PROLONGATIONS OF G-STRUCTURES TO TANGENT BUNDLES

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To Professor Kiyoshi Noshiro on the occasion of his 60th birthday.

§ Introduction and notations.

The purpose of the present paper is to study the prolongations of G-structures on a manifold M to its tangent bundle $T(M)$, G being a Lie subgroup of $GL(n, R)$ with $n = \dim M$. Recently, K. Yano and S. Kobayashi [9] studied the prolongations of tensor fields on M to $T(M)$ and they proposed the following question: Is it possible to associate with each G-structure on M a naturally induced G'-structure on $T(M)$, where G' is a certain subgroup of $GL(2n, R)$? In this paper we give an answer to this question and we shall show that the prolongations of some special tensor fields by Yano-Kobayashi —— for instance, the prolongations of almost complex structures —— are derived naturally by our prolongations of the classical G-structures. On the other hand, S. Sasaki [5] studied a prolongation of Riemannian metrics on M to a Riemannian metric on $T(M)$, while the prolongation of a (positive definite) Riemannian metric due to Yano-Kobayashi is always pseudo-Riemannian on $T(M)$ but never Riemannian. We shall clarify the circumstances for this difference and give the reason why the one is positive definite Riemannian and the other is not.

The crucial starting point for our study is the following simple fact (§ 1): The tangent bundle (space) $T(R^n)$ of the n-dimensional real euclidean space R^n is also a vector space and the tangent bundle $T(GL(n, R))$ of the general linear group can be identified to a subgroup of $GL(2n, R)$, the tangent bundle $T(G)$ of a Lie group G being a Lie group by the natural group multiplication. From this fact we can show that, if we denote by $F(M)$ the bundle of frames of M, $T(F(M))$ can be imbedded canonically into $F(T(M))$ (§ 2). Using this imbedding and the above identification of $T(GL(n, R))$ to a subgroup of $GL(2n, R)$, we can associate with each G-

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structure P on M (i.e. P is a G-subbundle of $F(M)$), a canonically induced $\tilde{G}$-structure $\tilde{P}$ on $T(M)$, where $\tilde{G}$ is a Lie subgroup of $GL(2n, R)$, $\tilde{G}$ being isomorphic to $T(G)$ (§ 3). We will call $\tilde{P}$ the prolongation of P. In § 4, we shall prove that, if a diffeomorphism f of a manifold M onto another which induces an isomorphism of a G-structure P on M to a G-structure P' on M', then the induced diffeomorphism Tf of $T(M)$ onto $T(M')$ is an isomorphism of $\tilde{P}$ onto $\tilde{P}'$ and vice versa. In § 5, we shall see that a G-structure P on M is integrable (cf. Def. 5. 1) if and only if the prolongation $\tilde{P}$ is integrable. In § 6, we shall consider some classical G-structures and see that a certain geometric structure on M induces canonically a geometric structure of the same kind on $T(M)$. In § 7, we shall consider the relations of our prolongations of G-structures with the prolongations of tensor fields due to Yano-Kobayashi and Sasaki. In particular, we shall see that an almost complex structure on M induces an almost complex structure on $T(M)$ and in fact, this structure coincides with the one given in [9]. At the end of § 3 we shall show that, for a G-structure P on M, we can associate with each connection Γ on the pricipal fibre bundle P a naturally induced G_0-structure P_Γ on $T(M)$, where G_0 is a subgroup of $GL(2n, R)$ which is isomorphic to G itself, more precisely G_0 is the subgroup consisting of the matrices $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ for $a \in G$. Applying this fact for $G = 0(n)$, we see that, with each Riemannian metric g on M and a connection Γ on the orthogonal frame bundle P on M, we can associate a Riemannian metric g_Γ on $T(M)$. At the end of § 7 we see that the associated Riemannian metric g_Γ with the Riemannian connection Γ induced by the metric g is exactly the same metric studied by Sasaki mentioned above. In § 8, we shall give some remarks about the relations between the G-structure P and the induced G_0-structure P_Γ.

In this paper, all manifolds and mappings are assumed to be differentiable of class C^∞, unless otherwise stated. We denote by $T(M)$ the tangent bundle of a manifold M, $T_x(M)$ being the tangent space of M at $x \in M$. For manifolds M and N, $T(M \times N)$ is often identified with $T(M) \times T(N)$. We shall denote by $F(M)$ the frame bundle of M, i.e. the set of all linear isomorphisms $z : R^n \longrightarrow T_x(M)$ with $n = \dim M$. The isomorphism z will be identified with the frame $(z(e_1), \cdots, z(e_n))$, where $e_i = (\delta_i^1, \cdots, \delta_i^n) \in R^n$, δ_i^j being the Kronecker symbol. For a map $f : M \longrightarrow N$, we denote by Tf or $T(f)$ the induced map of $T(M)$ into $T(N)$, which will be sometimes called the tangential map of f. If f is a diffeomorphism, we denote by Ff or $F(f)$ the induced map of $F(M)$ to $F(N)$, i.e. $(F\ f)(z) = (T\ f)\circ z$ for $z \in F(M)$. For several maps $f : M' \longrightarrow M''$, $g : M \longrightarrow M'$, $f_i : M_i \longrightarrow N_i$ $(i = 1, 2)$ we have the following formulas which can be verified by the definitions:

$$T(f\circ g) = T\ f\circ T\ g,$$
$$T(1_M) = 1_{T(M)},\quad F(1_M) = 1_{F(M)},$$
$$T(f_1\times f_2) = T\ f_1 \times T\ f_2,$$

where 1_M stands for the identity map of M. We denote $1 = 1_M$ if the manifold M is clear from the context.

For a coordinate neighborhood U in M with a local coordinate system $\{x^1, \cdots, x^n\}$ we can define canonically a coordinate system $\{x^1, \cdots, x^n, \dot{x}^1, \cdots, \dot{x}^n\}$ on $T(U)$, i.e. a tangent vector $\sum_{i=1}^n \dot{x}^i\left(\frac{\partial}{\partial x^i}\right)_x$ has the coordinates $(x^1, \cdots, x^n, \dot{x}^1, \cdots, \dot{x}^n)$ if the point $x \in U$ has the coordinates $(x^1, \cdots, x^n)$. We will call this local coordinate system $\{x^1, \cdots, x^n, \dot{x}^1, \cdots, \dot{x}^n\}$ the induced local coordinate system on $T(U)$ by $\{x^1, \cdots, x^n\}$. Similarly we can define the induced local coordinate system $\{x^1, \cdots, x^n; \cdots, y^i_j, \cdots\}$ on $F(U)$, i.e. a frame $z = \left(\cdots, \sum_{i=1}^n y^i_j\left(\frac{\partial}{\partial x^i}\right)_x, \cdots\right)$ has the coordinates $(x^1, \cdots, x^n; \cdots, y^i_j, \cdots)$. We shall sometimes omit the summation notation $\sum$ for repeated indices, for instance $\sum_{i=1}^n y^i_j\left(\frac{\partial}{\partial x^i}\right)_x = y^i_j\left(\frac{\partial}{\partial x^i}\right)_x$.

If $f : M \longrightarrow N$ is a map of a set M into N and A is a subset of M, we often denote by f itself the restriction $f|A$ of f to A, if there is no confusion.

In the following, R^n always denotes the n dimensional real number space. The group of all linear automorphisms of R^n will be denoted by $GL(n, R)$, $GL(R^n)$ or simply by $GL(n)$. If $a^i_j \in R$ for $i, j = 1, 2, \cdots, n$, we denote by (a^i_j) the matrix of degree n whose (i, j)-entry is a^i_j.

