\mathfrak{G}'=\{\sigma\in\mathfrak{G}(B''/B)\ ;\ B'\sigma=B'\}$. Then, noting that $\mathfrak{G}(B'/B)\subseteq\mathfrak{G}'|B'$, Th. 5 together with Lemma 6 and Prop. 2 proves that $N_\alpha[B']/B$ is Galois. Hence, A'/B is locally Galois, and so it is Galois by Th. 7, for $[V_{A'}(B) : V_{A'}(A')]=[V_{A'}(H) : V_{A'}(A')]\leq[A' : H]_l<\infty$ (Prop. 1). And, by the fact used just above, the converse part will be an easy consequence of Prop. 8.

(b) If B' is an intermediate simple ring of A'/B with $[B' : B]_l<\infty$ and $V_A(B')=V_A(A')$, then A/B' is locally Galois. And so, by (a), for each finite subset F of A there exists a simple intermediate ring A'' of $A/V_A^2(B')[F]$ such that A''/B' is Galois and $[V_{A''}(B') : V_{A''}(A'')]\leq[A'' : V_A^2(B')]_l<\infty$. Prop. 8 implies then that A''/B' is $\mathfrak{G}(A''/B')$-locally Galois. Since A''/A' is h-Galois and locally finite by Cor. 8, A''/A' is locally Galois again by Prop. 8. We have proved therefore A/A' is locally Galois.

4. $(*_f)$-Regular Subgroups. By the validity of Ths. 4, 9 and Cor. 2 (and Lemma 3 if necessary), the proofs of Lemmas 2, 3 of [10] are applicable without any change to those of the following lemmas.

LEMMA 8. *Let A be h-q-Galois and left locally finite over B, and $\mathfrak{G}'$ a $(*_f)$-regular subgroup of $\mathfrak{G}$. If $A'=J(\mathfrak{G}', A)$ then $[A' : H\cap A']_l<\infty$.*

LEMMA 9. *Let A be h-q-Galois and left locally finite over B, and V' a simple subring of V with $[V : V']_r<\infty$. If $V_A(V_A(V')[F])\subseteq V'$ for some finite subset F of A then $V_A(V')$ is a simple ring.*

The first assertion of the following theorem contains [10, Th. 2].

THEOREM 11. *Let A be h-q-Galois and left locally finite over B, and $\mathfrak{G}'$ a $(*_f)$-regular subgroup of $\mathfrak{G}$ with $A'=J(\mathfrak{G}', A)$.*

(a) *$\mathfrak{G}'$ is f-regular (i.e. A' is simple) and dense in $\mathfrak{G}(A')$.*

(b) $\tilde{V}\cdot\mathrm{Cl}\,\mathfrak{G}'$[6] $=\mathfrak{G}(H\cap A')$.

(c) *If $\mathfrak{H}$ is an open subgroup of $\mathfrak{G}$ then $(\mathrm{Cl}\mathfrak{G}' : (\mathfrak{H}\cap\mathrm{Cl}\mathfrak{G}')\tilde{V}_{\mathfrak{G}'})<\infty$.*

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[6] $\mathrm{Cl}\,\mathfrak{G}'$ is the topological closure of $\mathfrak{G}'$ in $\mathfrak{G}$.

