A GENERALIZATION OF THE HAHN-BANACH THEOREM

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Recently the Hahn-Banach theorem for normed spaces over non-archimedean valued fields was treated by A. F. Monna [1], I. S. Cohen [2], A. W. Ingleton [3], and the writer [4]. In [3] and [4] very essential use was made of an idea of L. Nachbin [5].

In the present note, we shall further generalize the H-B theorem, in fact in two directions (§3, Theorem 3). One is to remove the commutativity of the ground field and the other is to introduce semi-linear functionals. The effect of the generalization will then be seen in the lattice characterization of normed spaces over division algebras over the p-adic number field (§4, Theorem 4).

Once the H-B theorem is transferred, many analogies to the case of usual theory may easily be established. It seems interesting that there exists much analogy between two antipodal topological fields containing rational numbers densely.

The writer wishes to express here his hearty thanks to Prof. T. Nakayama and Mr. G. Shimura for their advices.

§1. Automorphisms of non-archimedean valued sfields

In proving that the real number field $R_{p\infty}$ has only one (abstract) automorphism, use is made, usually, of the fact that the rational field R is dense in it and $\sqrt{a}$ is $\in R_{p\infty}$ for any $a(>0)\in R_{p\infty}$. Analogously we may infer the same for the Hensel p-adic number field R_p from the fact that for any natural number m, $\sqrt[m]{a}$ is $\in R_p$ where $a\in R_p$ is sufficiently close to 1,[1] as has been kindly communicated to the writer by G. Shimura. Here we shall apply Shimura's method to a division algebra k over R_p. Let n be the rang of k over R_p. Then, the valuation $|\ |$ of R_p is prolonged to a such $|\ |_k$ of k uniquely by $|\alpha|_k=\sqrt[n]{|N_{k/Rp}\alpha|}$ $(\alpha\in k)$, where the norm refers to the regular representation of k over R_p.[2] For such $|\ |_k$ we have

THEOREM 1. *Every automorphism of k is valuation preserving.*

Proof. From the avove form of $|\ |_k$, it is sufficient to show that every automorphism σ leaves R_p elementwise fixed. Let $\alpha(\neq 0)$ be in R_p. Then $\alpha=\varepsilon p^{\nu}$, where ε is prime to p, and we have $\alpha^{\sigma}=\varepsilon^{\sigma}p^{\nu}$. Define rational integers

Received August 2, 1953.

1) See [6] p. 418.

2) See [7] p. 53.

a, b such that $\varepsilon \equiv a(\mathrm{mod.}\ p)$, $ab \equiv 1(\mathrm{mod.}\ p)$. Then $\eta = \varepsilon b \equiv ab \equiv 1(\mathrm{mod.}\ p)$. Put $f_n(x) = x^n - \eta$ for any n such that $(p,\ n) = 1$. Then $f_n(x) \equiv x^n - 1(\mathrm{mod.}\ p)$ has no multiple root, and so $f_n(x) = (x - \xi_n)g_n(x)$ follows from Hensel's lemma.[3] Thus we get $\eta = \xi_n^n$, so $\eta^\sigma = (\xi_n^\sigma)^n$. Considering the (corresponding) exponential valutation ν_k, we have $\nu_k(\eta^\sigma) = n \cdot \nu_k(\xi_n^\sigma)$ for all n, $(n,\ p) = 1$. So $\nu_k(\eta^\sigma) = 0$ or $|\eta^\sigma|_k = |\varepsilon^\sigma|_k = 1$, hence it follows that $|\alpha^\sigma|_k = |\alpha|$ for all $\alpha \in R_p$. However, R is dense in R_p, so the last equality implies that $\alpha^\sigma = \alpha$.

Immediately we get the above mentioned

THEOREM 2. *R_p has only one automorphism.*

§ 2. σ-linear functionals

Let k be a sfield with non-trivial valuation and S be a normed space over k.[4] A mapping f is called σ-linear if $f(\alpha x + \beta y) = \alpha^\sigma f(x) + \beta^\sigma f(y)$ for all $x,\ y \in S$, $\alpha,\ \beta \in k$, where σ is an automorphism of k.[5] If f is continuous, moreover, it is called a *σ-linear functional*. For a valuation preserving σ, a σ-linear mapping f is continuous if and only if it is bounded. For, suppose f is not bounded. Since $|\ \cdot\ |$ is non-trivial, we can select a sequence $\{\alpha_n\}$ such that $|\alpha_n| \to \infty$. To each α_n there exists $x_n \in S$ such that $|f(x_n)| > |\alpha_n^2| \|x_n\|$. Put $y_n = (\alpha_n / f(x_n))x_n$. Then $\|y_n\| = |\alpha_n / f(x_n)| \|x_n\| < 1/|\alpha_n|$; hence $y_n \to 0$. On the other hand, $|f(y_n)| = |(\alpha_n / f(x_n))^\sigma f(x_n)| = |\alpha_n / f(x_n)||f(x_n)| = |\alpha_n| \to \infty$, and so f is not continuous at the origin, which proves the necessity assertion. The sufficiency is almost evident.