§1. Imbedding of $T(GL(n, R))$ into $GL(2n, R)$.

Let M be a manifold. As usual, we denote by $X + Y$ and cX the sum of tangent vectors $X, Y \in T_x(M)$ and the scalar multiplication of X by $c \in R$.

Let $\tau_x : R^n \longrightarrow R^n$ be the translation of R^n by $x \in R^n$, i.e. $\tau_x(y) = x + y$ for $y \in R^n$ and let $\sigma_c : R^n \longrightarrow R^n$ be the scalar multiplication by $c \in R$, i.e. $\sigma_c(x) = cx$ for $x \in R^n$.

DEFINITION 1. 1. Take two tangent vectors $X \in T_x(R^n)$ and $Y \in T_y(R^n)$. We define the sum $X \oplus Y$ of X and Y and the new scalar multiplication $c \circ X$ by $c \in R$ as follows:

$$X \oplus Y = (T\tau_y) \cdot X + (T\tau_x) \cdot Y ,$$
$$c \circ X = (T\sigma_c) \cdot X \cdot$$

PROPOSITION 1. 2. *The tangent bundle $T(R^n)$ of R^n is a vector space of dimension $2n$ with respect to the sum "$\oplus$" and the scalar multiplication "$\circ$"*

Proof. Let x^i be the i-th component of $x \in R^n$, then $X \in T_x(R^n)$ and $Y \in T_y(R^n)$ can be expressed as follows:

$$X = \sum a_i\Big(\frac{\partial}{\partial x^i}\Big)_x , \quad Y = \sum b_i\Big(\frac{\partial}{\partial x^i}\Big)_y$$

for some a_i, $b_i \in R$, $i = 1, 2, \cdots, n$. We see readily that

$$(T\tau_y) \cdot X = \sum a_i\Big(\frac{\partial}{\partial x^i}\Big)_{x+y} \quad \text{and so}$$
$$X \oplus Y = \sum (a_i + b_i)\Big(\frac{\partial}{\partial x^i}\Big)_{x+y} .$$

Similarly $c \circ X = \sum c \cdot a_i\Big(\frac{\partial}{\partial x^i}\Big)_{cx}$. From these expressions of the sum and the scalar multiplication, it is clear that $T(R^n)$ becomes a vector space of dimension $2n$, i.e. $T(R^n)$ has the induced "global" coordinate system $\{x^1, \cdots, x^n, \dot{x}^1, \cdots, \dot{x}^n\}$.

PROPOSITION 1. 3. *Let $f : R^n \longrightarrow R^m$ be a linear map. Then, the tangential map $T f : T(R^n) \longrightarrow T(R^m)$ is also a linear map of the vector space $T(R^n)$ into $T(R^m)$.*

Proof. Let x^i be the i-th component of $x \in R^n$ and y^j be the j-th component of $y \in R^m$. Let $f^j(x)$ be the j-th component of $f(x)$. Then $f^j(x)$ is expressed as follows: $f^j(x) = \sum_{i=1}^{n} a_i^j x^i$ for some $a_i^j \in R$. Then we have:

$$(T f)\Big(\sum c_i\Big(\frac{\partial}{\partial x^i}\Big)_x\Big) = \sum c_i \; T f\Big(\frac{\partial}{\partial x^i}\Big)_x$$
$$= \sum c_i \frac{\partial f^k}{\partial x^i} \cdot \Big(\frac{\partial}{\partial y^k}\Big)_{f(x)}$$

$$= \sum c_i \cdot a_i^k \cdot \left(\frac{\partial}{\partial y^k}\right)_{f(x)}$$

for $c_i \in R$, $i = 1, 2, \cdots, n$. From this expression, it follows that $T\ f$ is a linear map, for instance:

$$\begin{aligned}
&(T\ f)\left(\sum c_i\left(\frac{\partial}{\partial x^i}\right)_x \oplus \sum c_i'\left(\frac{\partial}{\partial x^i}\right)_{x'}\right) \\
&= (T\ f)\left(\sum (c_i + c_i')\left(\frac{\partial}{\partial x^i}\right)_{x+x'}\right) \\
&= \sum (c_i + c_i')a_i^k\left(\frac{\partial}{\partial y^k}\right)_{f(x+x')} \\
&= \sum c_i a_i^k\left(\frac{\partial}{\partial y^k}\right)_{f(x)} \oplus \sum c_i' a_i^k\left(\frac{\partial}{\partial y^k}\right)_{f(x')} \\
&= (T\ f)\left(\sum c_i\left(\frac{\partial}{\partial x^i}\right)_x\right) \oplus (T\ f)\left(\sum c_i'\left(\frac{\partial}{\partial x^i}\right)_{x'}\right).
\end{aligned}$$

The following proposition is easily verified:

PROPOSITION 1. 4. *For any finite dimensional vector space V the tangent bundle $T(V)$ becomes a vector space with respect to the natural sum and the scalar multiplication "$\oplus$, $\circ$". If V is the direct sum of two subspaces W and W' then $T(V)$ is isomorphic canonically to the direct sum of $T(W)$ and $T(W')$.*

DEFINITION 1. 5. We denote by ρ: $GL(n, R) \times R^n \longrightarrow R^n$ the natural operation of $GL(n, R)$ on R^n, i.e. $\rho(y, x) = y \cdot x$ for $y \in GL(n, R)$ and $x \in R^n$.

It is well known that if G is a Lie group, then $T(G)$ is also a Lie group by taking $T\mu$ as the group multiplication, where $\mu : G \times G \longrightarrow G$ is the group multiplication of G.

PROPOSITION 1. 6. *By the tangential map $T\rho$: $T(GL(n)) \times T(R^n) \longrightarrow T(R^n)$ of ρ, the Lie group $T(GL(n))$ operates effectively on $T(R^n)$ as a linear group.*

Proof. (i) Take two tangent vectors $X_0 \in T_{x_0}(R^n)$, $Y_0 \in T_{y_0}(GL(n))$ and $c \in R$. We shall prove:

$$T\rho(Y_0,\ c \circ X_0) = c \circ T\rho(Y_0,\ X_0). \tag{1. 1}$$

First, we define the functions $\tilde{x}^j : R^n \longrightarrow R$, $\bar{y}_k^j : GL(n) \times R^n \longrightarrow R$, $\tilde{y}_k^j$: $GL(n) \longrightarrow R$ and $\bar{x}^k : GL(n) \times R^n \longrightarrow R$, as follows: $\tilde{x}^j(x) = x^j$, $\bar{y}_k^j(y, x) = y_k^j$, $\tilde{y}_k^j(y) = y_k^j$, $\bar{x}^k(y, x) = x^k$ for $x \in R^n$, $y = (y_k^j) \in GL(n)$, $j, k = 1, 2, \cdots, n$.

