William W. Adams
Nagoya Math. J.
Vol. 42 (1971), 79–87

SIMULTANEOUS ASYMPTOTIC DIOPHANTINE APPROXIMATIONS TO A BASIS OF A REAL NUMBER FIELD

WILLIAM W. ADAMS

1. Introduction

The purpose of this paper is to prove the following result.

THEOREM 1. *Let K be a real algebraic number field of degree $m = n + 1$. Let $1, \beta_1, \cdots, \beta_n$ be a basis of K. For a given constant $C > 0$ set $\lambda_B = \lambda_B(\beta_1, \cdots, \beta_n, C)$ equal to the number of solutions in integers $q, p_1, \cdots, p_n$ of the inequalities*

$$* \qquad 0 < q\beta_i - p_i < C/q^{\frac{1}{n}} \qquad (1 \leq i \leq n)$$

$$** \qquad 1 \leq q \leq B.$$

Then $\lambda_B = 0(1)$ or there is a $C' > 0$ such that $\lambda_B \sim C' \log B$ $(B \to \infty)$.

There is a dual theorem.

THEOREM 2. *With $\beta_1, \cdots, \beta_n, C$ as in Theorem 1 set Λ_B equal to the number of solutions in integers $q_1, \cdots, q_n, p$ of the inequalities*

$$0 < q_1\beta_1 + \cdots + q_n\beta_n - p < C/q^n$$

$$1 \leq q_1, \cdots, q_n \leq B$$

where $q = \max(q_1, \cdots, q_n)$. Then either $\Lambda_B = 0(1)$ or there is a $C'' > 0$ such that $\Lambda_B \sim C'' \log B$ $(B \to \infty)$.

These results generalize the results of [2, 3]. However the work of [2, 3] had the advantage that the constant C' was more precisely defined. For related work see [1, 4, 5].

The author would like to thank W.M. Schmidt and S. Kuroda for not believing that the results of [2] could not be generalized to higher dimensions.

Received April 28, 1970.

The proof of the present result closely parallels the proof in [2]. Hence this paper will sometimes be sketchy.

2. The reduction to counting units

Denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$ the integers, rationals, reals and complexes. Let $\tau_0 =$ identity, $\tau_1, \cdots, \tau_n$ denote the distinct embeddings of K into $\boldsymbol{C}$. Assume $\tau_0, \cdots, \tau_r$ are the real embeddings. For $\alpha \in K$, set $\tau_i \alpha = \alpha^{(i)}$ and assume as usual $n = r + 2s$ and $\alpha^{(r+i)} = \bar{\alpha}^{(r+s+i)}$ $(1 \leq i \leq s)$.

LEMMA 1. *The dual basis $\alpha_0, \alpha_1, \cdots, \alpha_n$ of K (with respect to the trace) has the following properties*

$$\alpha_0^{(i)} + \alpha_1^{(i)}\beta_1 + \cdots + \alpha_n^{(i)}\beta_n = 0 \ (1 \leq i \leq n) \tag{1}$$

$$\alpha_0 + \alpha_1\beta_1 + \cdots + \alpha_n\beta_n = 1 \tag{2}$$

$$A = (\alpha_j^{(i)}) \quad (1 \leq i, j \leq n) \text{ is nonsingular.} \tag{3}$$

For the proof see [2, 5].

Let M be the free $\boldsymbol{Z}$-module generated by $\alpha_0, \cdots, \alpha_n$. Let $\mathcal{O}$ be the associated order. Let U be the group of units of $\mathcal{O}$. Then by the Dirichlet Unit Theorem there are units $\zeta_1, \cdots, \zeta_{r+s} > 1$ in U such that $U = \{\pm \zeta_1^{\nu_1} \cdots \zeta_{r+s}^{\nu_{r+s}}\}$. For $\nu = (\nu_1, \cdots, \nu_{r+s}) \in \boldsymbol{Z}^{r+s}$ set $\zeta^\nu = \zeta_1^{\nu_1} \cdots \zeta_{r+s}^{\nu_{r+s}}$ so $U = \{\pm \zeta^\nu\}$ $(\nu \in \boldsymbol{Z}^{r+s})$.

For $\xi_1, \xi_2 \in M$ write $\xi_1 \sim \xi_2$ if there is a $\zeta \in U$ such that $\xi_1 = \zeta\xi_2$ (this is an equivalence relation). If $\Omega \subseteq M$ is an equivalence class, then for all $\xi \in M$, $|\boldsymbol{N}\xi|$ is the same and we denote this by $\boldsymbol{N}\Omega$ ($\boldsymbol{N}$ denotes the norm of $K/\boldsymbol{Q}$).

