HOMOLOGICAL INVARIANTS OF LOCAL RINGS

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Introduction

In this paper R is a commutative noetherian local ring with unit element 1 and M is its maximal ideal. Let K be the residue field R/M and let $\{t_1, t_2, \ldots, t_n\}$ be a minimal system of generators for M. By a complex $R<T_1, \ldots, T_\rho>$ we mean an R-algebra* obtained by the adjunction of the variables $T_1, \ldots, T_\rho$ of degree 1 which kill $t_1, \ldots, t_\rho$. The main purpose of this paper is, among other things, to construct an R-algebra resolution of the field K, so that we can investigate the relationship between the homology algebra H ($R<T_1, \ldots, T_n>$) and the homological invariants of R such as the algebra $\mathrm{Tor}^R(K, K)$ and the Betti numbers $B_\rho = \dim_K \mathrm{Tor}^R_\rho(K, K)$ of the local ring R. The relationship was initially studied by Serre [5]. Then Tate [6] gave the correct lower bound for the Betti numbers of a nonregular local ring. In his M. I. T. lecture (See a footnote of [6]) Eilenberg proves that

$$B_2 = \binom{n}{2} + \binom{n}{0}b_1 \text{ and } B_3 \geq \binom{n}{3} + \binom{n}{1}b_1,$$

where $b_1 = \dim_K H_1(R<T_1, \ldots, T_n>)$. In this paper these results of Eilenberg are generalized as follows:

$$B_3 = \binom{n}{3} + \binom{n}{1}b_1 + \varepsilon_2,$$

$$B_4 = \binom{n}{4} + \binom{n}{2}b_1 + \binom{n}{0}b_1^2 - \binom{b_1}{2} + \varepsilon_2\binom{n}{1} + \varepsilon_3\binom{n}{0},$$

and so forth, where $\varepsilon_2 = \dim_K H_2(\Lambda)/H_1(\Lambda)^2$, $\varepsilon_3 = \dim_K H_3(\Lambda)/H_1(\Lambda)\cdot H_2(\Lambda)$, and $\Lambda = R<T_1, \ldots, T_n>$. As corollaries of the above computation we obtain part of the results by Tate [6],

$$B_\rho \geq \binom{n}{\rho} + \binom{n}{\rho-2} + \binom{n}{\rho-4} + \cdots, \text{ for } \rho \leq 4,$$

Received May 14, 1962.

* For definition, see a paper of Tate [6]. Throughout the paper the numbers in square brackets refer to the papers of the bibliography at the end of the paper.

if R is not regular.

If R is a complete intersection, we have

$$B_3=\binom{n}{3}+\binom{n}{1}b_1,$$

$$B_4=\binom{n}{4}+\binom{n}{2}b_1+\binom{n}{0}b_1^2-\binom{b_1}{2}.$$

§ 1. **The complex $R<T_1, \cdots, T_p>$**

Let us consider a filtered complex $\Lambda=R<T_1, \ldots, T_n>$ with an increasing sequence of subcomplexes $R\subset R<T_1>\subset R<T_1, T_2>\subset\cdots\subset R<T_1, \ldots, T_p>\subset\cdots\subset\Lambda$.

Then the graded differential algebra Λ over R (in the sequel we shall call it simply "R-algebra" in the sense of Tate) has the increasing filtration $\{R<T_1, \ldots, T_p>\}$ such that $R<T_1, \ldots, T_p>$ is an R-subalgebra. Defining R-modules

$$D_{p,q}=H_{p+q}(R<T_1, \ldots, T_p>)$$
$$E_{p,q}=H_{p+q}(R<T_1, \ldots, T_p>/R<T_1, \ldots, T_{p-1}>),$$

we have the usual exact sequence

$$\cdots\xrightarrow{k}D_{p-1,q+1}\xrightarrow{i}D_{p,q}\xrightarrow{j}E_{p,q}\xrightarrow{k}D_{p-1,q}\xrightarrow{i}\cdots$$

for each pair $(R<T_1, \ldots, T_p>, R<T_1, \ldots, T_{p-1}>)$.

