

ON HOLOMORPHIC MAPS INTO A TAUT COMPLEX SPACE

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Introduction. The purpose of this paper is to study the extension problem of holomorphic maps of a complex manifold into a taut complex space, which is defined by analogy with a taut complex manifold given by H. Wu ([11]).

Let D be a domain in a complex manifold M and f be a holomorphic map of D into a taut complex space. We can construct the existence domain of f as in the case of holomorphic functions. We shall first prove the following theorem, which is essentially due to the Docquier-Grauert's theorem ([2]).

THEOREM A. *If D is an (unramified) Riemann domain over a Stein manifold M , the existence domain of f is a Stein manifold.*

Using Theorem A, we can easily prove that, for domains D and D' ($D \subset D'$) in a Stein manifold, if every holomorphic function on D has a holomorphic extension to D' , then every holomorphic map of D into a taut complex space X can be extended to a holomorphic map of D' into X .

For holomorphic maps defined on a complex manifold minus an analytic set of codimension one, we have the following improvement of Theorem 5 in [9], p. 18.

THEOREM B. *Let S be an irreducible analytic subset of codimension one in a domain D in \mathbb{C}^n and f be a holomorphic map of $D - S$ into a taut complex space X . If f has a cluster value in X at some regular point of S , then f can be extended to a holomorphic map of D into X .*

In connection with Theorem B, we give the following generalization of the big Picard theorem, whose proof is essentially due to [3].

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THEOREM C. *If S is a regular thin analytic subset of a domain D in \mathbf{C}^n , then every holomorphic map of $D - S$ into the N -dimensional complex projective space $P_N(\mathbf{C})$ minus $2N + 1$ hyperplanes in general position can be extended to a holomorphic map of D into $P_N(\mathbf{C})$.*

In case of holomorphic maps defined on a complex space, the analogous extension theorem is not valid in general. We shall construct a normal complex space Y of dimension ≥ 2 such that there exists a holomorphic map of Y minus one point into a taut complex space which has no holomorphic extension to Y .

§ 1. Local extension of holomorphic maps. In this paper, a complex space means a reduced complex space and all complex spaces and manifolds are assumed to be σ -compact and connected unless stated to the contrary.

For complex spaces M and X , we denote the space of all holomorphic maps of M into X endowed with the compact-open topology by $\text{Hol}(M, X)$. A sequence $\{f_\nu\}$ in $\text{Hol}(M, X)$ is said to be compactly divergent if, for any compact sets K in M and L in X , there is some ν_0 such that $f_\nu(K) \cap L = \emptyset$ for any $\nu \geq \nu_0$.

DEFINITION 1.1. A complex space X is said to be *taut* if $\text{Hol}(M, X)$ is normal for any complex manifold M , i.e., any sequence in $\text{Hol}(M, X)$ has a subsequence which is convergent in $\text{Hol}(M, X)$ or compactly divergent.

For example, a complete hyperbolic complex manifold in the sense of S. Kobayashi [8] is taut. In particular, a Riemann surface which is hyperbolic in the classical sense is a taut complex manifold. Moreover, if a relatively compact subdomain D of a Stein space X can be written $D = \{x \in X; u(x) < c\}$ with a plurisubharmonic function $u(x)$ on X and a constant c , then D is a taut complex space (W. Kaup [6], Satz 1.2, p. 306 and Satz 4.5, p. 318).

Now, we give the following definition for the convenience of description.

DEFINITION 1.2. Let D be a domain in a complex manifold M and x_0 be a boundary point of D . We shall say that D satisfies *the condition (C)* at x_0 if there is a sequence of maps $\{\varphi_\nu; \nu = 1, 2, \dots\}$ of $B := \{\zeta \in \mathbf{C}; |\zeta| < 1\}$ into M such that (1) $\{\varphi_\nu\}$ converges to φ in $\text{Hol}(B, M)$, (2) $\varphi_\nu(\zeta) \in D$ for any

ν and $\zeta \in B$, (3) $\varphi(0) = x_0$ and $\varphi(\zeta) \in D$ for any ζ with $0 < |\zeta| < 1$.