Now for $\zeta \in M$ we have unique $q, p_1, \cdots, p_n \in \boldsymbol{Z}$ such that $\xi = q\alpha_0 + p_1\alpha_1 + \cdots + p_n\alpha_n$. In this way we view $\xi \in M$ as being in 1-1 correspondence with possible solutions to * and **. Moreover by (1) we see

$$-\xi^{(i)} = \alpha_1^{(i)}(q\beta_1 - p_1) + \cdots + \alpha_n^{(i)}(q\beta_n - p_n) \quad (1 \leq i \leq n). \tag{4}$$

LEMMA 2. *There are only finitely many classes $\Omega \subseteq M$ which yield solutions to* *.

Proof. Combine * and (4) with the known fact that there are only finitely many Ω with $\boldsymbol{N}\Omega$ below a given value.

It suffices to show the following: *if for a fixed class $\Omega \subseteq M$ there are infinitely many solutions to * then there is a $C_1 > 0$ such that the number of $\xi \in \Omega$ satisfying * and ** is asymptotic to $C_1 \log B$.*

We now need some notation. Set $\Omega = \{\pm \zeta^\nu \xi_0\}$ $(\nu \in \boldsymbol{Z}^{r+s})$ for any $\xi_0 > 0$ in Ω. Let Ω^+ (Ω^-) denote the positive (negative) elements for Ω. Let

$\xi_\nu = \zeta^\nu \xi_0$ ($\nu \in \boldsymbol{Z}^{r+s}$) denote a typical element of Ω^+. Set

$$\xi_\nu = q_\nu \alpha_0 + p_{1\nu}\alpha_1 + \cdots + p_{n\nu}\alpha_n.$$

Write

$$\gamma_{i\nu} = q_\nu \beta_i - p_{i\nu} \quad (1 \leq i \leq n) \tag{5}$$

and so (4) is

$$(\xi_\nu^{(1)}, \cdots, \xi_\nu^{(n)})^t = -A(\gamma_{1\nu}, \cdots, \gamma_{n\nu})^t. \tag{6}$$

For $q_\nu > 0$ let $\delta_{i\nu} = q_\nu^{\frac{1}{n}} \gamma_{i\nu}$. Then we wish to count $\lambda_B(\Omega^+)$, the number of $\xi_\nu \in \Omega^+$ such that

$$0 < \delta_{1\nu}, \cdots, \delta_{n\nu} < C \tag{7}$$

$$1 \leq q_\nu \leq B \tag{8}$$

(the corresponding $\lambda_B(\Omega^-)$ will be shown to be bounded). For $q_\nu \neq 0$ set

$$\kappa_\nu' = \alpha_0 + \frac{p_{1\nu}}{q_\nu}\alpha_1 + \cdots + \frac{p_{n\nu}}{q_\nu}\alpha_n$$

so that

$$\xi_\nu = q_\nu \kappa_\nu' \tag{9}$$

$$\kappa_\nu' = 1 + \varepsilon_\nu / q_\nu \tag{10}$$

where

$$\varepsilon_\nu = \alpha_1 \gamma_{1\nu} + \cdots + \alpha_n \gamma_{n\nu}. \tag{11}$$

For $q_\nu > 0$

$$\delta_{i\nu} = (\xi_0 \kappa_\nu'^{-1})^{\frac{1}{n}} \zeta^{\frac{\nu}{n}} \gamma_{i\nu}. \tag{12}$$

Set

$$\eta_{i\nu} = \zeta^{\frac{\nu}{n}} \gamma_{i\nu} \tag{13}$$

$$\kappa_\nu = (\xi_0 \kappa_\nu'^{-1})^{\frac{1}{n}} \tag{14}$$

so that

$$\delta_{i\nu} = \kappa_\nu \eta_{i\nu}. \tag{15}$$

LEMMA 3. *Let $\lambda_B'(\Omega^+)$ be the number of solutions, v, of*

i) ξ_ν *is sufficiently large*

ii) $0 < max(\eta_{1\nu}, \cdots, \eta_{n\nu}) < 10\xi_0^{-\frac{1}{n}} C = C_2$

iii) $0 < \eta_{1\nu}, \cdots, \eta_{n\nu} < C\kappa_\nu^{-1}$

iv) $1 \leq \xi_\nu \leq 2B$.