Thus the exact couple $C(\Lambda)=<D, E; i, j, k>$ is associated with R-algebra Λ, where

$$D=\sum_{p,q}D_{p,q} \quad\text{and}\quad E=\sum_{p,q}E_{p,q}.$$

LEMMA 1.1.

$$E_{p,q}\simeq D_{p-1,q}$$

Proof. It is sufficient to show chain equivalences λ and μ

$$R<T_1, \ldots, T_p>/R<T_1, \ldots, T_{p-1}>\underset{\mu}{\overset{\lambda}{\rightleftarrows}}R<T_1, \ldots, T_{p-1}>$$

such that $\lambda\mu=1$ and $\mu\lambda=1$. Let x be a homogeneous element of degree $p+q$ in $R<T_1, \ldots, T_p>$. Then $x=x_1+x_2\cdot T_p$, where x_1 and x_2 are homogeneous elements of $R<T_1, \ldots, T_{p-1}>$ with degrees $p+q$ and $p+q-1$ respectively.

Obviously the residue class $\bar{x}$ is represented by $x_2 \cdot T_p$. Define $\lambda(\bar{x}) = x_2$. It is immediate to verify that λ is well defined and is a chain mapping. Defining μ by

$$\mu(y) = \overline{y \cdot T_p},$$

we see by straightforword computation that λ and μ are chain equivalences. This completes the proof.

By replacing the E-terms by the corresponding isomorphic D-terms, the exact couple $C(\Lambda)$ can be developed into a "lattice-like" diagram

$$\begin{tikzcd}
D_{n,0} \arrow[r,"j"] \arrow[d,equal] & \cdots \arrow[r] & D_{2,0} \arrow[r,"j"] \arrow[d,"i"] & D_{1,0} \arrow[r,"{k = -t_2}"] & D_{1,0} \arrow[r,"j"] \arrow[d,"i"] & D_{0,0} \arrow[r,"{k = \times t_1}"] & D_{0,0} = R \arrow[d,"i"] & \\
H_n(\Lambda) & \cdots \arrow[r] & D_{3,-1} \arrow[r,"j"] \arrow[d] & D_{2,-1} \arrow[r,"{k = -t_3}"] & D_{2,-1} \arrow[r,"j"] \arrow[d,"i"] & D_{1,-1} \arrow[r,"{k = \times t_2}"] & D_{1,-1} \arrow[r] \arrow[d,"i"] & 0 \\
 & & \vdots & {} \arrow[r] & D_{3,-2} \arrow[r,"j"] \arrow[d] & D_{2,-2} \arrow[r,"{k = \times t_3}"] & D_{2,-2} \arrow[r] \arrow[d] & 0 \\
C(\Lambda): & & \vdots \arrow[d,"i"] & & \vdots \arrow[d,"i"] & & \vdots & \\
 & & D_{n,-n+2} \arrow[r,"j"] \arrow[d,equal] & D_{n-1,-n+2} \arrow[r,"{k = -t_n}"] & D_{n-1,-n+2} \arrow[d,"i"] & & & \\
 & & H_2(\Lambda) & & D_{n,-n+1} \arrow[r,"j"] \arrow[d,equal] & D_{n-1,-n+1} \arrow[r,"{k = \times t_n}"] & D_{n-1,-n+1} \arrow[r] \arrow[d] & 0 \\
 & & & & H_1(\Lambda) & & D_{n,-n} \arrow[r] \arrow[d,equal] & 0 \\
 & & & & & & H_0(\Lambda) = M &
\end{tikzcd}$$

The steps from upper left to lower right are exact sequences. It is easy to see that $k_{p,q}$; $D_{p,q} \longrightarrow D_{p,q}$ is the multiplication by $(-1)^{p+q} t_{p+1}$. This diagram provides us with the whole story about the following known results which have been proved by several authors [2], [6].