Then we have: $\tilde{x}^j \circ \rho(y,x) = \tilde{x}^j(y \cdot x) = \tilde{x}^j((\sum y^i_k\, x^k)_i) = \sum y^j_k \cdot x^k = \sum (\bar{y}^j_k \cdot \tilde{x}^k)(y,x)$. Therefore, it follows that

$$\begin{aligned}(T\rho(Y_0, c\circ X_0))\cdot \tilde{x}^j &= (Y_0, c\circ X_0)(\tilde{x}^j\circ\rho)\\ &= (Y_0, c\circ X_0)\sum \bar{y}^j_k\cdot\tilde{x}^k\\ &= \sum (Y_0, c\circ X_0)(\bar{y}^j_k\cdot\tilde{x}^k)\\ &= \sum (Y_0, c\circ X_0)\,\bar{y}^j_k\cdot\tilde{x}^k(y_0, c\cdot x_0)\\ &+ \sum \tilde{y}^j_k(y_0, cx_0)\cdot(Y_0, c\circ X_0)\tilde{x}^k\\ &= \sum Y_0\,\tilde{y}^j_k\cdot c\ \ x^k_0 + \sum (y_0)^j_k\cdot cX_0\,\tilde{x}^k\\ &= c\cdot\sum (Y_0\,\tilde{y}^j_k\cdot x^k_0 + (y_0)^j_k\cdot X\ \tilde{x}^k).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(c\circ(T\rho(Y_0, X_0)))\cdot\tilde{x}^j &= T\sigma_c(T\rho(Y_0, X_0))\cdot\tilde{x}^j\\ &= (T\rho(Y_0, X_0))(\tilde{x}^j\circ\sigma_c)\\ &= c\cdot T\rho(Y_0, X_0)\tilde{x}^j\\ &= c\cdot\sum (Y_0\,\tilde{y}^j_k\cdot x^k_0 + (y_0)^j_k\cdot X_0\,\tilde{x}^k).\end{aligned}$$

Thus, $(T\rho(Y_0, c\circ X_0)\cdot\tilde{x}^j = (c\circ(T\rho(Y_0, X_0)))\cdot\tilde{x}^j$ for $j = 1, 2, \cdots, n$, hence (1, 1) holds.

(ii) Take $Y_0 \in T_{y0}(GL(n))$, $X_i \in T_{x_i}(R^n)$, $i = 1, 2$.
We shall prove:

$$(1.\ 2)\qquad T\rho(Y_0, X_1\oplus X_2) = T\rho(Y_0, X_1)\oplus T\rho(Y_0, X_2).$$

Keeping the notations as in (i), we calculate as follows:

$$\begin{aligned}T\rho(Y_0, X_1\oplus X_2)\tilde{x}^j &= (Y_0, X_1\oplus X_1)\tilde{x}^j\circ\rho\\ &= (Y_0, X_1\oplus X_2)\sum\bar{y}^j_k\cdot\tilde{x}^k\\ &= \sum Y_0\,\tilde{y}^j_k\cdot(x^k_1 + x^k_2) + \sum (y_0)^i_k\cdot(X_1\oplus X_2)\tilde{x}^k\\ &= \sum Y_0\,\tilde{y}^j_k\cdot(x^k_1 + x^k_2)\sum (y_0)^j_k(X_1\tilde{x}^k + X_2\tilde{x}^k)\\ &= \sum (Y_0\,\tilde{y}^j_k\cdot x^k_1 + (y_0)^j_k\cdot X_1\tilde{x}^k)\\ &+ \sum (Y_0\,\tilde{y}^j_k\cdot x^k_2 + (y_0)^j_k\cdot X_2\tilde{x}^k)\\ &= T\rho(Y_0, X_1)\tilde{x}^j + T\rho(Y_0, X_2)\tilde{x}^j\\ &= (T\rho(Y_0, X_1)\oplus T\rho(Y_0, X_2))\tilde{x}^j,\end{aligned}$$

where we have used following calculations in the above fourth equality: $(X_1\oplus X_2)\tilde{x}^k = (T\tau_{x2}X_1 + T\tau_{x1}X_2)\tilde{x}^k = X_1(\tilde{x}^k\circ\tau_{x2}) + X_2(\tilde{x}^k\circ\tau_{x1}) = X_1(\tilde{x}^k + x^k_2) +$

$X_2(\tilde{x}^k + x_1^k) = X_1\tilde{x}^k + X_2\tilde{x}^k$. Hence we have $T\rho(Y_0, X_1 \oplus X_2)\tilde{x}^j = (T\rho(Y_0, X_1) \oplus T\rho(Y_0, X_2))\tilde{x}^j$ for $j = 1, 2, \cdots, n$, which shows that (1, 2) holds.

(iii) For every $y_0 \in GL(n)$, $x_0 \in R^n$, $r_{ij} \in R$ and $c_j \in R$, we shall prove the following:

$$(1.3)\qquad T\rho\Big(\sum r_{ij}\Big(\frac{\partial}{\partial y_j^i}\Big)_{y_0},\ \sum c_j\Big(\frac{\partial}{\partial x^j}\Big)_{x_0}\Big)$$
$$= \sum_k \Big(\sum_j r_{kj}\cdot x_0^j + \sum_l c_l(y_0)_l^k\Big)\Big(\frac{\partial}{\partial x^k}\Big)_{y_0\cdot x_0}$$

In fact, from the calculation in (i) we proved the following:

$$T\rho(Y_0, X_0)\tilde{x}^j = \sum_l (Y_0\tilde{y}_l^j \cdot x_0^l + (y_0)_l^j \cdot X_0\tilde{x}^l).$$

Applying this equality for $Y_0 = \sum r_{ij}\Big(\frac{\partial}{\partial y_j^i}\Big)_{y_0}$ and $X_0 = \sum c_j\Big(\frac{\partial}{\partial x^j}\Big)_{x_0}$ we obtain the following:

$$T\rho\Big(\sum r_{ij}\Big(\frac{\partial}{\partial y_j^i}\Big)_{y_0},\ \sum c_j\Big(\frac{\partial}{\partial x^j}\Big)_{x_0}\Big)\tilde{x}^j$$
$$= \sum_l\Big(\sum_{i,p} r_{ip}\Big(\frac{\partial}{\partial y_p^i}\Big)_{y_0}\tilde{y}_l^j\cdot x_0^l + (y_0)_l^j\cdot\sum c_p\Big(\frac{\partial}{\partial x^p}\Big)_{x_0}\tilde{x}^l\Big)$$
$$= \sum_l\Big(\sum_{j,p} r_{i,p}\delta_i^j\delta_p^l x_0^l + (y_0)_l^j\cdot\sum_p c_p\delta_l^p\Big)$$
$$= \sum_l (r_{jl}x_0^l + (y_0)_l^j c_l).$$

On the other hand we have:

$$\sum_k\Big(\sum_l r_{kl}x_0^l + \sum_l c_l(y_0)_l^k\Big)\Big(\frac{\partial}{\partial x^k}\Big)_{y_0\cdot x_0}\tilde{x}^j$$
$$= \sum_l (r_{jl}x_0^l + c_l(y_0)_l^j).$$

Hence, the values of both hand sides of (1.3) at $\tilde{x}^j$ are equal for any $j = 1, 2, \cdots, n$, which shows that (1.3) holds.

(iv) Take $Y, Y' \in T(GL(n))$ and $X \in T(R^n)$.

We shall prove:

$$T\rho(Y\cdot Y', X) = T\rho(Y, T\rho(Y', X)),$$

where $Y\cdot Y' = T\mu(Y, Y')$, μ being the group multiplication of $GL(n)$.

Since $GL(n)$ operates on R^n, we have

$$\rho \circ (\mu \times 1_{R^n}) = \rho \circ (1_{GL(n)} \times \rho) .$$

Hence, we calculate as follows:

$$\begin{aligned} T\rho(Y \cdot Y', X) &= T\rho(T\mu(Y, Y'), X) \\ &= T\rho \circ (T\mu \times 1_{T(R^n)}) (Y, Y', X) \\ &= (T\rho \circ T(\mu \times 1_{R^n})) (Y, Y', X) \\ &= T(\rho \circ (\mu \times 1_{R^n})) (Y, Y', X) \\ &= T(\rho \circ (1_{GL(n)} \times \rho)) (Y, Y', X) \\ &= T\rho(T 1_{GL(n)} \times T\rho) (Y, Y', X) \\ &= T\rho(Y, T\rho(Y', X)) . \end{aligned}$$

By (i) $\sim$ (iv) we proved that the group $T(GL(n))$ operates effectively on $T(R^n)$ as a linear group. Thus Proposition 1. 6 is proved.