LEMMA 1.3. *Assume that a domain D in a complex manifold M satisfies the condition (C) at a boundary point x_0 . Then, every holomorphic map f of D into a taut complex space X has a possibly many-valued holomorphic extension to a neighborhood U of x_0 , i.e., a holomorphic map $g: U \rightarrow X$ such that $f(x) = g(x)$ in some non-empty open subset of $U \cap D$.*

Proof. Let $\varphi(\zeta)$ and $\varphi_\nu(\zeta)$ be holomorphic maps with the properties (1), (2) and (3) in Definition 1.2. Then $\{f \cdot \varphi_\nu\}$ is a sequence in $\text{Hol}(B, X)$. And, since $\lim_{\nu \rightarrow \infty} f \cdot \varphi_\nu(\zeta_0) = f \cdot \varphi(\zeta_0) \in X$ for an arbitrarily fixed ζ_0 with $0 < |\zeta_0| < 1$, $\{f \cdot \varphi_\nu\}$ cannot have a compactly divergent subsequence and so has a convergent subsequence by Definition 1.1. There is no harm in assuming that $\{f \cdot \varphi_\nu\}$ itself has a limit h in $\text{Hol}(B, X)$. Obviously, $h(\zeta) = f \cdot \varphi(\zeta)$ if $0 < |\zeta| < 1$. Put $q_0 = h(0)$ and take a Stein neighborhood V of q_0 . Then, for a sufficiently small $\rho > 0$ and a sufficiently large ν_0 , we see $(f \cdot \varphi_\nu)(\zeta) \in V$ and $h(\zeta) \in V$ if $\nu \geq \nu_0$ and $|\zeta| \leq \rho$. Consider the open set $D' := D \cap f^{-1}(V)$. Obviously, $\varphi(\zeta) \in D'$ if $0 < |\zeta| \leq \rho$ and $\varphi_\nu(\zeta) \in D'$ if $\nu \geq \nu_0$ and $|\zeta| \leq \rho$. So, the set

$$K := \{\varphi(\zeta); |\zeta| = \rho\} \cup \bigcup_{\nu \geq \nu_0} \{\varphi_\nu(\zeta); |\zeta| = \rho\}$$

is compact in D' . As is easily seen, by the maximum principle, the set

$$\hat{K} := \{x \in D'; |h(x)| \leq \sup |h(K)| \text{ for any holomorphic function } h \text{ on } D'\}$$

includes $\bigcup_{\nu \geq \nu_0} \{\varphi_\nu(\zeta); |\zeta| \leq \rho\}$, whose closure contains x_0 . In this situation, by the well-known argument, we can find a neighborhood U of x_0 such that every holomorphic function on D' has a possibly many-valued holomorphic extension to U . We consider the restriction $f|_{D'}$ of f to D' which has the image in a Stein space V . Then, as is well-known, $f|_{D'}$ has a possibly many-valued holomorphic extension $g: U \rightarrow V$, which is also considered as an extension of f . This concludes the proof of Lemma 1.3.

LEMMA 1.4. *Let D be a domain in the (z_1, z_2, \dots, z_n) -space and $x_0 := (a_1, \dots, a_n)$ be a boundary point of it. Consider hyperspheres B with the center (a_1, a_2) and S whose boundary contains (a_1, a_2) in the (z_1, z_2) -space. If D includes the set of all points $(z_1, z_2, a_3, \dots, a_n)$ such that (z_1, z_2) is contained in the interior of B and in the exterior of S , then D satisfies the condition (C) at x_0 .*

Proof. Let (b_1, b_2) be the center of S . We may assume that $a_1 \neq b_1$ and

write

$$S : |z_1 - a_1|^2 + |z_2 - a_2|^2 + 2 \times \operatorname{Re}((z_1 - a_1)(\bar{a}_1 - \bar{b}_1) + (z_2 - a_2)(\bar{a}_2 - \bar{b}_2)) \leq 0.$$

For our purpose, it suffices to take the maps defined as follows;

$$\varphi_\nu(\zeta) : z_1 = a_1 + \frac{1}{\bar{a}_1 - \bar{b}_1} \left(\frac{1}{\nu + \nu_0} - \rho \cdot \zeta (\bar{a}_2 - \bar{b}_2) \right), \quad z_2 = a_2 + \rho \cdot \zeta, \quad z_i = a_i \quad (3 \leq i \leq n),$$

and

$$\varphi(\zeta) : z_1 = a_1 - \rho \cdot \zeta \frac{\bar{a}_2 - \bar{b}_2}{\bar{a}_1 - \bar{b}_1}, \quad z_2 = a_2 + \rho \cdot \zeta, \quad z_i = a_i \quad (3 \leq i \leq n)$$

for a sufficiently large ν_0 and a sufficiently small $\rho > 0$, which have the properties (1), (2) and (3) in Definition 1.2.

The following proposition on local extensions of holomorphic maps is used later to give a general global extension theorem.

PROPOSITION 1.5. *In \mathbf{C}^n , consider the domains*

$$D := \{\delta < |z_1| < \rho_1, |z_2| < \rho_2, \dots, |z_n| < \rho_n\} \\ \cup \{|z_1| < \rho_1, |z_2| < \rho'_2, \dots, |z_n| < \rho'_n\}$$

and

$$D' := \{|z_1| < \rho_1, \dots, |z_n| < \rho_n\},$$

where $0 < \delta < \rho_1$, $0 < \rho'_i \leq \rho_i$ ($2 \leq i \leq n$). Then, every holomorphic map f of D into a taut complex space X can be extended to a holomorphic map of D' into X .

