A PROPERTY OF SOME OPEN RIEMANN SURFACES AND ITS APPLICATION

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Introduction

As L. Sario [8] and the others have shown, an important method to investigate the properties of an open Riemann surface F is the use of an exhaustion of F consisting of compact domains on F. But, for the same purpose, the investigation of properties of a non-compact region on F is also important.

Recently Lauri Myrberg [5] gave remarkable results for some harmonic functions and Ullemar [12], [13] gave interesting theorems on a symmetric Fuchsian or fuchsoid group without any elliptic transformation and of genus zero. These results concern with the property of a non-compact region.

Let G be a non-compact domain on F whose relative boundary C consists of at most an enumerable number of compact or non-compact analytic curves clustering nowhere in F[1]. We can construct an open Riemann surface $\hat{G}$ by the process of symmetrization. There is given an indirectly conformal mapping of $\hat{G}$ on itself which leaves every point on C fixed. The image of a point $p \in \hat{G}$ is denoted by $\tilde{p}$, the image of G by $\tilde{G}$. If $t = x + iy$ is the local parameter at $p \in \hat{G}$, the local parameter at $\tilde{p}$ is given by $t = x - iy$.

In this article, we shall investigate some properties of such a surface $\hat{G}$ and give some remarks on the results obtained by Myrberg and Ullemar as its application.

I. Properties of $\hat{G}$

1. We denote by HB or HD the class of single-valued bounded or Dirichlet bounded[2] harmonic functions respectively. If any function of HB (or HD) in G, which equals to zero on C and is continuous on $G \smile C$, equals identically to zero, we may say that G belongs to the class SO_{HB} (or SO_{HD}). Further, we call that G belongs to the class NO_{HB} (or NO_{HD}), if each function of HB (or HD) in G whose normal derivative at every point on C vanishes and which is continuous on $G \smile C$, reduces to a constant.

First we shall prove the following

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[1] Simply we call such a domain G a non-compact region throughout this article.

[2] We may say that the function is Dirichlet bounded in a region if the Dirichlet integral of it over the region is finite.

THEOREM 1. *If G is a non-compact region on an open Riemann surface F with null boundary, then $G \in SO_{HB}$, $\in SO_{HD}$, $\in NO_{HB}$ and $\in NO_{HD}$.*

Proof. It is an immediate consequence from the maximum and minimum principle that $G \in SO_{HB}$ and $\in SO_{HD}$. We shall give a proof of $G \in NO_{HB}$.

Let $\{F_n\}$ $(n=0, 1, \ldots)$ be an exhaustion of F satisfying the condition $\bar{F}_0 \subset G$ and Γ_n be the relative boundary of F_n. We denote by u the harmonic function in $F_n - \bar{F}_0$ $(n \geqq 1)$ which equals to zero on Γ_0 and to $\log \mu_n$ on Γ_n, where $\log \mu_n$ is the harmonic modulus[3] of $F_n - \bar{F}_0$, and by v the conjugate function of u. Then, from the definition of the harmonic modulus,

$$\int_{\Gamma_n} dv = 2\pi.$$

Let Γ_λ be the part of a niveau curve $u = \lambda (0 < \lambda \leqq \log \mu_n)$ contained in G.

If $G \notin NO_{HB}$, there exists a non-constant function $U(p)$ of HB on G whose normal derivative $\frac{\partial U}{\partial \nu}$ vanishes at every point on the relative boundary C of G. If $D(\lambda)$ is the Dirichlet integral of $U(p)$ taken over the compact open set bounded by Γ_λ and C and consisting of a finite number of compact domains, we have

$$D(\lambda) = \int_{\Gamma_\lambda} U \frac{\partial U}{\partial u} dv,$$

for $\frac{\partial U}{\partial \nu}$ equals to zero on C. By the Schwarz inequality, we get

$$\begin{aligned} D^2(\lambda) &\leqq \int_{\Gamma_\lambda} U^2 dv \int_{\Gamma_\lambda} \left(\frac{\partial U}{\partial u}\right)^2 dv \\ &\leqq M^2 \int_{\Gamma_\lambda} dv \int_{\Gamma_\lambda} \left(\frac{\partial U}{\partial u}\right)^2 dv \\ &\leqq 2\pi M^2 \frac{dD(\lambda)}{d\lambda}, \end{aligned}$$

provided that $|U(P)| \leqq M$, whence it follows that

$$d\lambda \leqq 2\pi M^2 \frac{dD(\lambda)}{D^2(\lambda)}.$$

Integrating the both sides from $\lambda = 0$ to $\lambda = \log \mu_n$, we obtain

$$\begin{aligned} \log \mu_n &\leqq 2\pi M^2 \left[\frac{1}{D_0} - \frac{1}{D_n}\right] \\ &< 2\pi M^2 \frac{1}{D_0}, \end{aligned}$$

[3] The notion of the harmonic modulus was introduced by Sario [8] and Pfluger [7].

where D_n is the Dirichlet integral of $U(p)$ taken over $G \cap F_n$. Therefore, it is immediate that $\lim_{n\to\infty} \mu_n < \infty$.