Then $\lambda_B(\Omega^+) + 0(1) \leq \lambda_B'(\Omega^+) \leq \lambda_{4B}(\Omega^+) + 0(1)$.

Proof. By i) and ii) we see $q_\nu \neq 0$ since $q_\nu = 0$ implies for $\eta_{i\nu} > 0$

$$C_2 > \eta_{i\nu} = \zeta^{\frac{\nu}{n}}(-p_{i\nu}) \geq \zeta^{\frac{\nu}{n}} = \xi_0^{-\frac{1}{n}} \xi_\nu^{\frac{1}{n}}$$

which violates i). Now by i) ζ^ν is large so by ii) and (13) $\gamma_{i\nu}$ is small so by (11) ε_ν is small and so from (10) we may assume $\frac{1}{2} \leq \kappa_\nu' \leq \frac{3}{2}$. From this and (9) we have q_ν is large and positive. In this situation (7) and iii) are equivalent (see (15)). Finally by iv) $q_\nu = \xi_\nu \kappa_\nu'^{-1} \leq 4B$. Thus the right hand inequality is true. Conversely assume (7) and (8). We may assume q_ν is large. So from (7), (10), (11), (12) we see that κ_ν' is close to 1 and so by (9) ξ_ν is large. Again (7) and iii) are equivalent. Here iii) implies ii). Finally by (8), $\xi_\nu = q_\nu \kappa_\nu' \leq 2B$ and so we are done.

It follows from the above argument that solutions of * are such that ξ_ν and q_ν have the same mangitude; in particular $\xi_\nu > 0$ and so $\lambda_B(\Omega^-) = 0(1)$.

So now we know it suffices to show that if there are an infinite number of solutions, $\lambda_B'(\Omega^+) \sim C_1 \log B$ (with $C_1 > 0$).

3. Counting the relevant units

We now essentially prove the theorem except that instead of counting $1 \leq q_\nu \leq B$ we count $1 \leq \nu_1 \leq N$. We then put B back.

For $1 \leq i \leq r + s$ set

$$X_i = (\log|\zeta_i^{\frac{1}{n}} \zeta_i^{(2)}|, \cdots, \log|\zeta_i^{\frac{1}{n}} \zeta_i^{(r+s)}|, \arg \zeta_i^{(r+1)}, \cdots, \arg \zeta_i^{(r+s)})^t$$

where we assume for $Z \in \boldsymbol{C}$ that the argument of Z satisfies $0 \leq \arg Z < 2\pi$. So $X_i \in \boldsymbol{R}^{n-1}$. Further for $r + s + 1 \leq i \leq n$ set $X_i = (0, \cdots, 0, 2\pi, 0, \cdots, 0)^t$ with 2π in the i^{th} spot.

Now $\det(X_1, \cdots, \hat{X}_i, \cdots, X_n) = 0$ for $i = 1, 2, \cdots, r + s$ violates the known rank for the regulator matrix. So without loss of generality (relabeling $\zeta_1, \cdots \zeta_{r+s}$) we may assume

$$\det(X_2, \cdots, X_n) \neq 0. \tag{16}$$

LEMMA 4. *Let $\#'_N$ be the number of solutions of the inequalities*

$$0 < \eta_{1\nu}, \cdots, \eta_{n\nu} < C_3 \tag{17}$$
$$1 \leq \nu_1 \leq N \quad (\text{or } 1 \leq -\nu_1 \leq N).$$

Then

$$\#'_N \sim C_4 N \quad (N \to \infty)$$

where $C_4 > 0$ is some constant.

Proof. Set

$$A_1 = -\begin{pmatrix} \xi_0^{(1)-1} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \cdot\ \xi_0^{(n)-1} \end{pmatrix} A$$

with A as in (3). Also set $\rho_{i\nu} = \zeta^{\overline{\frac{\nu}{2}}} \zeta^{(i)\nu} (1 \leq i \leq n)$. Then by (6) and (13)

$$A_1(\eta_{1\nu}, \cdots, \eta_{n\nu})^t = (\rho_{1\nu}, \cdots, \rho_{n\nu})^t.$$

Now $\rho_{r+s+i,\nu} = \bar{\rho}_{r+i,\nu}$ $(1 \leq i \leq s)$ so we omit the last s coordinates in the vector of ρ's.