Proof. One may remark here that $H' = H \cap A'$ is f-regular (Th. 4 and Prop. 2). As $[V : V_{\mathfrak{G}'}]_r < \infty$ and $V_{\mathfrak{G}'} = V_A^2(V_{\mathfrak{G}'})$, $V_A^2(A') = V_A(V_{\mathfrak{G}'})$ is simple by Lemma 9. Further, by Lemma 8, there holds $[A' : H']_l < \infty$. Since A/H' is locally finite (Th. 9), $V_{V_A^2(A')}(A')$ coincides with the center of $V_A^2(A')$ and $J(\mathfrak{G}' | V_A^2(A'), V_A^2(A')) = A'$, [10, Lemma 1] proves that A' is simple. And so, A/A' is h-q-Galois and locally finite (Th. 9). If T is an arbitrary intermediate ring of $A/A'[E]$ with $[T : A'] < \infty$, then A is T-A-irreducible and $[T : V_A^2(A') \cap T] = [V_A(A') : V_A(T)]$ (Th. 8). Hence, A/A' is h-Galois and $\mathfrak{G}'$ is dense in $\mathfrak{G}(A')$ by Prop. 3 (a), which completes the proof of (a). Recalling here that $[T : H']_l = [T : A']_l \cdot [A' : H']_l < \infty$ (Lemma 8), for each $\sigma \in \mathrm{Cl}(\tilde{V} \cdot \mathrm{Cl}\mathfrak{G}')$ we can find such an element $\tau \in \tilde{V} \cdot \mathrm{Cl}\,\mathfrak{G}'$ that $\tau | T = \rho | T$. And then $\sigma\tau^{-1}$ is contained in $\mathfrak{G}(T) \subseteq \mathfrak{G}(A') = \mathrm{Cl}\,\mathfrak{G}'$ by (a). Hence, σ is contained in $\tilde{V} \cdot \mathrm{Cl}\,\mathfrak{G}'$, which means that $\tilde{V} \cdot \mathrm{Cl}\,\mathfrak{G}'$ is a closed $(*_f)$-regular subgroup of $\mathfrak{G}$ with $J(\tilde{V} \cdot \mathrm{Cl}\,\mathfrak{G}', A) = H'$. Accordingly, (b) is a consequence of (a). Finally, we shall prove (c). Since $J(\mathrm{Cl}\,\mathfrak{G}', A) = A'$ and $V_{\mathrm{Cl}\,\mathfrak{G}'} = V_{\mathfrak{G}'}$, it suffices to prove our assertion for closed $\mathfrak{G}' = \mathfrak{G}(A')$. Moreover, without loss of generality, we may assume that $\mathfrak{H} = \mathfrak{G}(B')$ for some intermediate ring B' of $A/B[E]$ with $[B' : B]_l < \infty$. If $T = A'[B']$ (finite over A') then $\mathfrak{G}'(T)$ is a closed $(*_f)$-regular subgroup of $\mathfrak{G}'$ with $J(\mathfrak{G}'(T), A) = T$ by Cor. 1 or [5, Theorem 1]. And so, by (b), it follows $(\mathfrak{H} \cap \mathfrak{G}')\tilde{V}_{\mathfrak{G}'} = \mathfrak{G}'(T)\widetilde{V_A(A')} = \mathfrak{G}'(V_A^2(A') \cap T)$. Hence, by Th. 4 and Prop. 2 (c), we obtain $(\mathfrak{G}' : (\mathfrak{H} \cap \mathfrak{G}')\tilde{V}_{\mathfrak{G}'}) = (\mathfrak{G}' : \mathfrak{G}'(V_A^2(A') \cap T)) = \#(\mathfrak{G}' | V_A^2(A') \cap T) = [V_A^2(A') \cap T : A'] < \infty$.

As a direct consequence of Th. 11 (a) and Cors. 1, 8, we readily obtain the following theorem.

THEOREM 12. *If A is h-q-Galois and left locally finite over B and $[A : H]_l \leq \aleph_0$ then there exists a* 1-1 *dual correspondence between closed* $(*_f)$*-regular subgroups and f-regular intermediate rings of A/B, in the usual sense of Galois theory.*

Remark 3. Evidently, Th. 12 is nothing but [2, Th. 5], and the assumption cited in Th. 12 is the best one obtained by now to allow the existence of Galois correspondence.

Let A/B be h-q-Galois and left locally finite. If T is an intermediate ring of A/B left finite over B such that A is T-A-irreducible and $J(\mathfrak{G}(T), A) = T$, then T is a simple ring by Th. 11 (a). In particular, if A/B is h-Galois then the assumption $J(\mathfrak{G}(T), A)=T$ is automatically enjoyed by [5, Th. 1] (cf. [2, Cor. 6]). The next will be an easy consequence of the above remark, Th. 1 and [4, Lemma 1.1].

PROPOSITION 9. *Let A/B be locally h-Galois and left locally finite. If V is a division ring then every intermediate ring of A/B is simple.*

Remark 4. Let A be left algebraic over B (that is, $[B[a]:B]_l<\infty$ for every $a\in A$). If every intermediate ring of A/B left finite over B is a simple ring then V is a division ring. In fact, for an arbitrary non-zero element $v\in V$, $B[v]$ is a simple ring, and so the center of $B[v]$ is a field. Hence, v belonging to the center of $B[v]$ is regular and v^{-1} is contained in V.

We shall conclude our study with the following (cf. [2, Th. 2]).