DEFINITION 1. 7. For $Y \in T(GL(n))$ and $X \in T(R^n)$ we define $Y \cdot X$ by

$$Y \cdot X = T\rho(Y, X) .$$

More generally we see, by the same argument as in the proof (iv) in Prop. 1. 6, that, if G operates on a manifold M then $T(G)$ operates on $T(M)$.

DEFINITION 1. 8. We denote by $R_a : GL(n) \longrightarrow GL(n)$ the right translation of $GL(n)$ by $a \in GL(n)$, i.e. $R_a(y) = y \cdot a$ for $y \in GL(n)$. Take a tangent vector $Y \in T_a(GL(n))$. Then $B = TR_{a^{-1}}(Y)$ is a tangent vector of $GL(n)$ at the unit element e of $GL(n)$, namely B is an element of the Lie algebra $\mathfrak{gl}(n)$ of $GL(n)$. Conversely for any pair $a \in GL(n)$ and $B \in \mathfrak{gl}(n)$, we obtain a tangent vector $Y \in T_a(GL(n))$ by $Y = TR_a(B)$. We express this vector Y by : $Y = [a, B]$. On the other hand any $X \in T_x(R^n)$ is expressed by $X = \sum v_i \left(\frac{\partial}{\partial x^i}\right)_x$. We shall denote: $X = (x, v) \in T(R^n) = R^{2n}$.

PROPOSITION 1. 9. *We have the following equality:*

$$[a, B] \cdot (x, v) = (a \cdot x,\ B \cdot a \cdot x + a \cdot v)$$

for any $a \in GL(n)$, $B \in \mathfrak{gl}(n)$ *and* $x, v \in R^n$.

Proof. First, we shall show that

$$TR_a\left(\left(\frac{\partial}{\partial y^i_j}\right)_b\right) = \sum a^j_l \left(\frac{\partial}{\partial y^i_l}\right)_{b \cdot a} \tag{1. 4}$$

for $a=(a^i_j)\in GL(n)$, $b\in GL(n)$ and $i,j=1,2,\cdots,n$. In fact, denoting by $f^k_l(\cdots,y^i_j,\cdots)=\sum_m y^k_m\cdot a^m_l$ the (k,l)-entry of $R_a(y)$, we see that

$$TR_a\Big(\Big(\frac{\partial}{\partial y^i_j}\Big)_b\Big)=\sum_{k,l}\Big[\frac{\partial f^k_l}{\partial y^i_j}\Big]_{y=b}\cdot\Big(\frac{\partial}{\partial y^k_l}\Big)_{b\cdot a}$$
$$=\sum\delta^i_k\delta^j_m a^m_l\Big(\frac{\partial}{\partial y^k_l}\Big)_{b\cdot a}$$
$$=\sum_l a^j_l\Big(\frac{\partial}{\partial y^i_l}\Big)_{b\cdot a}.$$

By using (1. 4) and several definitions and by putting $B=\sum b^m_p\Big(\frac{\partial}{\partial y^m_p}\Big)_e$ we calculate as follows:

$$[a,B](x,v)=(TR_a\cdot B)\cdot\sum v_i\Big(\frac{\partial}{\partial x^i}\Big)_x$$
$$=TR_a\Big(\sum b^m_p\Big(\frac{\partial}{\partial y^m_p}\Big)_e\Big)\cdot\sum v_i\Big(\frac{\partial}{\partial x^i}\Big)_x$$
$$=\Big(\sum b^m_p\sum a_l\Big(\frac{\partial}{\partial y^m_l}\Big)_a\Big)\cdot\sum v_i\Big(\frac{\partial}{\partial x^i}\Big)_x$$
$$=T\rho\Big(\sum_{m,p,l}b^m_p a^p_l\Big(\frac{\partial}{\partial y^m_l}\Big)_a,\ \sum_i v_i\Big(\frac{\partial}{\partial x^i}\Big)_x\Big)$$
$$=\sum_k\Big(\sum_{p,i,j}b^i_p a^p_j x^j\delta^i_k+\sum_l v_l a^k_l\Big)\Big(\frac{\partial}{\partial x^k}\Big)_{a\cdot x}$$
$$=\sum_k\Big(\sum_{p,j}b^k_p a^p_j x^j+\sum v_l a^k_l\Big)\Big(\frac{\partial}{\partial x^k}\Big)_{a\cdot x}$$
$$=(a\cdot x,\ B\cdot a\cdot x+a\cdot v).$$

Thus proposition 1. 9 is proved.

Remark 1. 10. We can easily verify the following

$$[a,B]\cdot[a',B']=[aa',B+ad(a^{-1})B']$$

for $a,\ a'\in GL(n)$ and $B,\ B'\in\mathfrak{gl}(n)$.

DEFINITION 1. 11. We shall denote by

$$j_n: T(GL(n,R))\longrightarrow GL(2n,R)$$

the operation of $T(GL(n,R))$ on $R^{2n}=T(R^n)$ proved by Proposition 1. 6, i.e. $(j_n(Y))\cdot X=Y\cdot X=T\rho(Y,X)$.

By Proposition 1. 9 we have proved the following

PROPOSITION 1. 12. *For any $a \in GL(n)$ and $B \in \mathfrak{gl}(n)$ we have the following equality:*

$$j_n([a, B]) = \begin{pmatrix} a & 0 \\ Ba & a \end{pmatrix},$$

$$j_n([a, 0]) = Ta.$$

§ 2. Imbedding of $T(F(M))$ into $F(T(M))$.

DEFINITION 2. 1. Let $P(M, \pi, G)$ be a principal fibre bundle with bundle space P, base space M, projection π and structure group G. If $\{U_\alpha\}$ is an open covering of M, P being trivial bundle over U_α, and if $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$ is the transition function of P, we express this fibre bundle by

$$P = \{U_\alpha, \ g_{\alpha\beta}\}.$$

If G is a subgroup of a group G' and $j : G \longrightarrow G'$ is the injection map, then there is a fibre bundle $P' = \{U_\alpha, \ j \circ g_{\alpha\beta}\}$ and the injection $j : P \longrightarrow P'$ which is a bundle homomorphism, i.e.

$$j(z \cdot a) = j(z) \cdot a$$

for any $z \in P$ and $a \in G$. (cf. [7])

DEFINITION 2. 2. If $P(M, \pi, G)$ is a principal fibre bundle, then $T(P)(T(M), T\pi, T(G))$ is again a principal fibre bundle (cf. [4]).