§ 3. Hahn-Banach theorem

Hereafter, we impose on ground sfields k and normed spaces S over k the following conditions:

A. *if $|\ |$ is archimedean, k is complete, and so it is one of the real, complex and quaternion sfields.*[6]

NA. *if $|\ |$ is non-archimedean, k is complete and discrete and S is non-archimedean.*

For such normed spaces, we have the following generalized Hahn-Banach

THEOREM 3. *Let σ be a valuation preserving automorphism of k. Then, a normed space S over k satisfying the above conditions has the extension property for σ-linear functionals.*

Proof. Case NA. As we did not use the commutativity of k in our previous paper,[7] our sfield k has Nachbin's *binary intersection property*. For our purpose,

[3] See [6] p. 412.

[4] In accordance with [4] terms e.g. 'normed space,' 'bounded,' 'norm of f' etc. are defined similarly, so we do not give here full details.

[5] For brevity we avoid here the habitual term 'semi-linear.'

[6] See [8] p. 34.

[7] See [4] p. 2 lemma.

it is sufficient to show that the extension is capable for the adjunction of a 1-dimensional space. Let S_0 be a proper subspace of S and f_0 ($\neq 0$) be a σ-linear functional of S_0. Following Nachbin, we take $z \in S - S_0$ and put $\rho(\alpha) = \|f_0\|$ dist $(z, f_0^{-1}(\alpha))$, to get a family of circles $C_\alpha = \{\beta ; |\beta - \alpha| \leqq \rho(\alpha)\}$ with index set k. From the binary intersection property of such family $\{C_\alpha\}$, we have

$$(*)\quad |\gamma - f_0(x)| \leqq \|f_0\| \|z - x\| \quad \text{for all} \quad x \in S_0 \quad \text{and} \quad \gamma \in \bigcap_{\alpha \in k} C_\alpha .$$

Let S' be the space spanned by S_0 and z. Then, $x' \in S'$ is written as $x' = x + \alpha z$, $x \in S_0$, $\alpha \in k$. Using γ as above, put $f'(x') = f_0(x) + \alpha^\sigma \gamma$. Then f' is a σ-linear mapping of S'. For $\alpha (\neq 0) \in k$, we have $|\gamma + f_0(x/\alpha)| \leqq \|f_0\| \|z + x/\alpha\|$ from $(*)$. As $|\alpha^\sigma| = |\alpha|$, multiplying the left and the right hand sides by $|\alpha^\sigma|$ and $|\alpha|$ respectively, we get $|\alpha^\sigma \gamma + f_0(x)| \leqq \|f_0\| \|\alpha z + x\|$. This inequality holds even for $\alpha = 0$. Hence, for all $x' \in S'$, we have $|f'(x')| \leqq \|f_0\| \|x'\|$. So $\|f'\| \leqq \|f_0\|$ and f' is continuous. This settles the Case NA.

Case A. If k is the real field, σ must be the identity, so the Theorem is the well known one.[8] Next, if k is the complex field, then the valuation preserving σ is either the identity or the complex conjugate $\alpha^\sigma = \bar{\alpha}$. In the former case, the Theorem is known.[9] As for the latter case, we can reduce it to the former on considering the conjugate functional $\bar{f}$ insted of f. Lastly, let k be the quaternion field $\mathfrak{Q}$. Let $\mathfrak{R}$, $\mathfrak{K}$ be respectively the real and the complex number field. As the valuation preserving automorphism σ of $\mathfrak{Q}$ leaves $\mathfrak{R}$ elementwise fixed, such a σ is inner: $\alpha^\sigma = \tau^{-1} \alpha \tau$ ($\alpha \in \mathfrak{Q}$) with an element τ of $\mathfrak{Q}$.[10] Here, $\sigma = 1$ if and only if $\tau \in \mathfrak{R}$. So we consider separately the following two cases.