PROPOSITION 1.2. *The following statements are equivalent.*

i) $H_1(\Lambda) = 0$

ii) $H_\rho(R< T_1, \ldots, T_p>) = 0$ *for any* $\rho \geq 1$ *and for any* $p (n \geq p \geq 0)$.

iii) $\{t_1, t_2, \ldots, t_n\}$ *is an R-sequence.*

iv) R *is regular.*

Proofs. i) $\rightarrow$ ii) Since $H_1(A)=0$, $k_{n-1,-n+2}$ which is the multiplication by $-t_n$, is onto. It follows that any element $x \in D_{n-1,-n+2}$ belongs to $\bigcap_{\rho=0}^{\infty} M^{\rho} \cdot D_{n-1,-n+2}$. By virtue of Krull (for example see [7]) x vanishes, because $D_{n-1,-n+2}=H_1(R<T_1, \ldots, T_{n-1}>)$ is a noetherian module over R. By the repeated use of the same argument, we can prove that $H_1(R<T_1, \ldots, T_p>) = D_{p,-p+1}$ vanishes for all $p(n \geq p \geq 1)$. Then $i_{p,-p+2}$: $D_{p,-p+2} \rightarrow D_{p+1,-p+1}$ are all onto, because of the exactenss of the diagram $C(A)$. Since $D_{2,0}$ vanishes*, all $H_2(R<T_1, \ldots, T_p>)$ vanish. By repeating this process the proof of i) $\rightarrow$ ii) is established. ii) $\rightarrow$ iii) It is immediate by definition that $D_{p,-p}=H_0(R<T_1, \ldots, T_p>)=R/(t_1, \ldots, t_p)$. Since $k_{p,-p}$ is isomorphic, t_{p+1} is a non zero divisor for $R/(t_1, \ldots, t_p)$. This completes the proof.

iii) $\rightarrow$ iv) It is immediate by definition.

iv) $\rightarrow$ i) Without loss of generality we may assume that $\{t_1, \ldots, t_n\}$ is an R-sequence. Then all $k_{p,-p}$ are isomorphic so that all $i_{p,-p+1}$ are onto. Since $D_{1,0}=0$ in this case, we have $H_1(A)=0$.

§2. Construction of a minimal algebra resolution

Let us denote by b_ρ $\dim_K H_\rho(A)$ and let 1-cycles $\mathfrak{Z}_1^1, \ldots, \mathfrak{Z}_{b_1}^1$ represent the homology classes $Z_1^1, \ldots, Z_{b_1}^1 \in H_1(A)$ respectively. Then by adjoining $S_1, \ldots, S_{b_1}$ of degree 2 which kill the cycles $\mathfrak{Z}_1^1, \ldots, \mathfrak{Z}_{b_1}^1$ we obtain an R-algebra

$$A^{(2)}=A<S_1, \ldots, S_{b_1}>;\ \partial_2^{(2)} S_i=\mathfrak{Z}_i^1,$$

satisfying the following conditions:

a) $A^{(2)} \supset A=A^{(1)}$, and $A_\lambda^{(2)}=A_\lambda$ for $\lambda<2$,

b) $H_1(A^{(2)})=0$.

Let

$$V_\rho=H_\rho(A)/(H_{\rho-1}(A) \cdot H_1(A)+H_{\rho-2}(A) \cdot H_2(A)+\cdots+H_{\rho-\lambda}(A) \cdot H_\lambda(A))$$

for $\rho \geq 2$, where $\lambda=\frac{\rho}{2}$ if ρ is even and $\lambda=\frac{\rho-1}{2}$ if ρ is odd, and let $\varepsilon_\rho=\dim_K V_\rho$. Selecting ρ-cycles $\mathfrak{Z}_1^\rho, \ldots, \mathfrak{Z}_{\varepsilon_\rho}^\rho$ representing the homology classes $Z_1^\rho, \ldots, Z_{\varepsilon_\rho}^\rho \in V_\rho$ and adjoining $U_1^{\rho+1}, \ldots, U_{\varepsilon_\rho}^{\rho+1}$ of degree $\rho+1$, we have an R-algebra

* For t_1 is a non-zero divisor for R.