PROPOSITION 2. 3. *If $P = \{U_\alpha, g_{\alpha\beta}\}$, then $T(P) = \{T(U_\alpha), Tg_{\alpha\beta}\}$.*

Proof. Let $\Phi_\alpha : U_\alpha \times G \longrightarrow \pi^{-1}(U_\alpha)$ be the trivialization of P over U_α. Then, by definition,

$$\Phi_\alpha^{-1} \circ \Phi_\beta(x, g) = (x, g_{\alpha\beta}(x) \cdot g)$$

for $x \in U_\alpha \cap U_\beta$ and $g \in G$. Since $(T\pi)^{-1}(T(U_\alpha)) = T(\pi^{-1}(U_\alpha))$, it is sufficient to prove the following:

$$T\Phi_\alpha^{-1} \circ T\Phi_\beta(X, L) = (X, Tg_{\alpha\beta}(X) \cdot L),$$

for $(X, L) \in (T(U_\alpha) \cap T(U_\beta)) \times T(G)$. To prove this it is sufficient to prove the following assertion:

Let $f : U \longrightarrow G$ be a map, for which we define the map $\Psi : U \times G \longrightarrow U \times G$ by

$$\Psi(x,g) = (x, f(x)\cdot g)$$

for $(x,g) \in U \times G$. Then $T\Psi(X,L) = (X, Tf(X)\cdot L)$ for $(X,L) \in T(U) \times T(G)$. To prove this assertion we denote by $\pi_1 : U \times G \longrightarrow U$, $\pi_2 : U \times G \longrightarrow G$ the projections and $\mu : G \times G \longrightarrow G$ the group multiplication.
Since $\pi_2 \circ \Psi(x,g) = f(x)\cdot g = (\mu \circ (f \times 1_G))(x,g)$, we have $\pi_2 \circ \Psi = \mu \circ (f \times 1_G)$, and hence

$$\begin{aligned} T\pi_2 \circ T\Psi &= T(\pi_2 \circ \Psi) = T(\mu \circ (f \times 1_G)) \\ &= T\mu \circ T(f \times 1_G) = T\mu \circ (Tf \times 1_{T(G)}). \end{aligned}$$

Therefore,

$$T\pi_2 \circ T\Psi(X,L) = T\mu(Tf(X), L) = Tf(X)\cdot L.$$

On the other hand, $T\pi_1 \circ T\Psi(X,L) = T(\pi_1 \circ \Psi)(X,L) = T\pi_1(X,L) = X$. Thus, $T\Psi(X,L) = (X, Tf(X)\cdot L)$ and hence Proposition 2. 3 is proved.

THEOREM 2. 4. *For any manifold M, there is a canonical injection* $T(F(M)) \subset F(T(M))$.

Proof. Let $M = \cup U_\alpha$ be the open covering of M by coordinate neighborhoods U_α with a local coordinate system $\{x^1, \cdots, x^n\}$. We denote by $J_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(n,R)$, $n = \dim M$, the Jacobian matrix with respect to the local coordinate systems $\{x_\alpha^1, \cdots, x_\alpha^n\}$ and $\{x_\beta^1, \cdots, x_\beta^n\}$, i.e.

$$J_{\alpha,\beta}(x) = \frac{\partial(x_\beta^1, \cdots, x_\beta^n)}{\partial(x_\alpha^1, \cdots, x_\alpha^n)}$$

for $x \in U_\alpha \cap U_\beta$. Then $F(M) = \{U_\alpha, J_{\alpha,\beta}\}$, $F(M)$ being the bundle of frames of M. By Proposition 2. 3 and the remark in Definition 2. 1 we have:

$$(2.\ 1) \qquad T(F(M)) = \{T(U_\alpha),\ TJ_{\alpha\beta}\} \subset \{T(U_\alpha),\ j_n \circ TJ_{\alpha\beta}\},$$

j_n being defined in Definition 1. 11. Now, we denote by $\{x_\alpha^1, \cdots, x_\alpha^n, \dot{x}_\alpha^1, \cdots, \dot{x}_\alpha^n\}$ the induced local coordinate system on $T(U_\alpha)$, and by $\tilde{J}_{\alpha\beta} : T(U_\alpha) \cap T(U_\beta) \longrightarrow GL(2n,R)$ the Jacobian matrix with respect to the local coordinate systems $\{x_\alpha^1, \cdots, x_\alpha^n, \dot{x}_\alpha^1, \cdots, \dot{x}_\alpha^n\}$ and $\{x_\beta^1, \cdots, x_\beta^n, \dot{x}_\beta^1, \cdots, \dot{x}_\beta^n\}$, i.e.

$$\tilde{J}_{\alpha\beta}(x, \dot{x}) = \frac{\partial(x_\beta^1, \cdots, x_\beta^n, \dot{x}_\beta^1, \cdots, \dot{x}_\beta^n)}{\partial(x_\alpha^1, \cdots, x_\alpha^n, \dot{x}_\alpha^1, \cdots, \dot{x}_\alpha^n)}.$$

We shall prove

$$\tilde{J}_{\alpha\beta} = j_n \circ TJ_{\alpha\beta} \text{ on } T(U_\alpha) \cap T(U_\beta). \tag{2. 2}$$

Put $x^i = x^i_\alpha$, $y^i = x^i_\beta$, $U = U_\alpha$, $U' = U_\beta$, $J = J_{\alpha\beta}$ and $\tilde{J} = \tilde{J}_{\alpha\beta}$. Then $y^i = f^i(x)$, where f^i is a function on $U \cap U'$ for $i = 1, 2, \cdots, n$ and so $J(x) = \left(\frac{\partial f^i}{\partial x^j}\right)$. Since $\left(\frac{\partial}{\partial x^i}\right)_x = \sum \frac{\partial f^k}{\partial x^i} \cdot \left(\frac{\partial}{\partial y^k}\right)_x$ we have $\dot{y}^k = \sum \dot{x}^i \frac{\partial f^k}{\partial x^i}$ if $\sum \dot{y}^i \left(\frac{\partial}{\partial y^i}\right)_x = \sum \dot{x}^i \left(\frac{\partial}{\partial x^i}\right)_x$.

From these relations between two local coordinate systems: $y^i = f^i(x)$ and $\dot{y}^k = \sum \dot{x}^i \frac{\partial f^k}{\partial x^i}$, we obtain:

$$\tilde{J}(x, \dot{x}) = \begin{pmatrix} \left(\frac{\partial f^i}{\partial x_j}\right) & 0 \\ \left(\sum_{i=1}^{n} \dot{x}^i \frac{\partial^2 f^k}{\partial x^i \partial x^j}\right) & \left(\frac{\partial f^k}{\partial x^i}\right) \end{pmatrix}. \tag{2. 3}$$

Now, for any map $g : U \longrightarrow GL(n, R)$ we have

$$Tg\left(\frac{\partial}{\partial x^i}\right)_x = \sum_{k,l} \frac{\partial g^k_l}{\partial x^i} \left(\frac{\partial}{\partial y^k_l}\right)_{g(x)}, \tag{2. 4}$$

where $x \in U$ and $g(x) = (g^k_l(x))$. We calculate as follows:

$$\begin{aligned}
j_n \circ TJ(x, \dot{x}) &= j_n TJ\left(\sum \dot{x}^i \left(\frac{\partial}{\partial x^i}\right)_x\right) \\
&= j_n \sum \dot{x}^i TJ\left(\left(\frac{\partial}{\partial x^i}\right)_x\right) \\
&= j_n\left(\sum \dot{x}^i \frac{\partial^2 f^k}{\partial x^l \partial x^i} \cdot \left(\frac{\partial}{\partial y^k_l}\right)_{J(x)}\right) \\
&= j_n\left[J(x), \sum \dot{x}^i \frac{\partial^2 f^k}{\partial x^l \partial x^i} TR_{J(x)^{-1}}\left(\frac{\partial}{\partial y^k_l}\right)_{J(x)}\right] \\
&= j_n\left[J(x), \sum \dot{x}^i \frac{\partial^2 f^k}{\partial x^l \partial x^i} (J(x)^{-1})^l_m \left(\frac{\partial}{\partial y^k_m}\right)_e\right] \\
&= \begin{pmatrix} J(x) & 0 \\ \left(\sum \dot{x}^i \frac{\partial^2 f^k}{\partial x^l \partial x^i} (J(x)^{-1})^l_m\right) \cdot J(x) & J(x) \end{pmatrix} \\
&= \begin{pmatrix} \left(\frac{\partial f^i}{\partial x^j}\right) & 0 \\ \left(\sum \dot{x}^i \frac{\partial^2 f^k}{\partial x^l \partial x^i}\right) & \left(\frac{\partial f^i}{\partial x^j}\right) \end{pmatrix} = \tilde{J}(x, \dot{x}),
\end{aligned}$$

where we have used (1. 4) in the fifth equality and used (2. 4) in the third equality. Since we proved (2. 2), we see, by (2. 1), that $T(F(M)) \subset \{T(U_\alpha), \tilde{J}_{\alpha\beta}\} = F(T(M))$. Thus Theorem 2. 4 is proved.