Case 1. $\sigma \neq 1$. Since $\tau \notin \mathfrak{R}$, the field $\mathfrak{R}(\tau)$ may be regarded as the complex field: $\mathfrak{R}(\tau) = \mathfrak{K}$. Let $i \in \mathfrak{K}$ be the imaginary unit. For this i, there exists a $j \in \mathfrak{Q}$ such that $jij^{-1} = -i$. Since $\tau^\sigma = \tau^{-1} \tau \tau = \tau$, σ leaves $\mathfrak{K}$ elementwise fixed. So we get $(j^\gamma)^{-1} i j^\gamma = -i$ and it follows that $j(j^\gamma)^{-1} i j^\gamma j^{-1} = -jij^{-1} = i$, hence $j(j^\gamma)^{-1} \in \mathfrak{K}$. As σ is valuation preserving, we have $j^\sigma = j\varepsilon$, where $\varepsilon \in \mathfrak{K}$ has the absolute value 1. For $\alpha \in \mathfrak{K}$ we have obviously $j\alpha = \bar{\alpha} j$. In particular $j^{\gamma^{-1}} = j\bar{\varepsilon}$. Thus we apply the method 'real to complex' in [10] to the somewhat complicated case 'complex to quaternion.' Let f_0 be a σ-linear functional of S_0 ($\subset S$). Put $f_0(x) = g_0(x) + jh_0(x)$. Then g_0 is a linear functional of S_0 (considered as a normed space over $\mathfrak{K}$) and we have $\|g_0\| \leqq \|f_0\|$. Since $g_0(j^{\gamma^{-1}} x) + jh_0(j^{\gamma^{-1}} x) = f_0(j^{\gamma^{-1}} x) = jf_0(x) = jg_0(x) - h_0(x)$, whence $h_0(x) = -g_0(j^{\gamma^{-1}} x) = -g_0(j\bar{\varepsilon} x) = -g_0(\varepsilon j x) = -\varepsilon g_0(jx)$, we have $f_0(x) = g_0(x) - j\varepsilon g_0(jx)$. Let g be the extension of g_0 (con-

[8] See Banach's book [9] p. 55, or [5]. $R_{p\infty}$ satisfies the binary intersection property together with R_p, so Nachbin's method is useful to treat the both fields simultaneously.

[9] See [10].

[10] For the properties of normal simple algebras mentioned here, see M. Deuring [11] IV. § 4.

sidering S as a space over $\mathfrak{R}$ and put $f(x)=g(x)-j\varepsilon g(jx)$. Then, this f is the desired extension. Obviously, f is an additive extension of f_0. Next, let $\alpha=\alpha_1+j\alpha_2$. Then $f(\alpha x)=g(\alpha x)-j\varepsilon g(j\alpha x)=g(\alpha_1 x+j\alpha_2 x)-j\varepsilon g(j\alpha_1 x-\alpha_2 x)=g(\alpha_1 x+\bar{\alpha}_2 jx)-j\varepsilon g(\bar{\alpha}_1 jx-\alpha_2 x)=\alpha_1 g(x)+\bar{\alpha}_2 g(jx)-j\varepsilon\bar{\alpha}_1 g(jx)+j\varepsilon\alpha_2 g(x)=(\alpha_1+j\varepsilon\alpha_2)g(x)-(\bar{\alpha}_2-j\varepsilon\bar{\alpha}_1)jjg(jx)=\alpha^\sigma g(x)-(\bar{\alpha}_2 j-j\varepsilon\bar{\alpha}_1 j)jg(jx)=\alpha^\sigma g(x)-(j\alpha_2+\bar{\varepsilon}\alpha_1)\varepsilon\bar{\varepsilon}jg(jx)=\alpha^\sigma g(x)-(j\alpha_2\varepsilon+\alpha_1)j\varepsilon g(jx)=\alpha^\sigma g(x)-\alpha^\sigma j\varepsilon g(jx)=\alpha^\sigma f(x)$. Thus f is σ-linear. Lastly, multiplying $\eta\in\mathfrak{Q}(|\eta|=1)$, we get $\eta f(x)=r\in\mathfrak{R}$. It follows that $|f(x)|=|\eta f(x)|=|f(\eta^{\sigma^{-1}}x)|=|g(\eta^{\sigma^{-1}}x)|\leqq\|g\|\|\eta^{\sigma^{-1}}x\|=\|g\|\|x\|$, as σ is valuation preserving. Hence, $\|f\|\leqq\|g\|=\|g_0\|\leqq\|f_0\|$. This settles the Case 1.

Case 2. ($\sigma=1$) is treated similarly; we have only to retrace our proof for the Case 1 with arbitrarily selected quaternion units *i, j*.

As immediate consequences of this theorem we have the following two corollaries:

COROLLARY 1. *Let σ be a valuation preserving automorphism of k. To each $a(\neq 0)\in S$, there exists a σ-linear functional f of S such that $f(a)=1$ and $\|f\|=1/\|a\|$.*

Proof. Let S_0 be the line spanned by a and f_0 be the σ-linear functional of S_0 defined by $f_0(\alpha a)=\alpha^\sigma$ ($\alpha\in k$). Then the extension f of f_0 is the desired functional.