Proof. By virtue of Lemma 1.4, Proposition 1.5 can be proved by the argument as in the proof of the equivalence of two different types of definitions for pseudoconcave sets in Tadokoro [10]. On account of the possibility of multivalence of the extended function, we need some careful checks. To prove Proposition 1.5, we may assume that $\rho_i = \rho'_i$ ($3 \leq i \leq n$). Indeed, if it is proved in this case, the proof for the general case is easily given by mathematical induction. Take an arbitrary point $z' := (z'_1, a'_2, \dots, a'_n)$ with $|z'_1| = \frac{\delta + \rho_1}{2}$, $|a'_2| < \rho_2, \dots, |a'_n| < \rho_n$ and consider the straight line $L : z_1 = t \cdot z'_1, z_2 = a'_2, \dots, z_n = a'_n$ ($0 \leq t \leq 1$). If f is analytically continuable along L just before $a' = (a'_1, \dots, a'_n) \in L$ and not continuable to a' itself, then we call the point a' an α -point with respect to z' .

Let E be the set of all α -points in D' . For our purpose, it suffices to show $E = \emptyset$. Assume the contrary. Then, by the same manner as in [10],

pp. 284~285, it can be proved that, for a suitable a_0 in E and local coordinates w_1, \dots, w_n in a neighborhood of a_0 (let $a_0 = (\alpha_1, \dots, \alpha_n)$), in the (w_1, w_2) -space we can find a sufficiently small hypersphere B with the center (α_1, α_2) and a hypersphere S whose boundary contains (α_1, α_2) such that the set of all points $(w_1, w_2, \alpha_3, \dots, \alpha_n)$ with the property that (w_1, w_2) is contained in the interior of B and in the exterior of S does not intersect E . By the definition of E , f can be extended to a single-valued holomorphic map of a neighborhood of E into X . In this situation, f has a possibly many-valued extension g to a neighborhood of $E \cap U$. This contradicts the definition of α -points.

§ 2. Existence domains of holomorphic maps. Let M be an (unramified) Riemann domain over a complex manifold N with projection map $\pi : M \rightarrow N$ and X be an arbitrary complex space. By \mathcal{O}^X we denote the sheaf of germs of holomorphic maps defined on open subsets of N into X . The set \mathcal{O}^X has a canonically defined structure of complex manifold and the projection $\tilde{\pi} : \mathcal{O}^X \rightarrow N$ is locally biholomorphic. Then, by putting $v(f_x) = f(x)$ for each $f_x \in \mathcal{O}_x^X (x \in N)$, we can define a continuous map $v : \mathcal{O}^X \rightarrow X$.

Now, let f be a holomorphic map of M into X . We consider the map $\sigma : M \rightarrow \mathcal{O}^X$ which assigns the germ of $f \cdot (\pi|U)^{-1}$ at $\pi(x)$ to each point x in M , where U is a neighborhood of x such that $\pi|U : U \rightarrow \pi(U)$ is biholomorphic. Obviously, $v \cdot \sigma = f$, $\tilde{\pi} \sigma = \pi$ and σ is continuous. So, $\sigma(M)$ is connected. By $H^f(M)$ we denote the connected component of \mathcal{O}^X which includes $\sigma(M)$. Then, the map $\tilde{\pi}|H^f(M) : H^f(M) \rightarrow N$ may be considered to define a Riemann domain over N which includes M as a Riemann domain. Moreover, the map $f' := v|H^f(M) : H^f(M) \rightarrow X$ is a holomorphic extension of f to $H^f(M)$ because $f' \cdot \sigma = f$. As is easily seen, $H^f(M)$ is the largest one among Riemann domains over N which includes M as a Riemann domain and to which f can be holomorphically extended. Modeling after the case of functions, we give

DEFINITION 2.1. For a holomorphic map $f : M \rightarrow X$, we call the Riemann domain $H^f(M)$ over N constructed as the above *the existence domain of f* .

THEOREM 2.2. *Let M be a Riemann domain over a Stein manifold and f be a holomorphic map of M into a taut complex space. Then, the existence domain of f is a Stein manifold.*

For the proof we use the Docquier-Grauert's result on the Levi problem for Riemann domains over a Stein manifold. In their paper [2], many equivalent definitions of convexity for a Riemann domain were given. Among them, we shall use here particularly the notion of " p_r -convexity". For a Riemann domain M over N with the projection map $\pi : M \rightarrow N$, we denote the set of all (accessible) boundary points of M by ∂M and put $\check{M} = M \cup \partial M$. The set \check{M} has a canonically defined Hausdorff topology and π has a continuous extension $\check{\pi} : \check{M} \rightarrow N$. The definition of p_r -convexity is given as follows:

Let $\mathcal{D} := \{(z_1, \dots, z_n); |z_1| \leq 1, |z_k| < 1, 2 \leq k \leq n\}$, $\delta\mathcal{D} := \{z \in \mathcal{D}; |z_1| = 1\}$, $\overset{\circ}{\mathcal{D}} := \{z \in \mathcal{D}; |z_1| < 1\}$ and $\bar{\mathcal{D}} := \{|z_k| \leq 1, 1 \leq k \leq n\}$ in \mathbf{C}^n . A Riemann domain M of dimension n over a Stein manifold N with projection map $\pi : M \rightarrow N$ is said to be p_r -convex if and only if there is no continuous map $\varphi : \bar{\mathcal{D}} \rightarrow \check{M}$ with the property that 1) $\varphi(\delta\mathcal{D}) \in M$, $\varphi(\overset{\circ}{\mathcal{D}}) \subset M$, 2) $\varphi(\bar{\mathcal{D}}) \cap \partial M \neq \emptyset$ and 3) $\check{\pi}\varphi$ is the restriction to $\bar{\mathcal{D}}$ of a biholomorphic map of a neighborhood of $\bar{\mathcal{D}}$ onto an open subset of N .

The Docquier-Grauert's result which we need here is the following

THEOREM ([2], Satz 10, p. 113). *Any p_r -convex Riemann domain over a Stein manifold is Stein.*

Proof of Theorem 2.2. Without loss of generality, we may assume $H'(M) = M$. It suffices to show that M is p_r -convex. Assume that there is a continuous map $\varphi : \bar{\mathcal{D}} \rightarrow M$ satisfying the above conditions 1), 2) and 3). By the condition 3), $\check{\pi}\varphi$ is extended to a biholomorphic map ψ of a neighborhood U of $\bar{\mathcal{D}}$ onto an open set W in N . If we put $G := \varphi^{-1}(M) \cap U$, the map $g := \varphi|_G : G \rightarrow M$ is holomorphic because $\varphi|_G$ is a biholomorphic map onto an open set in M . Moreover, since $\delta\mathcal{D} \in G$ and $\overset{\circ}{\mathcal{D}} \subset G$ by the condition 1), we can find real numbers δ, ρ_1, ρ'_1 and ρ_i ($2 \leq i \leq n$) with $0 < \delta < 1 < \rho_1$, $0 < \rho'_i < 1 < \rho_i$ ($2 \leq i \leq n$) such that

$$D := \{\delta < |z_1| < \rho_1, |z_i| < \rho_i \ (2 \leq i \leq n)\} \cup \{|z_1| < \rho_1, |z_i| < \rho'_i \ (2 \leq i \leq n)\}$$

is included in G . In this situation, Proposition 1.5 implies that the map $g|_D : D \rightarrow M$ has a holomorphic extension $h : D' := \{|z_i| < \rho_i \ (1 \leq i \leq n)\} \rightarrow M$. Here, we may assume $U = D$. On the other hand, by the condition 2), there exists a point $z \in \bar{\mathcal{D}}$ such that $\varphi(z) \in \partial M$. If we take connected neighborhoods $U' (\subset U)$ of z and $W' (\subset W)$ of $\check{\pi} \cdot \varphi(z)$, then a suitable connected component W'_i of $\check{\pi}^{-1}(W') \cap M$ gives a neighborhood of $\varphi(z)$ in \check{M}

by the definition of the topology of \tilde{M} and $\pi|W'_i$ is a biholomorphic map onto an open subset \tilde{W}' of W' . The holomorphic map $f \cdot \pi^{-1}|_{\tilde{W}'} : \tilde{W}' \rightarrow X$ has a holomorphic extension $h \cdot \psi^{-1}|_{W'} : W' \rightarrow X$. It follows from Definition 2.1 that $\pi|W'_i : W'_i \rightarrow W'$ is a biholomorphic map. This contradicts the fact $\varphi(z) \in \partial M$ and completes the proof.

As a result of Theorem 2.2, we have the following global extension theorem.

THEOREM 2.3. *Let M be a Riemann domain over a Stein manifold and $H(M)$ be its envelope of holomorphy. Then, for any taut complex space X , every holomorphic map $f : M \rightarrow X$ can be extended to a holomorphic map $f : H(M) \rightarrow X$.*

Proof. By the definition of the envelope of holomorphy, we have a holomorphic map $\tau : M \rightarrow H(M)$ with $\pi = \tilde{\pi}\tau$, where π and $\tilde{\pi}$ are projection maps of Riemann domains M and $H(M)$ respectively. On the other hand, the existence domain $H^f(M)$ is Stein by Theorem 2.1. By the well-known argument the map $\sigma : M \rightarrow H^f(M)$ can be extended to a holomorphic map $\tilde{\sigma} : H(M) \rightarrow H^f(M)$ with $\tilde{\sigma}\tau = \sigma$. The map $f' \cdot \tilde{\sigma} : H(M) \rightarrow X$ is a holomorphic extension of f to $H(M)$, where f' is a holomorphic extension of f to $H^f(M)$.