$$A^{(\rho+1)} = A^{(\rho)} < U_1^{\rho+1}, \dots, U_{\varepsilon_\rho}^{\rho+1} > ; \ \partial_{\rho+1}^{(\rho+1)} U_i^{\rho+1} = \mathfrak{Z}_i^{\rho}$$

satisfying

$$\begin{aligned} &\text{a) } A^{(\rho+1)} \supset A^{(\rho)},\ A_\lambda^{(\rho+1)} = A_\lambda^{(\rho)} \text{ for } \lambda < \rho+1 \\ &\text{and } A_{\rho+1}^{(\rho+1)} = A_{\rho+1}^{(\rho)} \oplus RU_1^{\rho+1} \oplus \cdots \oplus RU_{\varepsilon_\rho}^{\rho+1} \\ &\text{b) } H_\rho(A^{(\rho+1)}) = H_\rho(A^{(\rho)})/RZ_1^\rho + \cdots + RZ_{\varepsilon_\rho}^\rho \\ &\qquad = H_\rho(A^{(\rho)})/V_\rho \end{aligned}$$

Letting $X_\rho = A_\rho^{(\rho)}$ and defining $\partial_{\rho+1} : X_{\rho+1} \to X_\rho$ by $\partial_{\rho+1} = \partial_{\rho+1}^{(\rho+1)}$, we obtain an R-algebra $X = \bigcup\limits_\rho X_\rho$

$$X\colon\ \longrightarrow X_{\rho+1} \xrightarrow{\partial_{\rho+1}} X_\rho \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\varepsilon} K \longrightarrow 0$$

where $X_0 = R$ and the mapping ε is the augmentation homomorphism.

Defining vector spaces over K, $D_{p,q} = H_{p+q}(A^{(p)})$ and $E_{p,q} = H_{p+q}(A^{(p)}/A^{(p-1)})$, we obtain a spectral sequence

$$\begin{tikzcd}
\cdots \arrow[r] & D_{1,3} = H_4(A) \arrow[d, "i_{13}"] & & & & & & & \\
\cdots \arrow[r] & D_{2,2} \arrow[r] \arrow[d, "i_{22}"] & E_{2,2} \arrow[r, "k_{22}"] & D_{1,2} = H_3(A) \arrow[d, "i_{12}"] & & & & & \\
\cdots \arrow[r] & D_{3,1} \arrow[r] \arrow[d, "i_{31}"] & E_{3,1} \arrow[r, "k_{3,1}"] & D_{2,1} \arrow[r] \arrow[d, "i_{21}"] & E_{2,1} \arrow[r, "k_{21}"] & D_{1,1} = H_2(A) \arrow[d, "i_{11}"] & & & \\
\cdots \arrow[r] & D_{4,0} \arrow[r] \arrow[d, "i_{40}"] & E_{4,0} \arrow[r] & D_{3,0} \arrow[r] \arrow[d, "i_{30}"] & E_{3,0} \arrow[r] & D_{2,0} \arrow[r] \arrow[d, "i_{20}"] & E_{2,0} \arrow[r] & D_{1,0} = H_1(A) \arrow[d] & \\
\cdots \arrow[r] & D_{5,-1} \arrow[r] \arrow[d, equal] & 0 \arrow[r] & D_{4,-1} \arrow[r] \arrow[d, equal] & 0 \arrow[r] & D_{3,-1} \arrow[r] \arrow[d, equal] & 0 \arrow[r] & D_{2,-1} \arrow[r] & 0 \\
& H_4(X) & & H_3(X) & & H_2(X) & & H_1(X) &
\end{tikzcd}$$

By virtue of the construction of X it is seen that $D_{\rho+1,-1} = H_\rho(A^{(\rho+1)}) = H_\rho(X)$ for $\rho \geq 1$, $H_1(X) = H_1(A^{(2)}) = 0$, and $H_\rho(X) = D_{\rho,0}/V_\rho$. If we can prove $D_{\rho,0} \simeq V_\rho$, X is aspherical so that we have a desired R-algebra minimal resolution of K. In this paper we contend

PROPOSITION 2.1.

i) $D_{2,0} \simeq H_2(A)/H_1(A)^2 = V_2$,

ii) $D_{3,0} \simeq H_3(A)/H_2(A) \cdot H_1(A) = V_3$.