COROLLARY 2. *Let S_0 be a proper subspace of S such that for a vector $a(\neq 0)\notin S_0$, dist $(a, S_0)=d>0$. Then, for any valuation preserving σ, there exists a σ-linear functional of S such that $f(a)=1, f(x)=0$ for all $x\in S_0$ and $\|f\|=1/d$.*

Proof. Let S_1 be the space spanned by S_0 and a and let f_1 be the σ-linear fuuctional of S_1 defined by $f_1(x)=\alpha^\sigma$ where $x=x_0+\alpha a$, $x_0\in S_0$, $\alpha\in k$. Then the extension f of f_1 satisfies our requirement.

§ 4. Some applications

1°. First we deal with *the Mackey-Kakutani lattice characterization of normed spaces.*[11] Let S, S' be normed spaces over k whose dimensions are $\geqq 3$ and P, P' be the lattices of closed subspaces of S and S' respectively. If there exists a lattice isomorphism φ between P and P', then, putting $\varphi(S^n)=S'^n$, φ induces a collineation between P^{n-1} and P'^{n-1}, where S^n means the n-dimensional subspace ($n\geqq 3$) and P^{n-1} is the $(n-1)$-dimensional projective space corresponding to S^n. If $x_1, \ldots, x_n$ is a basis of S^n and if we denote the line spanned by $x(\neq 0)\in S$ by $[x]$, we can select a basis $x_1', \ldots, x_n'$ of S'^n such that $\varphi([\sum_{i=1}^n \alpha_i x_i])=[\sum_{1=i}^n \alpha_i^\sigma x_i']$, where σ is an automorphism of k.[12] Now, it is

[11] See Mackey [12] and Kakutani [13].

[12] See M. Abe [14].

easily verified that the σ does not depend on the subspace S^n. Thus, we write such an automorphism determined by φ uniquely σ_φ and call it the automorphism induced by φ. After these remarks, we get the following generalization of Mackey-Kakutani's Theorem:

THEOREM 4. (*Lattice characterization*) *Let S, S' and P, P' be the same as above and φ be a lattice isomorphism between P and P'. If the automorphism σ_φ induced by φ is valuation preserving, then there exists a homeomorphical isomorphism T from S onto S' such that $[Tx]=\varphi[x]$ for all $x\in S$.*

Remark 1. The Mackey's method for his lemma 2.1 in [12] is not applicable to *non-commutative k*. But this lemma is, as it seems to the writer, more than what is needed to show the closedness of S^n in S.[13] On the other hand, Artin's method, in [15], is free from the commutativity of k, and we know the closedness of S^n directly.[14]

Remark 2. The analytic lemmas referred to Banach's book [9] in [12] are easily verified to hold for our spaces.

Remark 3. According to Theorem 1 in §1, if k is the real number-field (Mackey case) or a division algebra over R_p, then we may remove the clumsy condition: 'if σ_φ is valuation preserving,' to get

THEOREM 5. *A normed space over R_{p_∞} or a division algebra over R_p can be characterized by its lattice of closed subspaces.*

2°. Next we impose on k *the commutativity besides the conditions mentioned in the beginning of* §3. Let S be a normed space over k. Then the totality $\mathfrak{A}(S)$ of continuous k-homomorphism of S into itself forms a normed algebra over k,[15] if we define operations and norms as follows: $(T+T')x=Tx+T'x$, $(\alpha T)x=\alpha(Tx)$, $(TT')x=T(T'x)$ and $\|T\|=\sup_{x\neq 0}\|Tx\|/\|x\|$. On this algebra $\mathfrak{A}(S)$, using Corollary 1 in §3, we have the generalization of Hille's theorem:

THEOREM 6. *The algebra $\mathfrak{A}(S)$ is semi-simple.*[16]

Similarly we get the generalization of Eidelheit-Kawada's theorem:

THEOREM 7. (*Ring characterization*) *Let S, S' be normed spaces of dimension $\geqq 2$ and $\mathfrak{A}(S)$, $\mathfrak{A}(S')$ be the corresponding algebras. If φ is a homeomorphical k-isomosphism between $\mathfrak{A}(S)$ and $\mathfrak{A}(S')$, then there exists a homeomorphical k-*

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[13] [12], lemma 2.2 p. 247.

[14] See Artin [15] Chapt. pp. 18-20.

[15] Normed algebra A over k is a structure which is an algebra as well as a normed space over k and which, in addition, satisfies $\|xy\|\leqq\|x\|\,\|y\|$. See Hille [16].

[16] See [16] p. 479 Theorem 22.13.7.

isomorphism U between S and S' such that $\varphi(T) = UTU^{-1}$ for all $T \in \mathfrak{A}(S)$.[17]

As an immediate consequence of this theorem, we have the following

THEOREM 8. *Every automrophism of $\mathfrak{A}(S)$ that leaves k elementwise fixed is inner.*

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