THEOREM 13. *Let A be h-q-Galois and left locally finite over B, and $\mathfrak{G}'$ an N-regular subgroup of $\mathfrak{G}$. Then, $\mathfrak{G}'$ is $(*_f)$-regular if and only if $[V:I(\mathfrak{G}')]_r<\infty$, $V_A^2(I(\mathfrak{G}'))=I(\mathfrak{G}')=I(\mathrm{Cl}\,\mathfrak{G}')$ and $(\mathrm{Cl}\,\mathfrak{G}':(\mathfrak{H}\cap \mathrm{Cl}\,\mathfrak{G}')\widetilde{I(\mathfrak{G}')})<\infty$ for every open subgroup $\mathfrak{H}$ of $\mathfrak{G}$.*

Proof. If $\mathfrak{G}'$ is $(*_f)$-regular then $I(\mathfrak{G}')$ coincides with $V_{\mathfrak{G}'}$, so that the only if part is obvious by Th. 11. To prove the if part, we may restrict our proof to the case that $\mathfrak{G}'$ is closed. By Th. 11 (a), $V_A(I(\mathfrak{G}'))$ is simple and there exists a finite subset F of $V_A(I(\mathfrak{G}'))$ with $V_A(B[F])=I(\mathfrak{G}')$. If we set $\mathfrak{H}=\mathfrak{G}(B[F])$, $\mathfrak{G}^*=\mathfrak{H}\cap\mathfrak{G}'$ is a subgroup of $\mathfrak{H}$ containing $\widetilde{I(\mathfrak{G}')}$. And so, there holds $B[F]\subseteq J(\mathfrak{G}^*, A)\subseteq V_A(I(\mathfrak{G}'))$, which implies $I(\mathfrak{G}')=V_A(B[F])\supseteq V_{\mathfrak{G}}\supseteq V_A^2(I(\mathfrak{G}'))=I(\mathfrak{G}')$. We see therefore $\mathfrak{G}^*$ is a closed $(*_f)$-regular subgroup of $\mathfrak{G}$ with $V_{\mathfrak{G}^*}=I(\mathfrak{G}')$. By assumption, $(\mathfrak{G}':\mathfrak{G}^*)<\infty$: $\mathfrak{G}'=\cup_1^m\mathfrak{G}^*\sigma_i$. Now, we set $A^*=J(\mathfrak{G}^*, A)$ and $A'=J(\mathfrak{G}', A)$. Then $\mathfrak{G}^*=\mathfrak{G}(A^*)$ and A is h-Galois and locally finite over A^* (Th. 11 (a) and its proof). And hence, by Th. 4 and Prop. 2, $A^{**}=A^*[\cup_1^m A^*\sigma_i]$ is a $\mathfrak{G}'$-invariant simple ring as an intermediate ring between $V_A^2(A^*)=V_A(V_{\mathfrak{G}^*})=V_A(I(\mathfrak{G}'))$ and A^*. If an element $\sigma\in\mathfrak{G}'$ induces an inner automorphism in A^{**} : $\sigma|A^{**}=\tilde{v}|A^{**}(v\in V_{A^{**}}(A'))$ then $\sigma|\mathfrak{H}\cap A^*=1$, and so σ is contained in $\mathfrak{G}(H\cap A^*)=\mathfrak{G}^*\tilde{V}$ (Th. 11 (b)) : $\sigma=\tau\tilde{u}(\tau\in\mathfrak{G}^*, \tilde{u}\in\tilde{V})$. But then, $\tau^{-1}\sigma=\tilde{u}\in\mathfrak{G}'\cap\tilde{V}=\widetilde{I(\mathfrak{G}')}$ implies $\sigma\in\tau\widetilde{I(\mathfrak{G}')}\subseteq\mathfrak{G}^*$, so that v is contained in $V_{A^{**}}(A^*)=V_{A^{**}}(A^{**})$. Hence, $\sigma|A^{**}=\tilde{v}|A^{**}=1$, which means $\mathfrak{G}'|A^{**}$ is an outer group of finite order. Accordingly, as is well-known, A^{**} is outer Galois and finite over the simple ring A'. Moreover, noting that $\mathfrak{G}^*=\mathfrak{G}^*(\tilde{V}$

$\cap \mathfrak{G}') = \mathfrak{G}^*\tilde{V} \cap \mathfrak{G}' = \mathfrak{G}(H \cap A^*) \cap \mathfrak{G}' = \mathfrak{G}'(H \cap A^*)$, we obtain $[A^* : A'] = \# (\mathfrak{G}'|A^*) = (\mathfrak{G}' : \mathfrak{G}^*) = \# (\mathfrak{G}'| A'[H \cap A^*]) = [A'[H \cap A^*] : A']$ by Prop. 2 (c), whence there holds $A^* = A'[H \cap A^*]$. We see therefore our assertion $I(\mathfrak{G}') = V_{\mathfrak{G}^*} = V_{\mathfrak{G}'}$.

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