DEFINITION 2. 5. We denote by $j_M : T(F(M)) \longrightarrow F(T(M))$ the injection obtained by Theorem 2. 4. The injection j_M is independent of the choice of the open covering $\{U_\alpha\}$ of M with coordinate neighborhoods $\{U_\alpha\}$.

If U is a coordinate neighborhood in M, then there is the canonical trivialization $\Phi_U : U \times GL(n) \longrightarrow F(U)$ and $\Psi_U : T(U) \times GL(2n) \longrightarrow F(T(U))$ of $F(M)$ and $F(T(M))$ over U and $T(U)$ respectively. By virtue of (2. 2) in the proof of Theorem 2. 4 we see readily the follwing

PROPOSITION 2. 6.

$$j_M | T(F(U)) = \Psi_U \circ (1_{T(U)} \times j_n) \circ (T\Phi_U)^{-1}.$$

The following proposition is also clear from our construction of j_M:

PROPOSITION 2. 7. *Let $\pi : F(M) \longrightarrow M$, $\tilde{\pi} : F(T(M)) \longrightarrow T(M)$ be the projections. Then the map j_M is a bundle homomorphism of $T(F(M))$ into $F(T(M))$ with respect to j_n, i.e.*

$$j_M(X \cdot Y) = j_M(X) \cdot j_n(Y)$$

for $X \in T(F(M))$, $Y \in T(GL(n))$ and we have the following commutative diagram:

$$\begin{array}{ccc} T(F(M)) & \xrightarrow{j_M} & F(T(M)) \\ {\scriptstyle T\pi}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ T(M) & \xrightarrow{1_{T(M)}} & T(M) . \end{array}$$

§ 3. Prolongations of G-structures.

DEFINITION 3. 1. Let G be a Lie subgroup of $GL(n, R)$. We shall denote by $\tilde{G}$ the image of $T(G)$ by $j_n : \tilde{G} = j_n(T(G))$ (cf. Def. 1. 11). Clearly $\tilde{G}$ is a Lie subgroup of $GL(2n, R)$ and isomorphic to $T(G)$.

DEFINITION 3. 2. Let M be a manifold of dimension n and G be a Lie subgroup of $GL(n, R)$. A G-structure on M is, by definition, a G-subbundle $P(M, \pi, G)$ of the frame bundle $F(M)$ of M. Therefore, a G-structure on M is nothing but a reduction of the structure group $GL(n, R)$ of $F(M)$ to the subgroup G of $GL(n, R)$. (For the general theory of G-structures see, for instance [1], [2], [8]).

THEOREM 3. 3. *If a manifold M has a G-structure P, then $T(M)$ has a canonical $\tilde{G}$-structure $\tilde{P}$.*

Proof. Let $M = \cup U_\alpha$ be an open covering of M by U_α, over which the bundle $P(M,\pi,G)$ is trivial and let $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$ be the transition function. Then, by proposition 2. 3, $T(P) = \{T(U_\alpha), Tg_{\alpha\beta}\}$. Put $\tilde{P} = j_M(T(P))$. Then we see that $\tilde{P} = \{T(U_\alpha), j_n \circ Tg_{\alpha\beta}\}$. Since $j_n \circ Tg_{\alpha\beta}$ maps $T(U_\alpha) \cap T(U_\beta)$ into $\tilde{G}$, we obtained a $\tilde{G}$-structure $\tilde{P}$ on $T(M)$.

DEFINITION 3. 4. We shall call $\tilde{P}$ in Theorem 3. 3 the prolongation of the G-structure P on M to the tangent bundle $T(M)$.

We can prove the following well known fact:

COROLLARY 3. 5. *If a manifold M is completely parallelizable, then $T(M)$ is also completely parallelizable.*

Proof. Since M is completely parallelizable, M has a $\{e_n\}$-structure, where e_n is the unit matrix of $GL(n,R)$, $n = \dim M$. Then, by Theorem 3. 3, $T(M)$ has a $j_n(T(\{e_n\}))$-structure. Clearly $j_n(T(\{e_n\})) = \{e_{2n}\}$, which implies that $T(M)$ is completely parallelizable.

PROPOSITION 3. 6. *If a manifold M has a G-structure P, then $T(M)$ has a (not canonical) G_0-structure, where* $G_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| \; a \in G \right\}$.

Proof. Since G_0 is a closed subgroup of $\tilde{G}$ such that the quotient space $\tilde{G}/G_0$ is diffeomorphic to the Lie algebra of G, and hence $\tilde{G}/G_0$ is diffeomorphic to a euclidean space. Let $\tilde{P}/G_0$ be the space obtained by identifying z and $z \cdot a_0$ for $z \in \tilde{P}$ and $a_0 \in G_0$. Then $\tilde{P}/G_0$ is a fibre bundle with base $T(M)$, fibre $\tilde{G}/G_0$ and structure group $\tilde{G}$. Since the fibre $\tilde{G}/G_0$ is diffeomorphic to a euclidean space, this bundle $\tilde{P}/G_0$ has a global cross section σ and hence the structure group $\tilde{G}$ of $\tilde{P}$ is reduced to the subgroup G_0 (cf. [7]), which means that $T(M)$ has a G_0-structure $\tilde{P}_0$. We remark that $\tilde{P}_0$ is not canonical since the cross section σ is not canonical (cf. Remark 8. 4.). Thus Proposition 3. 6 is proved.

Remark 3. 7. Let $P(M,\pi,G)$ be a G-structure on M, and let Γ be a connection on the principal fibre bundle P, then $T(M)$ has a G_0-structure P_Γ canonically induced by Γ. In fact, the connection Γ defines a global cross section σ_Γ of $\tilde{P}/G_0$ and hence σ_Γ induces a G_0-structure on $T(M)$ as in the proof of Proposition 3. 6.

§ 4. Prolongations of isomorphisms of G-structures.

DEFINITION 4. 1. Let M and M' be manifolds of dimension n and G be a Lie subgroup of $GL(n, R)$ and let P and P' be G-structures on M and M' respectively. Let $f: M \longrightarrow M'$ be a diffeomorphism of M onto M'. We call f an isomorphism of P to P' if $(Ff)(P) = P'$. (cf. Introduction).