For the proposition we need the following two lemmas.

LEMMA 2.2.

$$i_{11},\ i_{21}\ \textit{and}\ i_{12}\ \textit{are onto.}$$

LEMMA 2.3.

a) $k_{21}(E_{2,1}) \simeq H_1(\Lambda)^2$,

b) $k_{22}(E_{2,2}) + i_{12}^{-1}k_{31}(E_{3,1}) \simeq H_2(\Lambda) \cdot H_1(\Lambda)$.

Proof of Proposition 2.1.

It is immediate from the exactness of the spectral sequence and the above two lemmas.

Proof of Lemma 2.2.

Let $Z \in D_{2,0}$, then Z is represented by a cycle

$$\mathfrak{Z} = c + \sum_{i=1}^{b_1} \lambda^i S_i,$$

where $c \in \Lambda_2$ and $\lambda^i \in R$. Since $0 = \partial_2 \mathfrak{Z} = \partial_2 c + \sum_{i=1}^{b_1} \lambda^i \mathfrak{Z}_i^1$, we have $\sum_{i=1}^{b_1} \bar{\lambda}^i Z_i^1 = 0$ where $\bar{\lambda}^i \in K$. Therefore $\lambda^i \in M$ for all i. Let $\lambda^i = \sum_{j=1}^{n} r^{ij} \cdot t_j$, then

$$\begin{aligned} \mathfrak{Z} &= c + \sum_{i,j} r^{ij} t_j S_i \\ &= (c + \sum_{i,j} r^{ij} T_j \mathfrak{Z}_i^1) + \partial_3 (\sum_{i,j} r^{ij} T_j S_i). \end{aligned}$$

The cycle $\mathfrak{Z}' = (c + \sum_{i,j} r^{ij} T_j \mathfrak{Z}_i^1)$ represents an element $Z' \in D_{1,1}$ whose image under i_{11} is Z. Therefore i_{11} is onto.

Secondly we wish to show that i_{21} and i_{12} are onto. Let $y \in X_3$ represent an element $Y \in D_{3,0}$. Then

$$y = d + \sum_{j=1}^{b_1} \sum_{i=1}^{n} \mu^{ij} (T_i \cdot S_j) + \sum_{k=1}^{\varepsilon_2} \nu^k U_k^3,$$

where $d \in \Lambda_3$.

$$0 = \partial_3 y = \sum_{j=1}^{b_1} (\sum_{i=1}^{n} \mu^{ij} t_i) S_j + (\partial_3 d - \sum_{i,j} \mu^{ij} T_i \mathfrak{Z}_j^1 + \sum_k \nu^k \mathfrak{Z}_k^2).$$

Thus we have

$$\sum_{i=1}^{n} \mu^{ij} t_i = 0 \qquad \text{for all } j,$$

so that $\sum_i \mu^{ij} T_i$ is 1-cycle of Λ and $\sum_j (\sum_i \mu^{ij} T_i) \mathfrak{Z}_j^1$ represents an element $Z'' \in$

$H_1(\Lambda)^2$. From this $Z''=\sum_{k=1}^{\varepsilon_2}\bar{\nu}^k Z_k^2$, and hence $\nu^k\in M$. Letting $\nu^k=\sum_{l=1}^{n}\nu^{kl}t_l$ and considering $\partial_4(\sum_{k,l}\nu^{kl}T_lU_k^3)=\sum_k\nu^kU_k^3-\sum_{k,l}\nu^{kl}T_l\mathfrak{Z}_k^2$, we find 2-cycle of $\Lambda^{(2)}$,

$$d+\sum_{j,i}\mu^{ij}(T_i\cdot S_j)+\sum_{k,l}\nu^{kl}T_l\mathfrak{Z}_k^2,$$

whose homology class Y' is mapped onto Y under i_{21}. From the analogous argument it is easy to see that i_{12} is onto. Thus the proof is omitted. This completes the proof of the Lemma.