That is, define the linear transformation A_2 by

$$\boldsymbol{R}^n \xrightarrow{A_1} \boldsymbol{R}^r \times \boldsymbol{C}^{2s} \xrightarrow{\text{proj.}} \boldsymbol{R}^r \times \boldsymbol{C}^s = \boldsymbol{R}^n \text{ so}$$
$$A_2(\eta_{1\nu}, \cdots, \eta_{n\nu})^t = (\rho_{1\nu}, \cdots, \rho_{r+s,\nu})^t.$$

We show A_2 is non singular. We know from (3) of Lemma 1 that A is non singular and so also A_1 is non sigular (i.e., $\det A_1 \neq 0$). Now if the row vectors of A_1 are $A^{(1)}, \cdots, A^{(n)}$ we see $A^{(r+i)} = \bar{A}^{(r+s+i)}$ $(1 \leq i \leq s)$ and that the row vectors of A_2 are $A^{(1)}, \cdots, A^{(r)}, Re(A^{(r+1)}), \cdots, Re(A^{(r+s)}), Im(A^{(r+1)}), \cdots, Im(A^{(r+s)})$. So we see that $\det A_2 = 2^{-s} \det A_1 \neq 0$, as desired.

Now

$$\begin{aligned} \rho_{1\nu}\rho_{2\nu} \cdots \rho_{n\nu} &= \rho_{1\nu} \cdots \rho_{r\nu} |\rho_{r+1,\nu}|^2 \cdots |\rho_{r+s,\nu}|^2 \\ &= (N\zeta_1)^{\nu_1} \cdots (N\zeta_{r+s})^{\nu_{r+s}} \\ &= \pm 1 \end{aligned}$$

So the images of the points $(\eta_{1\nu}, \cdots, \eta_{n\nu})$ under A_2 lie on the surface $\mathscr{S}$:

$u_1u_2 \cdots u_r|u_{r+1}|^2 \cdots |u_{r+s}|^2 = \pm 1$ in $\boldsymbol{R}^n = \boldsymbol{R}^r \times \boldsymbol{C}^s$ (so $(u_1, \cdots, u_r) \in \boldsymbol{R}^r$, $(u_{r+1}, \cdots, u_{r+s}) \in \boldsymbol{C}^s$). Define $\phi: \mathscr{S} \to \boldsymbol{R}^{n-1}$ by

$$\phi(u_1, \cdots, u_{r+s}) = (\log|u_2|, \cdots, \log|u_{r+s}|, \arg u_{r+1}, \cdots, \arg u_{r+s})^t.$$

Then on any part of $\mathscr{S}$ where the signs of $u_1, \cdots, u_r$ are fixed we see ϕ is 1-1 and has as image

$$\mathscr{D} = \{(x_1, \cdots, x_{n-1}) | 0 \leq x_i < 2\pi \text{ for } r+s \leq i < n\}.$$

Now our problem is to count the number of $(\eta_{1\nu}, \cdots, \eta_{n\nu})$ lying in an open box in $\boldsymbol{R}^n$. This then is equivalent to counting the number of $(\rho_{1\nu}, \cdots, \rho_{r+s,\nu})$ lying in some open parallelepiped in $\boldsymbol{R}^n = \boldsymbol{R}^r \times \boldsymbol{C}^s$ which meets $\mathscr{S}$ in a set open in $\mathscr{S}$. Then again restricting ourselves to a portion of $\mathscr{S}$ with the signs of $u_1, \cdots, u_r$ fixed we see the image $\mathscr{R}$ under ϕ *of this subset of* $\mathscr{S}$ *is a bounded open set together with some of its* boundary. Moreover it is clear that the volume of the boundaries of all the sets involved is zero. We wish to count the number of $\phi(\rho_{1\nu}, \cdots, \rho_{r+s,\nu})$ lying in $\mathscr{R}$.

We see readily

$$\phi(\rho_{1\nu}, \cdots, \rho_{r+s,\nu}) \equiv \nu_1 X_1 + \cdots + \nu_{r+s} X_{r+s} \pmod{2\pi}$$

where the congruence is read only in the last s coordinates. So when the last s coordinates are reduced mod 2π we want to count which $\nu_1 X_1 + \cdots + \nu_{r+s} X_{r+s}$ lies in $\mathscr{R}$. This is clearly the same as counting the number of $\nu_1 X_1 + \cdots + \nu_n X_n \in \mathscr{R}$ (see the definition of $X_{r+s+1}, \cdots, X_n$).