COROLLARY 2.4 (c.f. H. Wu, [11], p. 211). *If a Riemann domain M over a Stein manifold is taut, then M is Stein.*

Proof. The identity map $\text{id}_M : M \rightarrow M$ has a holomorphic extension $h : H(M) \rightarrow M$ with $h\tau = \text{id}_M$, where $\tau : M \rightarrow H(M)$ is a canonical holomorphic map with $\tilde{\pi}\tau = \pi$. Obviously, τh is the identity map of $H(M)$. This shows that M is biholomorphic with $H(M)$, whence M is Stein.

§ 3. A counter example. Theorem 2.2 shows that the simultaneous continuability of holomorphic functions leads to the simultaneous continuability of holomorphic maps defined on subdomains in a Stein manifold into a taut complex space. On the contrary, the analogous extension theorem for holomorphic maps defined on a complex space is not valid in general. In this section, we shall give a counter example.

Let P be a projective algebraic manifold of dimension ≥ 1 . Moreover, assume that P is hyperbolic in the sense of S. Kobayashi ([8], p. 465 and [9], p. 10). For example, a closed Riemann surface of genus at least two satisfies these conditions. According to H. Grauert [4], taking a suitable line

bundle F over P , we can find a relatively compact strongly pseudoconvex neighborhood X of the zero section Z of F . Then, there are a normal complex space M and a holomorphic map $\tau : X \rightarrow M$ such that, for a special point x_0 in M , $\tau^{-1}(x_0) = Z$ and $\tau|_{X-Z} : X - Z \rightarrow M - \{x_0\}$ is a biholomorphic map (c.f. [4], Satz 5, p. 340). Shrinking a neighborhood X of Z we may assume that M is complete hyperbolic. Then we want to prove that the space X is taut.

By d_P , d_M and d_X we denote the so-called Kobayashi pseudo-distance on P , M and X respectively (c.f. [8], p. 462 and [9], Definition, p. 10). We shall first show that X is hyperbolic. Assume the contrary. Then, there are two points $x, y \in X (x \neq y)$ such that $d_X(x, y) = 0$. By the distance-decreasing property of holomorphic maps with respect to the Kobayashi pseudo-distance, we have $d_M(\tau(x), \tau(y)) \leq d_X(x, y) = 0$. This implies that $\tau(x) = \tau(y)$ from the hyperbolicity of M . So, $\tau(x) = \tau(y) = x_0$, i.e., $x, y \in Z$, because τ is injective on $X - Z$. On the other hand, since F is a line bundle over P , the canonical projection map $\pi : F \rightarrow P$ gives a biholomorphic map $\pi|_Z : Z \rightarrow P$. Again using the distance-decreasing property, we see $d_P(\pi(x), \pi(y)) \leq d_X(x, y) = 0$. By the hyperbolicity of P , we have $\pi(x) = \pi(y)$ and hence $x = y$, which is a contradiction. Therefore, X is hyperbolic. Now, it is easily proved that X is complete with respect to the distance d_X because M is complete and $\tau : X \rightarrow M$ is proper. Accordingly, we come to the conclusion that X is taut.

Consider the map $f := (\tau|_{X-Z})^{-1}$. It is a holomorphic map of $M - \{x_0\}$ into a taut complex space X which obviously cannot be extended to a holomorphic map of M into X . This gives a desired counter example.

§ 4. Extensions across an analytic subset of codimension one.

Let S be a thin analytic subset of a domain D in \mathbb{C}^n and f be a holomorphic map of $D - S$ into a taut complex space X . If $\dim S \leq n - 2$, f is continuable to a holomorphic map of D into X by virtue of Theorem 2.3. In case of $\dim S = n - 1$, we can prove

THEOREM 4.1. *Assume that, for each irreducible component S_i of S , there is a sequence $\{a_\nu\}$ in $D - S$ which converges to some regular point of S_i such that $\{f(a_\nu)\}$ has a limit point in X . Then f can be extended to a holomorphic map of D into X .*

For the proof, we need the following result of M. H. Kwack in [9].

THEOREM. *Let X be a hyperbolic complex space and f be a holomorphic map of the domain $B^* := \{z \in \mathbf{C}; 0 < |z| < 1\}$ into X . If, for a suitable sequence $\{\alpha_\nu\}$ in B^* with $\lim_{\nu \rightarrow \infty} \alpha_\nu = 0$, $\{f(\alpha_\nu)\}$ converges to a point in X , then f can be extended to a holomorphic map of $B := \{|z| < 1\}$ into X .*

Remark. Though the above theorem is slightly modified from the original, we can prove it by the same argument as in the proof of Theorem 3 in [9], p. 14.