On the other hand, it is well known that $\lim_{n\to\infty} \mu_n = \infty$, if and only if F has a null boundary (cf. Kuroda [2]). Thus $G \in NO_{HB}$ if F has a null boundary.

By the similar arguments as above (cf. Kuroda [3], Tsuji [11]), we can prove the fact that $G \in NO_{HD}$ in the case of F with null boundary.

2. Here we shall state a theorem of Myrberg in the following form.[4)]

THEOREM 2. *Suppose that $G \in SO_{HB}$ (or $\in SO_{HD}$) and that $U(p)$ is a function of HB (or HD) on $\hat{G}$. Then $U(p) = U(\tilde{p})$ for any $p \in \hat{G}$.*

Proof. Putting $V(p) = U(p) - U(\tilde{p})$, we can see that $V(p)$ is also a function of HB (or HD) on $\hat{G}$ and so in G and $V(p) = 0$ on C. Since $G \in SO_{HB}$ (or $\in SO_{HD}$), $V(p)$ reduces to a constant zero in $\hat{G}$ and hence $U(p) = U(\tilde{p})$.

Now we can prove the following

THEOREM 3. *If $G \in SO_{HB}$ and $\in NO_{HB}$, then $\hat{G} \in O_{HB}$, in other words, there exists no non-constant function of HB on $\hat{G}$ and vice versa.*

Proof. Let $U(p)$ be a function of HB on $\hat{G}$. By Theorem 2, we get $U(p) = U(\tilde{p})$ on G. Hence it is immediate that the normal derivative of $U(p)$ at every point on C vanishes. Since $G \in NO_{HB}$ by the assumption, $U(p)$ must reduce to a constant, which proves the first part of our assertion.

Next we shall prove the converse. Suppose that $G \notin SO_{HB}$. Then there exists a non-constant function $U(p)$ of HB in G which equals to zero on C and is continuous on $G \cup C$. If we define a function $V(p)$ on $\hat{G}$ such that

$$V(p) = \begin{cases} U(p) & \text{for } p \in G \cup C \\ -U(p) & \text{for } p \in \tilde{G} \end{cases},$$

$V(p)$ belongs to HB and is non-constant. On the other hand, if $G \notin NO_{HB}$, there exists a non-constant function $U(p)$ of HB in G whose normal derivatives on C vanish and which is continuous on $G \cup C$. The function

$$V(p) = \begin{cases} U(p) & \text{for } p \in G \cup C \\ U(\tilde{p}) & \text{for } p \in \tilde{G} \end{cases}$$

on $\hat{G}$ is a non-constant function of HB. Thus the proof is complete.

By Nevanlinna's theorem [6], for a Riemann surface with finite genus, $F \in O_{HB}$ (or $\in O_{HD}$)[5)] is equivalent to that F has a null boundary. Hence we get

4) See Theorem 5.

5) The class O_{HD} means the class of Riemann surfaces on which there exists no non-constant function of HD.

COROLLARY. *If $G \in SO_{HB}$ and $\in NO_{HB}$ and, further, G is finitely connected, $\hat{G}$ has a null boundary.*

By the similar arguments as above, we have the followings.

THEOREM 3′. *If $G \in SO_{HD}$ and $\in NO_{HD}$, then $\hat{G} \in O_{HD}$, and vice versa.*

COROLLARY. *For a finitely connected region G, $\hat{G}$ has a null boundary under the same condition for G as in Theorem 3′.*

Remark. Under the condition of Theorem 3, $\hat{G}$ has not always a null boundary. This is observed from the example of a Riemann surface, due to Tôki [9], which has a positive boundary and belongs to O_{HB}.