Proof of Lemma 2.3.

Select 3-relative cycle $\mathfrak{Z}$ of $\Lambda^{(2)}/\Lambda$ representing an element $Z\in E_{2,1}$. Then $\mathfrak{Z}=x+\sum_{i,j}\lambda^{ij}T_i\cdot S_j$, where $x\in\Lambda_2$ and $\sum_{i=1}^{n}\lambda^{ij}T_i$ is 1-cycle of Λ. Since $k_{21}(Z)$ is represented by 2-cycle of $\sum_{j=1}^{b_1}(\sum_{i=1}^{n}\lambda^{ij}T_i)\mathfrak{Z}_j^1$, we have $k_{21}(Z)\in H_1(\Lambda)^2$. Conversely it is obvious that $H_1(\Lambda)^2\subset k_{21}(E_{2,1})$, beause $\mathfrak{Z}_i^1\mathfrak{Z}_j^1=\partial_3(-\mathfrak{Z}_i^1S_j)$ for any pair (i, j). This completes the proof of Lemma 2.3. a).

Let $Y\in E_{3,1}$ and y be 4-relative cycle of $\Lambda^{(3)}/\Lambda^{(2)}$ representing Y. Then we have

$$y=c+\sum_{i,j}\lambda^{ij}T_iU_j^3,$$

where $c\in\Lambda_3^{(2)}$ and $\sum_i\lambda^{ij}T_i$ is 1-cycle of Λ. By considering k_{31} and i_{12}, $i_{12}^{-1}k_{31}(Y)$ is represented by 3-cycle of Λ, $\sum_{j=1}^{\varepsilon_2}(\sum_{i=1}^{n}\lambda^{ij}T_i)\mathfrak{Z}_j^2$, whose homology class is in $H_1(\Lambda)\cdot V_2\subset H_1(\Lambda)\cdot H_2(\Lambda)$.

Let $\mathfrak{Z}$ be a relative 4-cycle representing an element $Z\in E_{2,2}$, and let

$$\mathfrak{Z}=a+\sum_{b_1\geq k>i\geq 1}\lambda^{ik}S_i\cdot S_k+\sum_{b_1\geq k\geq 1}\lambda^{kk}S_k^{(2)}+\sum_{\substack{n\geq j>i\geq 1\\ b_1\geq k\geq 1}}\mu^{ijk}\ (T_iT_jS_k),$$

where $a\in\Lambda_4$ and $1\cdot S_k^{(2)}$ is a generator of $\Lambda_4^{(2)}$, whose boundary is defined by $\mathfrak{Z}_k^1S_k$ (refer to [6]). Considering the boundary of $\mathfrak{Z}$, we have

$$\Lambda_3\ni\partial_4\mathfrak{Z}=(\partial_4a+\sum_{i,j,k}\mu^{ijk}(T_i\cdot T_j)\mathfrak{Z}_k^1)+\sum_{k=1}^{b_1}\{\sum_{i=1}^{k}\lambda^{ik}\mathfrak{Z}_i^1+\sum_{i=k+1}^{b_1}\lambda^{ki}\mathfrak{Z}_i^1+\partial_2(\sum_{i,j}\mu^{ijk}T_i\cdot T_j)\}S_k,$$

so that

$$\sum_{j=1}^{k}\lambda^{ik}\mathfrak{Z}_i^1+\sum_{i=k+1}^{b_1}\lambda^{ki}\mathfrak{Z}_i^1+\partial_2(\sum\mu^{ijk}T_i\cdot T_j)=0 \text{ for each } k.$$