Let Λ be the lattice in $\boldsymbol{R}^{n-1}$ spanned by $X_2, \cdots, X_n$ (a lattice by (16)). Thus we now see that we wish to count the number of $\nu_1 X_1$ $(1 \leq \nu_1 \leq N)$ lying in $\mathscr{R}$ mod Λ. It is a well known theorem in uniform distribution theory that if $\mathscr{R}$ is a set whose boundary has zero volume, then there is a $C_4 \geq 0$ such that the number of $\nu_1 X_1$ in $\mathscr{R}$ mod Λ, $1 \leq \nu_1 \leq N$ $(1 \leq -\nu_1 \leq N)$ is asymptotic to $C_4 N$.

It should be recalled that the region $\mathscr{R}$ depended on the sign of the coordinates $\rho_{1\nu}, \cdots, \rho_{r\nu}$. The sign of $\rho_{i\nu}$ depends only on the sign of $\zeta_1^{(i)\nu_1}, \cdots, \zeta_{r+s}^{(i)\nu_{r+s}}$ $(1 \leq i \leq r)$ and so for a fixed parity of $\nu_1, \cdots, \nu_{r+s}$ the region $\mathscr{R}$ does not change. Then the argument should be repeated as above for these 2^{r+s} cases.

This concludes the proof of Lemma 4.

We now solve for ξ_ν in terms of ν_1.

LEMMA 5. *There is a constant* $C_7 \neq 0$ *such that for all* ν *satisfying* i), ii), iii) *of Lemma* 3 *we have*

$$\log \xi_\nu = \nu_1 C_7 + 0(1) \qquad (\nu_1 \to \pm \infty). \tag{18}$$

Proof. As in the proof of Lemma 3 we see i) and ii) imply κ_ν' is close to 1 and so from (10) and (14)

$$C\kappa_\nu^{-1} = C\xi_0^{-\frac{1}{n}}(1+\varepsilon_\nu/q_\nu)^{\frac{1}{n}} = C_6 + 0(q_\nu^{-1}).$$

So we are interested in the ν such that

$$0 < \eta_{1\nu}, \cdots, \eta_{n\nu} < C_6 + 0(q_\nu^{-1}). \tag{19}$$

The C_7 will depend only on $\zeta_1, \cdots, \zeta_{r+s}$ so it suffices to show (18) for all ν such that

$$0 < \eta_{1\nu}, \cdots, \eta_{n\nu} < C_8.$$

Then as in Lemma 4 there is a bounded subset $\mathscr{R} \subseteq \boldsymbol{R}^{n-1}$ such that we want all $\nu_1 X_1 + \cdots + \nu_n X_n \in \mathscr{R}$. This says that for $2 \leq i \leq r+s$

$$\nu_2 \log|\zeta_2^{\frac{1}{n}}\zeta_2^{(i)}| + \cdots + \nu_{r+s}\log|\zeta_{r+s}^{\frac{1}{n}}\zeta_{r+s}^{(i)}| = -\nu_1 \log|\zeta_1^{\frac{1}{n}}\zeta_1^{(i)}| + 0(1).$$

Set $X_i' = (\log|\zeta_i^{\frac{1}{n}}\zeta_i^{(2)}|, \cdots, \log|\zeta_i^{\frac{1}{n}}\zeta_i^{(r+s)}|)$, $(1 \leq i \leq r+s)$. Then $\det(X_2, \cdots, X_n) \neq 0$, (16), implies $\det(X_2', \cdots, X_{r+s}') \neq 0$, Solving for ν_j $(2 \leq j \leq r+s)$ yields with $Y = -\nu_1 X_1' + 0(1)$

$$\begin{aligned}\nu_j &= \frac{\det(X_2', \cdots, X_{j-1}', Y, X_{j+1}', \cdots, X_{r+s}')}{\det(X_2', \cdots, X_{r+s}')} \\ &= (-1)^{j-1}\nu_1 \frac{\det(X_1', \cdots, \hat{X}_j', \cdots, X_{r+s}')}{\det(X_2', \cdots, X_{r+s}')} + 0(1).\end{aligned}$$

Thus

$$\begin{aligned}\log \xi_\nu &= \log \xi_0 + \nu_1 \log \zeta_1 + \cdots + \nu_{r+s}\log \zeta_{r+s} \\ &= \nu_1 \sum_{j=1}^{r+s}(-1)^{j-1}\frac{\det(X_1', \cdots, \hat{X}_j', \cdots, X_{r+s}')}{\det(X_2', \cdots, X_{r+s}')}\log \zeta_j + 0(1) \\ &= \nu_1 C_7 + 0(1).\end{aligned}$$

Now $C_7 = 0$ implies

$$\sum_{j=1}^{r+s}(-1)^{j-1}\det(X_1', \cdots, \hat{X}_j', \cdots, X_{r+s}')\log \zeta_j = 0$$

and this is the regulator of $\mathcal{O}$. So $C_7 \neq 0$.