Further, we get

THEOREM 4. *If $G \in SO_{HB}$ (or $\in SO_{HD}$), then $\hat{G} \in O_{AB}$ (or $\in O_{AD}$).*[6]

Proof. Let $f(p) = U(p) + iV(p)$ be a single-valued bounded (or Dirichlet bounded) analytic function on $\hat{G}$. Then, obviously, $U(p) \in HB$ (or $\in HD$) and $V(p) \in HB$ (or $\in HD$). By Theorem 2, $U(p) = U(\tilde{p})$ and $V(p) = V(\tilde{p})$ for any $p \in \hat{G}$. Therefore, the differential df must equal to zero at every point on C. Thus $f(p)$ reduces to a constant and so $\hat{G} \in O_{AB}$ (or $\in O_{AD}$).

Remark. Mori [4] pointed out the fact that, for a simply connected region G, the converse of Theorem 4 holds good.

II. Applications

3. Now we shall prove Myrberg's theorem [5] applying theorems stated above. We suppose that a non-compact region G is simply connected. Then G can be mapped on the upper half z-plane one to one conformally such that a point on the relative boundary C of G corresponds to $z = \infty$. The ideal boundary of G corresponds to a bounded closed linear set E on the real axis of the z-plane. If we denote by Ω the complementary domain of E with respect to the whole z-plane, Ω is equivalent to $\hat{G}$ conformally.

By Mori's remark [4], E is of measure zero, if and only if $G \in SO_{HB}$. And, if we notice Ahlfors-Beurling's theorem [1], it is easily seen that Ω is of span zero if and only if $G \in SO_{HD}$. Hence we have the following by Theorem 2.

THEOREM 5 (Myrberg [5]). *Let E be a bounded closed set on the real axis in the z-plane and let $U(z)$ be a function of HB in the complementary domain Ω of E with respect to the z-plane. If E is of measure zero, then $U(z) = U(\bar{z})$.*

[6] We denote by O_{AB} (or O_{AD}) the class of Riemann surfaces on which there exists no non-constant single-valued bounded (or Dirichlet bounded) analytic function.

And, if $V(z)$ is a function of HD in Ω and the span of Ω equals to zero, then $\overline{V(z)} = V(\bar{z})$.

4. We shall use the notation as in the previous section and prove the following

THEOREM 6. *A simply connected region G belongs to NO_{HB}, if and only if the set E is of logarithmic capacity zero.*

Proof. If E is of logarithmic capacity zero, then, of course, $\Omega \in O_{HB}$ and hence $\hat{G} \in O_{HB}$. Thus the sufficiency of our condition is obvious from Theorem 3. We shall state the proof of the necessity.

For its purpose, it is sufficient to prove that, if the logarithmic capacity of E is positive, there exists a non-constant function of HB in the upper half z-plane whose normal derivative at every point on the real axis vanishes excluding the set E. Since the logarithmic capacity of E is positive, there exist two closed subsets E_1 and E_2 of E such that they are disjoint each other and their logarithmic capacities are both positive. We denote by Ω' the complementary domain of $E_1 \cup E_2$ and choose an exhaustion $\{\Omega_n\}$ $(n = 1, 2, \ldots)$ of Ω' as follows:

The boundary Γ_n of Ω_n consists of two classes $\Gamma_n^{(1)}$ and $\Gamma_n^{(2)}$ of analytic closed curves such that $\{\Gamma_n^{(1)}\}$ $(n = 1, 2, \ldots)$ clusters to E_1 and $\{\Gamma_n^{(2)}\}$ $(n = 1, 2, \ldots)$ to E_2, and Ω_n is symmetric for the real axis.

Let ω_n be a harmonic function in Ω_n which equals to zero on $\Gamma_n^{(1)}$ and to 1 on $\Gamma_n^{(2)}$. Then we can easily find a suitable subsequence, say again $\{\omega_n\}$, such that it converges to a non-constant limiting function ω which is harmonic in Ω'.

Since each function ω_n is symmetric for the real axis, the limiting function ω is also symmetric for the real axis. From this fact, the normal derivative of ω vanishes at every point on the real axis except all points of E_1 and E_2. Thus our proof is complete.

In the above proof, choosing a suitable subsequence of $\{\omega_n\}$, we can get a limiting function ω Dirichlet bounded in Ω'. Hence we have easily

THEOREM 6′. *The simply connected non-compact region belongs to NO_{HD}, if and only if E is of logarithmic capacity zero.*

As we can map the whole z-plane on the whole w-plane by a linear transformation one to one conformally such that the upper half z-plane corresponds to the unit circle $|w| < 1$ and the ideal boundary of G corresponds to a closed set E on the circumference $|w| = 1$. Hence we can get the following.

THEOREM 7. *Let E be a closed set on the circumference of the unit circle on the complex plane and let* **C** *be the complementary set of E with respect to the circumference. In order that any function in the unit circle belonging to*

HB or HD, whose normal derivative at every point on C vanishes, reduces to a constant, it is necessary and sufficient that the set E is of logarithmic capacity zero.