Proof of Theorem 4.1. Without loss of generality, we may assume that S is irreducible. As is well-known, the set of all singularities of S is an analytic set of dimension $\leq n - 2$ in D . There is no harm in assuming that S is regular. Moreover, it may be assumed that $D := \{|z_1| < 1, \dots, |z_n| < 1\}$ and $S := \{z = (z_1, \dots, z_n) \in D; z_n = 0\}$. Indeed, if Theorem 4.1 is proved in this case, the set of all $x \in S$ such that f is continuable to a neighborhood of x is an open and closed subset of S and hence coincides with the whole set S .

Put $a_\nu = (a_1^{(\nu)}, \dots, a_{n-1}^{(\nu)}, a_n^{(\nu)})$ and $\lim_{\nu \rightarrow \infty} a_\nu = (a_1, \dots, a_{n-1}, 0)$. The condition $a_\nu \in D - S$ implies $0 < |a_n^{(\nu)}| < 1$. For each ν , we take an integer k_ν such that $2^{-(k_\nu+2)} \leq |a_n^{(\nu)}| < 2^{-(k_\nu+1)}$. Choosing a subsequence and changing indices if necessary, we may assume that $1 \leq k_1 < k_2 < \dots$. Moreover, $\{2^{k_\nu} a_n^{(\nu)}\}$ may be assumed to converge to a point b with $\frac{1}{4} \leq |b| \leq \frac{1}{2}$. Now, we define the holomorphic maps $f_\nu(z_1, \dots, z_{n-1}, z_n) := f\left(z_1, \dots, z_{n-1}, \frac{z_n}{2^{k_\nu}}\right)$ of $D - S$ into X . Then, for $a_\nu = (a_1^{(\nu)}, \dots, a_{n-1}^{(\nu)}, 2^{k_\nu} a_n^{(\nu)})$, $\{f_\nu(a_\nu)\}$ converges to a point in X and $\lim_{\nu \rightarrow \infty} a_\nu = (a_1, \dots, a_{n-1}, b) \in D - S$. The sequence $\{f_\nu\}$ in $\text{Hol}(D - S, X)$ cannot have a compactly divergent subsequence and hence it has a convergent subsequence because $\text{Hol}(D - S, X)$ is normal. We may assume that $\lim_{\nu \rightarrow \infty} f_\nu = g$ exists in $\text{Hol}(D - S, X)$. Take an arbitrary point z' in $D' := \{|z_1| < 1, \dots, |z_{n-1}| < 1\}$ and put $h(z_n) := f(z', z_n)$, which is considered as a map of $B^* := \{0 < |z_n| < 1\}$ into X . Then for $\alpha_\nu := \frac{1}{2^{k_\nu+1}} (\nu = 1, 2, \dots)$ $\lim_{\nu \rightarrow \infty} \alpha_\nu = 0$ and $\lim_{\nu \rightarrow \infty} h(\alpha_\nu) = \lim_{\nu \rightarrow \infty} f_\nu\left(z', \frac{1}{2}\right) = g\left(z', \frac{1}{2}\right) \in X$ exists. On the other hand, it was shown by P. Kiernan ([7]) that a taut complex manifold is always hyperbolic and this assertion can be easily generalized to the case of complex spaces. So, X is hyperbolic. According to the above M. H. Kwack's result, h is extended to a map of $B := \{|z_n| < 1\}$ into X .

Thus, we obtain a map $f(z_1, \dots, z_n) : D \rightarrow X$, which is an extension of f and holomorphic in z_n for each fixed $z' = (z_1, \dots, z_{n-1})$ in D' . To complete the proof of Theorem 4.1, it suffices to show that f is continuous on D . To this end, take an arbitrary sequence $z'_\nu = (z_1^{(\nu)}, \dots, z_{n-1}^{(\nu)})$ in D' with $\lim_{\nu \rightarrow \infty} z'_\nu = z'_0 = (z_1^{(0)}, \dots, z_{n-1}^{(0)}) \in \bar{D}'$ and consider the maps $h_\nu(z_n) := f(z'_\nu, z_n)$ in $\text{Hol}(B, X)$. Since $\lim_{\nu \rightarrow \infty} h_\nu(z_n) = f(z'_0, z_n) \in X$ for any fixed z_n with $0 < |z_n| < 1$, $\{h_\nu\}$ has no compactly divergent subsequence. By the normality of $\text{Hol}(B, X)$, $\{h_\nu\}$ has a sequence which converges in $\text{Hol}(B, X)$ and whose limit is necessarily equal to $h_0(z_n) := f(z_1^{(0)}, \dots, z_{n-1}^{(0)}, z_n)$. Accordingly, $\{h_\nu\}$ itself converges to h_0 . This shows that f is continuous and completes the proof of Theorem 4.1.