LEMMA 6. *Let* $\#_N$ *be the number of solutions of* i), ii), iii) *of Lemma* 3 *such that* $1 \leq \nu_1 \leq N$ $(1 \leq -\nu_1 \leq N)$. *Then for some* $C_9 > 0$, $\#_N \sim C_9 N$.

Proof. We just observed that we must count the number of ν such that ξ_ν and q_ν are large, (19) holds and $1 \leq \nu_1 \leq N$ $(1 \leq -\nu_1 \leq N)$. Then examining the proof of Lemma 4 we see that we wish to count the number of ν_1, $1 \leq \nu_1 \leq N$ $(1 \leq -\nu_1 \leq N)$ such that

$$\nu_1 X_1 + \cdots + \nu_n X_n \in \mathscr{R} + 0(q_\nu^{-1}).$$

Let $F_N(\varepsilon)$ be the number lying in $\mathscr{R}_\varepsilon$ (all $X \in \boldsymbol{R}^{n-1}$ no further from $\mathscr{R}$ than ε). Then $F_N(\varepsilon) \sim C(\varepsilon)N$ $(N \to \infty)$. Moreover from the uniform distribution theroy $\lim C(\varepsilon) = C(0)$ $(\varepsilon \to 0)$. Then given ε there is an $N_0(\varepsilon)$ such that $|\nu_1| > N_0(\varepsilon)$ implies $\mathscr{R} + 0(q_\nu^{-1}) \subseteq \mathscr{R}_\varepsilon$. Now clearly the q_ν yielding solutions to * are such that $q_\nu \to \infty$ so also $\xi_\nu \to \infty$ and so by Lemma 5, $\nu_1 \to \infty$. Hence there is a constant C_{10} independent of N and ε such that

$$\#_N \leq F_N(\varepsilon) + C_{10} N_0(\varepsilon).$$

Thus as $N \to \infty$

$$\limsup \#_N / N \leq \limsup \,(F_N(\varepsilon) + C_{10} N_0(\varepsilon))/N = C(\varepsilon).$$

Since this is true for all $\varepsilon > 0$ we see

$$\limsup \#_N / N \leq C(0).$$

Similarly for $\liminf$ and so $\lim \#_N / N = C(0)$.

We now prove the theorem. It follows from Lemma 5 that ξ_ν is large if and only if ν_1 is large and has the same sign as C_7; we now restrict ν_1 to these values. Set $N = |C_7|^{-1} \log B$. Then $1 \leq \xi_\nu \leq 2B$ implies by Lemma 5

$$0 \leq \log \xi_\nu = \nu_1 C_7 + 0(1) \leq \log B + 0(1)$$

or

$$0(1) \leq |\nu_1| \leq N + 0(1).$$

Thus

$$\lambda_B'(\Omega^+) \leq \#_{N+0(1)}.$$

The lower bound is similar and so there are constants C_{11} and C_{12} such that

$$\frac{\#_{N-C_{11}}}{N} \leq \frac{\lambda'_B(\Omega^+)}{|C_7|^{-1}\log B} \leq \frac{\#_{N+C_{12}}}{N}$$

and so letting N (hence B) tend to ∞ we have the desired result.

References

[1] W.W. Adams. Simultaneous Asymptotic Diophantine Approximations. Mathematika 14 (1967) 173–180.

[2] W.W. Adams. Simultaneous Asymptotic Diophantine Approximations to a Basis of a Real Cubic Number Field. J. of Number Theory 1 (1969) 179–194.

[3] S. Lang. Asymptotic Approximation to Quadratic Irrationals. Am. J. of Mathematics 87 (1965) 481–487.

[4] L.G. Peck. Simultaneous Rational Approximations to Algebraic Numbers. Bull. AMS 67 (1961) 197–201.

[5] W.M. Schmidt. Simultaneous Approximations to a Basis of a Real Algebraic Number Field. Am. J. of Mathematics 88 (1966) 517–527.

Department of Mathematics
University of Maryland