In fact, it is sufficient to consider the case where E is not identical to the circumference of the unit circle. Considering the complementary domain of E with respect to the whole plane as an open Riemann surface and using Theorems 6 and 6′, we arrive at the required.

Remark. Myrberg [5] proved only the sufficiency of the condition for HD. Further, the following complete form of Myrberg's theorem is also easily obtained.

THEOREM 7′. *Under the same notation as in Theorem 7, any single-valued positive harmonic function in the unit circle whose normal derivative at each point on C vanishes, reduces to a constant, if and only if E is of logarithmic capacity zero.*

For, there exists a non-constant function of HB in the unit circle whose normal derivative at every point on C vanishes, if the logarithmic capacity of E is positive. Hence there exists a single-valued non-constant positive harmonic function in the unit circle whose normal derivative at every point on C vanishes. Thus the necessity of our condition is obtained. The sufficiency is nothing but the result obtained by Myrberg.

5. From Theorems 6 and 6′, it is immediately seen that, for a simply connected region G, $G \in NO_{HB}$ is equivalent to $G \in NO_{HD}$. Further, we have

THEOREM 8 (Tsuji [10]). *Let G be a simply connected domain bounded by an analytic Jordan curve C and let E be a closed set on C with logarithmic capacity zero. If we map G on the unit circle one to one conformally and denote by E′ the closed set on the circumference of the unit circle corresponding to E, then E′ is of logarithmic capacity zero.*

In fact, the complementary domain of E with respect to the whole plane may be considered as an open Riemann surface with null boundary. Hence, by Theorem 1, we can see that $G \in NO_{HB}$ considering the set E as the ideal boundary of this Riemann surface. Thus our assertion is an immediate result of Theorem 6.

Remembering Mori's remark stated in §3 and using Theorem 6, we have

THEOREM 9. *The class NO_{HB} of simply connected regions is a proper subclass of SO_{HB} of simply connected regions.*

6. Consider a finite or an enumerable number of circular open arcs $\{\alpha_i\}$ $(i = 0, 1, 2, \dots)$ in the unit circle $|z| < 1$ which are orthogonal to the circumference $|z|=1$ and disjoint each others in $|z|<1$ and denote by D_0 the simply connected domain in $|z|<1$ bounded by $\{\alpha_i\}$ $(i=0, 1, 2, \ldots)$ and the closed set E on $|z|=1$. If $\tilde{D}_0$ is the reflection of D_0 with respect to an arc of $\{\alpha_i\}$, say α_0, then the domain $D_0 \cup \alpha_0 \cup \tilde{D}_0$ is a fundamental domain of a symmetric Fuchsian or fuchsoid group $\mathfrak{G}$ without any elliptic transformation and of genus zero. Conversely, such a symmetric group has a fundamental domain as stated above.

We denote by $\{\tilde{\alpha}_i\}$ $(i=0, 1, 2, \ldots)$ the boundary arcs of $\tilde{D}_0$ being equivalent to $\{\alpha_i\}$ $(i=0, 1, 2, \ldots)$ for $\mathfrak{G}$. Of course, $\tilde{\alpha}_0$ is identical to α_0. Identifying the equivalent points on α_i and $\tilde{\alpha}_i$ $(i=1, 2, \ldots)$, we get an open Riemann surface $\hat{D}$. This surface $\hat{D}$ can be decomposed by a relative boundary C into two parts D and $\tilde{D}$, each one of which is the image of the other by an indirectly conformal mapping. And $D \cup C$ (or $\tilde{D} \cup C$) is conformally equivalent to $D_0 \cup \bigcup_{i=0}^{\infty} \alpha_i$ (or $\tilde{D}_0 \cup \bigcup_{i=0}^{\infty} \tilde{\alpha}_i$).

We shall show the following

THEOREM 10. *If the set E is of logarithmic capacity zero, then $\hat{D}$ has a null boundary.*

Proof. The complementary domain of E with respect to the whole z-plane can be considered as an open Riemann surface with null boundary. Hence, by Theorem 1, the domain D_0 and so D belongs to SO_{HB} and to NO_{HB}. By the Corollary of Theorem 3, $\hat{D}$ has a null boundary.

Remark. The necessary and sufficient condition in order that $\hat{D} \in O_{AB}$ or $\in O_{AD}$ was obtained by Ullemar (=Uskila) [12], [13]. His results can be proved easily by using Theorem 4 and by remembering the remark stated in § 3.

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