Remark. In Theorem 4.1, we cannot omit the assumption that the limit of $\{a_\nu\}$ is a regular point of S_i . For example, consider the taut complex space X defined as the Riemann sphere minus $\{0, 1, \infty\}$ and the analytic subset $S := \{z_1 = 0\} \cup \{z_2 = 0\} \cup \{z_1 = z_2\}$ in \mathbb{C}^2 . Putting $f(z_1, z_2) = \frac{z_2}{z_1}$ on $\mathbb{C}^2 - S$, we have the holomorphic map of $\mathbb{C}^2 - S$ into X which cannot be extended to a holomorphic map of \mathbb{C}^2 into X but has an arbitrary point in X as a cluster value at the origin.

As direct consequences of Theorem 4.1, we have

COROLLARY 4.2 (M.H. Kwack). *Let S be a thin analytic subset of a domain D in \mathbb{C}^n . Then every holomorphic map of $D - S$ into a compact taut complex space X can be extended to a holomorphic map of D into X .*

COROLLARY 4.3. *Under the same assumption as in Corollary 4.2, if a holomorphic map of $D - S$ into a taut complex space X can be extended to an open subset of D which intersects with each irreducible component of S , it can be extended to the whole set D .*

Remark. Corollary 4.3 is also a consequence of Theorem 2.3. Because, as is well-known, if a domain D' with $D - S \subset D' \subset D$ intersects with each irreducible component of S , then the envelope of holomorphy of D' includes D .

COROLLARY 4.4. *Let X be a taut complex space and assume that there exists a negative real-valued continuous function $u(x)$ on X such that $u(x)$ is plurisubharmonic on $X - K$ for a suitable compact subset K of X and $\{x \in X; u(x) < c\}$ is relatively compact in X for any $c < 0$. If S is a thin analytic subset of a domain D in \mathbb{C}^n ,*

then every holomorphic map f of $D - S$ into X has a holomorphic extension to D .

Proof. We may assume that S is regular connected and of dimension $n - 1$. Suppose that f cannot be extended to S . By virtue of Theorem 4.1, $\{f(a_\nu)\}$ has no accumulation point in X for any sequence $\{a_\nu\}$ converging to a point in S . Therefore, an arbitrary point a_0 in S has a neighborhood U such that $f(U - S) \cap K = \emptyset$. Then, $v := u \cdot f$ is a non-positive plurisubharmonic function on $U - S \cap U$. Moreover, $\lim_{z \rightarrow a_0} v(z) = 0$. Putting $v(z) = 0$ for any $z \in S \cap U$, we have a plurisubharmonic function v on U (c.f. Grauert-Remmert [5]). This contradicts the maximum principle for plurisubharmonic functions. Thus we have Corollary 4.4.

§ 5. **A generalization of the big Picard theorem.** The extension problem of holomorphic maps into a taut complex space is closely related to the classical big Picard theorem. Here, we shall study holomorphic maps into the N -dimensional complex projective space $P_N(\mathbf{C})$ minus some hyperplanes.

In [3], J. Dufresnoy gave the following profound theorem.

THEOREM ([3], p. 18). *Let D be a domain in the complex plane and \mathcal{F} be a family of holomorphic maps of D into the complement of arbitrarily given $2N + 1$ hyperplanes in general position in $P_N(\mathbf{C})$. Then \mathcal{F} is relatively compact in $\text{Hol}(D, P_N(\mathbf{C}))$.*

As a consequence of this result, we have the following theorem, which gives an answer to the conjecture of H. Wu (e.g., [13], p. 216).

THEOREM 5.1. *For arbitrarily given $2N + 1$ hyperplanes $H_1, H_2, \dots, H_{2N+1}$ in general position in $P_N(\mathbf{C})$, the space $X := P_N(\mathbf{C}) - (\cup_{k=1}^{2N+1} H_k)$ is a taut complex space.*

Proof. Owing to the result of T.J. Barth [1], we have only to show that $\text{Hol}(B, X)$ is normal for the special domain $B := \{|z| < 1\}$ in \mathbf{C} . Take a sequence $\{f_\nu\}$ in $\text{Hol}(B, X)$. By the above theorem a suitable subsequence $\{f_{\nu_k}\}$ converges to a map g in $\text{Hol}(B, P_N(\mathbf{C}))$. For our purpose, it suffices to ascertain that $g(B) \subset X$ if $g(B) \cap X \neq \emptyset$. For each $i(1 \leq i \leq 2N + 1)$ we consider the set $E_i := \{z \in B; g(z) \in H_i\}$. Obviously, E_i is closed in B . On the other hand, for any $z_0 \in E_i$, we can choose neighborhoods U of z_0 and V of $g(z_0)$ such that $g(U) \subset V$ and $f_{\nu_k}(U) \subset V$ for almost all k and

$H_i \cap V$ is the set of all zeros of a non-zero linear form l on V . Since $l \cdot f_{\nu_k} \neq 0$ on V and $l \cdot g(z_0) = 0$, $\lim_{k \rightarrow \infty} l \cdot f_{\nu_k} = l \cdot g$ vanishes identically on V . This shows that E_i is open in B . Eventually, $E_i = D$ if $E_i \neq \phi$. Therefore, we see $g(D) \cap X = \phi$ if $E_i = D$ for some i and $g(D) \subset X$ if $E_i = \phi$ for any i . This completes the proof.