Therefore all $\lambda^{ik}\in M$ for any pair (i, k) satisfing $b_1\geq k\geq i\geq 1$. Letting $\lambda^{ik}=\sum_{j=1}^{n}\lambda^{ijk}t_j$, considering $\xi_k=\sum_{i=1}^{k}\sum_{j=1}^{n}\lambda^{ijk}T_j\mathfrak{Z}_i^1+\sum_{i=k+1}^{b_1}\sum_{j=1}^{n}\lambda^{kji}T_j\mathfrak{Z}_i^1$, we obtain a 2-cycle η_k

of Λ by

$$\eta_k - \xi_k = \sum_{i,j} \mu^{ijk}(T_i \cdot T_j),$$

because $\partial_2(\xi_k) = \sum_{i=1}^{k} \lambda^{ik}\mathfrak{z}_i^1 + \sum_{i=k+1}^{b_1} \lambda^{ki}\mathfrak{z}_i^1$. The straightforward computation shows $\sum_{k=1}^{b_1} \xi_k \mathfrak{z}_k^1 = 0$, so that we have

$$\sum_{i,j,k} \mu^{ijk}(T_i \cdot T_j)\mathfrak{z}_k^1 = \sum_{k=1}^{b_1} \eta_k \mathfrak{z}_k^1.$$

Since $k_{22}(Z)$ is represented by $\sum_{k=1}^{b_1} \eta_k \mathfrak{z}_k^1$, $k_{22}(E_{2,2}) \subset H_2(\Lambda) \cdot H_1(\Lambda)$. It is immediate to show that $H_2(\Lambda) \cdot H_1(\Lambda) \subset k_{22}(E_{2,2})$, because $\partial_4(\eta \cdot S_k) = \eta \cdot \mathfrak{z}_k^1$ for any 2-cycle η of Λ. This completes the proof of Lemma 2.3.

§3. Computation of B_ρ ($\rho \leq 4$)

PROPOSITION 3.1.

i) $B_1 = \binom{n}{1}$. $B_2 = \binom{n}{2} + b_1$,

ii) $B_3 = \binom{n}{3} + \binom{n}{1} \cdot b_1 + \varepsilon_2$

iii) $B_4 = \binom{n}{4} + \binom{n}{2} \cdot b_1 + \binom{n}{0} b_1^2 - \binom{b_1}{2} + \binom{n}{1}\varepsilon_2 + \binom{n}{0}\varepsilon_3$.

Proof.

In the previous section we have proved that the sequence

$$X_4 \xrightarrow{\partial_4} X_3 \xrightarrow{\partial_3} X_2 \xrightarrow{\partial_2} X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\varepsilon} K \longrightarrow 0$$

is exact. By definition $\mathrm{Tor}_\rho^R(K, K)$ is computed by $X_\rho \otimes_R K$ for all $\rho \leq 3$. Therefore we get i) and ii). From a general theory (for example, see [5] or [4]) we know that there exists $\tilde{X}_5$ such that $\tilde{X}_5 \xrightarrow{\tilde{\partial}_5} X_4 \xrightarrow{\partial_4} X_3$ is exact and $\tilde{\partial}_5(\tilde{X}_5) \subset MX_4$. Therefore B_4 can be computed as stated in 3.1. iii) without knowing explicitely a system of generators for $\tilde{X}_5$.

Note that $\tilde{X}_5$ may be considered as X_5 which we constructed in §2.

§4. Corollaries and a conjecture

COROLLARY 4.1.

If R is a complete intersection, we have

$$B_3 = \binom{n}{3} + \binom{n}{1} b_1$$

$$B_4 = \binom{n}{4} + \binom{n}{2} b_1 + \binom{n}{0} b_1^2 - \binom{b_1}{2}$$

COROLLARY 4.2.

$$B_\rho \geq \binom{n}{\rho} + \binom{n}{\rho-2} + \binom{n}{\rho-4} + \cdots$$

for $\rho \leq 4$, *if* R *is not regular.*

Proofs

By a Theorem of Assmus [1] R is a local complete intersection if and only if $H(A)$ is the exterior algebra on $H_1(A)$. Therefore we have $\varepsilon_2 = \varepsilon_3 = 0$ in this case. The corollary 4.1. coincides with a result of Tate [6]. The special case when $b_1 = 1$. $b_2 = b_3 = 0$, provides us with the proof of Corollary 4.2., which is the estimation of Tate [6].

Tate said in [6] that it is doubtful whether minimal R-algebra resolutions exist in all cases. It seems to the author that such resolution may be probable in view of the construction we consider in this paper.

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