ON GROUPS ALL OF WHOSE 2-BLOCKS HAVE THE HIGHEST DEFECTS

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1. Introduction

If G is a finite group and p a fixed prime number, the irreducible representations of G are distributed into disjoint systems, the p-blocks. These blocks have been investigated especially by R. Brauer. In this note we are concerned with the problem: What is the structure of G which has only one p-block? Or more generally, what is the structure of G all of whose p-blocks have the highest defects?

It is well known that if a finite group G has a non-trivial normal p-subgroup N such that the centralizer of N is contained in N, then G has only one p-block (see Curtis-Reiner [3], p. 634, Ex. 1). Therefore it is interesting to consider the converse of this theorem. For p-solvable groups, the converse is true. Indeed, since $O_{p'}(G)$ is a intersection of kernels of all the characters contained in the first p-block of G (R. Brauer [1]), if a p-solvable group G has only one p-block, then $O_{p'}(G) = 1$. Therefore Lemma 1. 2. 3 of P. Hall-G. Higman [6] shows that $C_G(O_p(G)) \subset O_p(G)$. Hence the converse holds for p-solvable groups. In general the converse is not true. In fact, the Mathieu group of degree 24 and that of degree 22 have only one 2-block (for the character tables of the Mathieu groups, see G. Forbenius [5], N. Burgoyne and P. Fong [2]). The purpose of this note is to prove the following theorem.

THEOREM 1. *Let G be a finite group with an abelian Sylow 2-subgroup S. Let $|G| = 2^a \cdot g'$, $(2, g')=1$. Then every 2-block of G has defect a if and only if S is normal in G.*

COROLLARY. *If a finite group G with an abelian Sylow 2-subgoup S has only one 2-block, then S is normal in G.*

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2. Proof of Theorem

We shall first prove two lemmas.

LEMMA 1. *Let G be a finite p-solvable group with an abelian Sylow p-subgroup S. If every p-block of G has the highest defect, then S is normal in G.*

Proof. Let $|G|=p^a g'$, $(p,g')=1$. We shall apply a result of P. Fong [4]. Since S is abelian, by Theorem (3C) of [4] every irreducible character has height 0. Since every p-block of G has defect a, every irreducible character of G has degree prime to p. By Lemma (3D) of [4], S is normal in G.

LEMMA 2. *If every p-block of a finite group G with an abelian Sylow p-subgroup has the highest defect, then the centralizer $C_G(x)$ of any p-element x of G has the same property as G.*

Proof. Suppose that $H=C_G(x)$ has a block B of defect $d<a$, where $|G|=p^a\cdot g'$, $(p,g')=1$. Let D be a defect group of B (in H). Then by a theorem of R. Brauer $C_H(D)\cdot D=C_H(D)$ has a block of defect d. Since $x\in D$, $C_H(D)=C_G(D)$. Hence $C_G(D)$ has a block of defect d. Since $C_G(D)$ has a Sylow p-subgroup of G, by a theorem of R. Brauer G has a block of defect d (see Curtis-Reiner [3], Theorem (88. 3) and Theorem (88. 8)). This is impossible.

Now we shall prove our theorem. Since a defect group of any 2-block of G contains $O_2(G)$, "if" part of the theorem is trivial.

We shall prove "only if" part of Theorem 1 by induction on the order of G. Suppose that "only if" part of Theorem 1 is not true. Let G be a minimal counter example.

LEMMA 3. *The centralizer $C_G(x)$ of any non-identity 2-element x of G is a proper subgroup of G.*

Proof. Suppose that $G=C_G(x)$ for some 2-element $x\neq 1$. Then by a theorem of R. Brauer ([1], p. 155), the blocks of G and the blocks of $G/<x>$ have one to one correspondence. Therefore every 2-block of $G/<x>$ has defect a', where $|G/<x>|=2^{a'}\cdot g'$, $(2,g')=1$. By induction a Sylow 2-subgroup of $G/<x>$ is normal. Hence a Sylow 2-subgroup of G is normal, which is a contradiction. Hence $G > C_G(x)$ for every non-identity 2-element x of G.

LEMMA 4. *The Sylow 2-subgroups of G are indenpendent.*

Proof. Let x be any non-identity 2-element of G. Then by Lemma 2, Lemma 3 and by induction, a Sylow 2-subgroup of $C_G(x)$ is normal in $C_G(x)$. If two Sylow 2-subgroups S_1, S_2 of G intersect non trivially, then the contralizer $C_G(x)$ of any non-identity element x of $S_1 \cap S_2$ contains S_1 and S_2. Since $C_G(x)$ is 2-closed, $S_1 = S_2$. Hence the Sylow 2-subgroups of G are independent.

By Lemma 1 and by a fundamental theorem of M. Suzuki [8], G has a series of normal subgroups

$$G \supset G_1 > G_2 \supset 1$$

such that G/G_1 and G_2 are groups of odd order and G_1/G_2 is isomorphic to $LF(2, 2^\alpha)$. Put $\bar{G} = G/G_2$ and $\bar{L} = G_1/G_2$. We shall show that $\bar{G}$ has a irreducible character X whose degree is divisible by 2^α. As is well known, the group $\bar{L} \cong LF(2, 2^\alpha)$ has a irreducible character ζ of degree 2^α. Let X be a irreducible component of the induced character $\zeta^{\bar{G}}$. Then by the Frobenius' reciprocity theorem $X|_{\bar{L}}$ contains ζ in its irreducible components. By a Clifford's theorem, $X|_{\bar{L}} = e \sum_g \zeta^g$, where e is an integer and the summation ranges over all the distinct conjugate character of ζ. Hence $X(1) \equiv 0 \pmod{2^\alpha}$. If we consider X as a character of G, X is a character of defect O. This contradicts with the assumption of Theorem 1. Thus we have proved "only if" part of our theorem. The proof is complete.

For an odd prime p, the following result of N. Ito [7] is known.

THEOREM 2. *Let G be a finite group all of whose p-blocks have the highest defects. If G has a permutation representation of degree p, then a Sylow p-subgroup of G is normal.*

Actually in [7], N. Ito has proved a theorem different from Theorem 2. But considering his theorem from our point of view, we get easily Theorem 2.

The author thanks to Prof. N. Ito who has suggested him the formulation of Theorem 1. Our original one is Corollary to Theorem 1.

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