Now, we shall prove the following generalization of the big Picard theorem.

THEOREM 5.2. *Let S be a regular thin analytic subset of a domain D in \mathbf{C}^n . Then every holomorphic map of $D - S$ into the complement X of $2N + 1$ hyperplanes $H_1, H_2, \dots, H_{2N+1}$ in general position in $P_N(\mathbf{C})$ can be extended to a holomorphic map of D into $P_N(\mathbf{C})$.*

Proof. Without loss of generality, we may assume that S is of dimension $n - 1$. Furthermore, it may be assumed that $D = \{|z_1| < 1, \dots, |z_n| < 1\}$ and $S := \{(z_1, \dots, z_n) \in D; z_n = 0\}$. As in the proof of Theorem 4.1, consider the holomorphic maps $f_\nu(z_1, \dots, z_{n-1}, z_n) := f\left(z_1, \dots, z_{n-1}, \frac{z_n}{2^\nu}\right)$ ($z := (z_1, \dots, z_{n-1}, z_n) \in D - S$ and $\nu \geq 1$) in $\text{Hol}(D - S, X)$. By the above result of Dufresnoy, a suitable subsequence $\{f_{\nu_k}\}$ converges to g in $\text{Hol}(D - S, P_N(\mathbf{C}))$. By the same argument as in the proof of Theorem 5.1, it holds either $g(D - S) \subset H_i$ or $g(D - S) \cap H_i = \phi$ for each i ($1 \leq i \leq 2N + 1$). Since H_i ($1 \leq i \leq 2N + 1$) are located in general position, we can choose some i_0 such that $g(D - S) \cap H_{i_0} = \phi$. Let $w_0 : w_1 : \dots : w_N$ be a system of homogeneous coordinates on $P_N(\mathbf{C})$ such that $\{w_0 = 0\} = H_{i_0}$. The space $P_N(\mathbf{C}) - H_{i_0}$ may be considered as the space \mathbf{C}^N and $w^{(1)} = \frac{w_1}{w_0}, \dots, w^{(N)} = \frac{w_N}{w_0}$ give the global coordinate system on \mathbf{C}^N . Put $f_k^{(i)} = w^{(i)} \cdot f_{\nu_k}$, $f^{(i)} = w^{(i)} \cdot f$ and $g^{(i)} = w^{(i)} \cdot g$ ($1 \leq i \leq N$). Since $\lim_{k \rightarrow \infty} f_k^{(i)} = g^{(i)}$ uniformly on the compact set $E := \left\{ |z_1| \leq \frac{1}{2}, \dots, |z_{n-1}| \leq \frac{1}{2}, |z_n| = \frac{1}{2} \right\}$ in $D - S$, we can find a real constant M such that $|f_k^{(i)}(z)| \leq M$ ($1 \leq i \leq N$) for any $z \in E$ and k . Therefore, if $|z_1| \leq \frac{1}{2}, \dots, |z_{n-1}| \leq \frac{1}{2}, |z_n| = \frac{1}{2^{\nu_k+1}}$, we have $|f^{(i)}(z_1, \dots, z_n)| = |f_k^{(i)}(z_1, \dots, z_{n-1}, 2^{\nu_k} z_n)| \leq M$. Then, considering the holomorphic functions $h^{(i)}(z_n) := f^{(i)}(z_1, \dots, z_{n-1}, z_n)$ in z_n for arbitrarily fixed z_1, \dots, z_{n-1} with $|z_i| \leq \frac{1}{2}$ ($1 \leq i \leq n - 1$) and applying the maximum principle, we obtain $|f^{(i)}(z_1, \dots, z_n)| \leq M$ if $\frac{1}{2^{\nu_k+1+1}} \leq |z_n| \leq \frac{1}{2^{\nu_k+1}}$ for any k . Eventually, $|f^{(i)}(z_1, \dots, z_n)| \leq M$ on $\left\{ |z_1| \leq \frac{1}{2}, \dots, |z_{n-1}| \leq \frac{1}{2}, 0 < |z_n| \leq \frac{1}{2} \right\}$. In this sit-

uation, Theorem 5.2 is an immediate consequence of the classical Riemann's theorem on removable singularities of bounded holomorphic functions.

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