THEOREM 4. 2. *Let f be a diffeomorphism of M onto M'. Then we have:*

$$(FTf) \circ j_M = j_{M'} \circ (TFf). \tag{4. 1}$$

Proof. Let $\{U_\alpha\}$ be an open covering of M by coordinate neighborhood U_α. Put $V_\alpha = f(U_\alpha)$. Then we have $M' = \cup V_\alpha$ and V_α is a coordinate neighborhood. Let

$$\Phi_\alpha : U_\alpha \times GL(n) \longrightarrow F(U_\alpha),$$
$$\Phi'_\alpha : V_\alpha \times GL(n) \longrightarrow F(V_\alpha)$$

be the trivialization of $F(M)$ and $F(M')$ over U_α and V_α respectively. Then we have the following commutative diagram:

$$\begin{array}{ccc} F(U_\alpha) & \xleftarrow{\Phi_\alpha} & U_\alpha \times GL(n) \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle f_\alpha} \\ F(V_\alpha) & \xleftarrow{\Phi'_\alpha} & V_\alpha \times GL(n), \end{array} \tag{4. 2}$$

where we have defined the diffeomorphism f_α by

$$f_\alpha = \Phi'^{-1}_\alpha \circ Ff \circ \Phi_\alpha.$$

The diagram (4. 2) induces the following commutative diagram:

$$\begin{array}{ccc} T(F(U_\alpha)) & \xleftarrow{T\Phi_\alpha} & T(U_\alpha) \times T(GL(n)) \\ {\scriptstyle TFf}\downarrow & & \downarrow{\scriptstyle Tf_\alpha} \\ T(F(V_\alpha)) & \xleftarrow{T\Phi'_\alpha} & T(V_\alpha) \times T(GL(n)). \end{array} \tag{4. 3}$$

On the other hand, let

$$\Psi_\alpha : T(U_\alpha) \times GL(2n) \longrightarrow F(T(U_\alpha))$$
$$\Psi'_\alpha : T(V_\alpha) \times GL(2n) \longrightarrow F(T(V_\alpha))$$

be the local trivializations of $F(T(M))$ (and $F(T(M')))$ over $T(U_\alpha)$ (resp. $T(V_\alpha)$ induced by the local coordinate system on U_α (resp. V_α). Then, we have the following commutative diagram:

$$
(4.4)\qquad
\begin{array}{ccc}
T(U_\alpha)\times GL(2n) & \xrightarrow{\Psi_\alpha} & F(T(U_\alpha)) \\
{\scriptstyle \tilde f_\alpha}\downarrow & & \downarrow{\scriptstyle FTf} \\
T(V_\alpha)\times GL(2n) & \xrightarrow{\Psi'_\alpha} & F(T(V_\alpha)),
\end{array}
$$

where we have defined the diffeomorphism $\tilde{f}_\alpha$ by

$$\tilde{f}_\alpha = \Psi'^{-1}_\alpha \circ FTf \circ \Psi_\alpha .$$

Let $j_\alpha = 1_{T(U_\alpha)} \times j_n$ and $j'_\alpha = 1_{T(V_\alpha)} \times j_n$. By virtue of Proposition 2.6, we have:

$$
(4.5)\qquad
\begin{cases}
j_M = \Psi_\alpha \circ j_\alpha \circ (T\Phi_\alpha)^{-1} & \text{on} \quad TF(U_\alpha), \\
j_{M'} = \Psi'_\alpha \circ j'_\alpha \circ (T\Phi'_\alpha)^{-1} & \text{on} \quad TF(V_\alpha).
\end{cases}
$$

To prove (4.1), it is now sufficient to prove that the following diagram

$$
(4.6)\qquad
\begin{array}{ccc}
T(U_\alpha)\times T(GL(n)) & \xrightarrow{j_\alpha} & T(U_\alpha)\times GL(2n) \\
{\scriptstyle Tf_\alpha}\downarrow & & \downarrow{\scriptstyle \tilde f_\alpha} \\
T(V_\alpha)\times T(GL(n)) & \xrightarrow{j'_\alpha} & T(V_\alpha)\times GL(2n)
\end{array}
$$

is commutative, since the diagrams (4.3) and (4.4) are commutative and since (4.5) holds.

To prove the commutativity of the diagram (4.6), we take the local coordinate system $\{x^1, \cdots, x^n\}$ on U_α and $\{y^1, \cdots, y^n\}$ on V_α. Then $T(U_\alpha)$ (and $T(V_\alpha)$) has the induced local coordinate system $\{x^1, \cdots, x^n, \dot{x}^1, \cdots, \dot{x}^n\}$ $\{y^1, \cdots, y^n, \dot{y}^1, \cdots, \dot{y}^n\}$ resp.). Similarly $F(U_\alpha)$ (and $F(V_\alpha)$) has the induced local coordinate system $\{x^1, \cdots, x^n, \cdots, w^i_j, \cdots\}$ ($\{y^1, \cdots, y^n, \cdots, z^i_j, \cdots\}$ resp.). Now this local coordinate system on $F(U_\alpha)$ induces the local coordinate system $\{x^1, \cdots, x^n, w^i_j, \dot{x}^1, \cdots, \dot{x}^n, \dot{w}^i_j\}$ on $T(F(U_\alpha))$ ($\{y^1, \cdots, y^n, z^i_j, \dot{y}^1, \cdots, \dot{y}^n, \dot{z}^i_j\}$ on $T(F(V_\alpha))$ resp.). Let $f : U_\alpha \longrightarrow V_\alpha$ be expressed by

$$y^i = f^i(x^1, \cdots, x^n)$$

in terms of the above local coordinate systems. Then the maps Tf, f_α, Tf_α are expressed as follows:

$$Tf : y^i = f^i(x),\ \dot{y}^i = \sum \frac{\partial f^i}{\partial x^k}\dot{x}^k,$$

$$f_\alpha : y^i = f^i(x),\ z_i^j = \sum w_i^k \frac{\partial f^j}{\partial x^k},$$

$$(4.7)\qquad Tf_\alpha : \begin{cases} y^i = f^i(x),\ z_j^i = \sum w_i^k \dfrac{\partial f^j}{\partial x^k}, \\ \dot{y}^i = \sum \dfrac{\partial f^i}{\partial x^k}\dot{x}^k, \\ \dot{z}_i^j = \sum w_i^k \dfrac{\partial^2 f_j}{\partial x^k \partial x^l}\dot{x}^l + \sum \dfrac{\partial f^j}{\partial x^k}\dot{w}_i^k. \end{cases}$$

On the other hand, the local coordinate system $\{x^1, \cdots, x^n, \dot{x}^1, \cdots, \dot{x}^n\}$ on $T(U_\alpha)$ ($\{y^1, \cdots, y^n, \dot{y}^1, \cdots, \dot{y}^n\}$ on $T(V_\alpha)$ resp.) induces the local coordinate system $\{x^1, \cdots, x^n, \dot{x}^1, \cdots, \dot{x}^n, \tilde{w}_\nu^\mu\}$ $\mu, \nu = 1, 2, \cdots, 2n$ on $F(T(U_\alpha))$ ($\{\dot{y}^1, \cdots, y^n, \dot{y}^1, \cdots, \dot{y}^n, \tilde{z}_\nu^\mu\}$ on $F(T(V_\alpha))$ resp.). Since $\tilde{z}_\nu^\mu = \sum_{\kappa=1}^{2n} \tilde{w}_\nu^\kappa \frac{\partial f^\mu}{\partial x^\kappa}$ by putting $f^{i+n}(x, \dot{x}) = \sum_k \frac{\partial f^i}{\partial x^k}\dot{x}^k$, and $x^{i+n} = \dot{x}^i$, we can express the map $\tilde{f}_\alpha$ by the following equations:

$$(4.8)\qquad \tilde{f}_\alpha : \begin{cases} y^i = f^i(x),\ \dot{y}^i = \sum\limits_{k=1}^n \dfrac{\partial f^i}{\partial x^k}\dot{x}^k, \\ \tilde{z}_\nu^j = \sum\limits_{k=1}^n \tilde{w}_\nu^k \dfrac{\partial f^j}{\partial x^k}, \\ \tilde{z}_\nu^{j+n} = \sum\limits_{k,l=1}^n \tilde{w}_\nu^k \dfrac{\partial^2 f^j}{\partial x^l \partial x^k}\dot{x}^l + \sum\limits_{k=1}^n \tilde{w}_\nu^{n+k} \dfrac{\partial f^j}{\partial x^k}, \end{cases}$$

for $i, j = 1, 2, \cdots, n$; $\nu = 1, 2, \cdots, 2n$. We now remark that, if $X \in T(GL(n))$ has the coordinates w_j^i and $\dot{w}_j^i$ then, we have:

$$(4.9)\qquad j_n(X) = \begin{pmatrix} (w_j^i) & 0 \\ (\dot{w}_j^i) & (w_j^i) \end{pmatrix}.$$

In fact, since $X = \sum \dot{w}_j^i \left(\frac{\partial}{\partial w_j^i}\right)_w$,

$$j_n(X) = j_n\left(\left[(w_j^i),\ TR_{w^{-1}} \sum \dot{w}_i^j \left(\frac{\partial}{\partial w_i^j}\right)\right]\right)$$

$$= j_n\left(\left[(w_j^i),\ \sum \dot{w}_i^j (w^{-1})_j^l \left(\frac{\partial}{\partial w_i^l}\right)_e\right]\right)$$

$$= \begin{pmatrix} (w^i_j) & 0 \\ (\dot{w}^j_i(w^{-1})^l_j)\cdot(w^j_i) & (w^i_j) \end{pmatrix} = \begin{pmatrix} (w^i_j) & 0 \\ (\dot{w}^i_j) & (w^i_j) \end{pmatrix},$$

where we have used the notations in Definition 1. 8 and the formula (1. 4).

Now take an element $\tilde{X} = (x, \dot{x}, w^j_i, \dot{w}^j_i) \in T(U_\alpha) \times T(GL(n))$, then by (4. 7) we have

$$Tf_\alpha(\tilde{X}) = (f^i(x), \sum \frac{\partial f^j}{\partial x^k} \dot{x}^k, (z^j_i), (\dot{z}^j_i)),$$

where z^j_i, $\dot{z}^j_i$ are given in (4. 7). Hence, by (4. 9) we have:

$$(4.10) \qquad j'_\alpha \circ Tf_\alpha(\tilde{X}) = \left((f^i(x), \sum \frac{\partial f^i}{\partial x^k} \dot{x}^k), \begin{pmatrix} (z^j_i) & 0 \\ (\dot{z}^j_i) & (z^j_i) \end{pmatrix} \right).$$

Now, by using (4. 9) and (4. 8) we obtain:

$$\tilde{f}_\alpha \circ j_\alpha(\tilde{X}) = \tilde{f}_\alpha \left((x^i, \dot{x}^i), \begin{pmatrix} (w^j_i) & 0 \\ (\dot{w}^j_i) & (w^j_i) \end{pmatrix} \right)$$
$$= \left((f^i(x), \sum \frac{\partial f^i}{\partial x^k} \dot{x}^k), \begin{pmatrix} (z^j_i) & 0 \\ (\dot{z}^j_i) & (z^j_i) \end{pmatrix} \right),$$

where z^j_i and $\dot{z}^j_i$ are the same as in (4. 10). Finally we obtain $j'_\alpha \circ Tf_\alpha = \tilde{f}_\alpha \circ j_\alpha$, whence Theorem 4. 2 is proved.

THEOREM 4. 3. *Let P and P' be G-structures on M and M' respectively. Let f be a diffeomorphism of M onto M'. Then f is an isomorphism of P to P' if and only if Tf is an isomorphism of $\tilde{P}$ to $\tilde{P}'$.*

Proof. By the definition of $\tilde{P}$ and $\tilde{P}'$ we have:

$$\tilde{P} = j_M T(P), \quad \tilde{P}' = j_{M'} T(P').$$

Suppose f is an isomorphism of P to P'. Then

$$FTf(\tilde{P}) = FTf(j_M(T(P))) = j_{M'}(TFf(T(P)))$$
$$= j_{M'}(T(P')) = \tilde{P}',$$

and hence Tf is an isomorphism of $\tilde{P}$ to $\tilde{P}'$.

Conversely, suppose Tf is an isomorphism of $\tilde{P}$ to $\tilde{P}'$. Then $FTf(\tilde{P}) = \tilde{P}'$, and hence $FTf(j_M(T(P))) = j_{M'}(T(P'))$ and so $j_{M'} TFf(T(P)) = j_{M'}(T(P'))$. Since $j_{M'}$ is injective, we have $TFf(T(P)) = T(P')$, which implies that $Ff(P)=P'$, whence f is an isomorphism of P to P'. Thus Theorem 4. 3 is proved.

COROLLARY 4. 4. *Let P be a G-structure on M and let f be a diffeomorphism of M onto itself. Then, f is an automorphism of P if and only if Tf is an automorphism of $\tilde{P}$.*

§ 5. Integrability of the prolongations of G-structures.

DEFINITION 5. 1. Let $P(M,\pi,G)$ be a G-structure on M. The G-structure P is called integrable (or flat) if for each point $p\in M$ there is a coordinate neighborhood U with local coordinate system $\{x^1,\cdots,x^n\}$ on U such that the frame

$$\Big(\Big(\frac{\partial}{\partial x^1}\Big)_x,\cdots,\Big(\frac{\partial}{\partial x^n}\Big)_x\Big)\in P$$

for any $x\in U$.

LEMMA 5. 2. *Let $\{x^1,\cdots,x^n\}$ be a local coordinate system on a neighborhood U in M and let $f: U\longrightarrow GL(m,R)$ be a map, $f^i_j(x)$ being the (i,j)-entry of $f(x)$ for $x\in U$. Then we have*

$$(5.\ 1)\qquad (j_m\circ Tf)(x,\dot{x})=\begin{pmatrix} (f^i_j(x)) & 0 \\ \Big(\sum_k \frac{\partial f^i_j}{\partial x^k}\dot{x}^k\Big) & (f^i_j(x))\end{pmatrix},$$

where $(x,\dot{x})=(x^1,\cdots,x^n,\ \dot{x}^1,\cdots,\dot{x}^n)$ is the induced local coordinate system on $T(U)$.

Proof. We have

$$Tf(x,\dot{x})=Tf\Big(\sum\dot{x}^k\Big(\frac{\partial}{\partial x^k}\Big)_x\Big)=\sum\dot{x}^k\frac{\partial f^i_j}{\partial x^k}\Big(\frac{\partial}{\partial w^i_j}\Big)_{f(x)}.$$

Applying (4. 9) for m, we see that (5. 1) holds.

PROPOSITION 5. 3. *Let $\{x^1,\cdots,x^n\}$ be a local coordinate system on a neighborhood U in M. Let ϕ be a cross section of $F(M)$ over U, which is expressed by:*

$$\phi(x)=\Big(\cdots,\sum\phi^i_j(x)\Big(\frac{\partial}{\partial x^i}\Big)_x,\cdots\Big)$$

for $x \in U$. *Define* $\tilde{\phi} = j_M \circ T\phi$. *Then* $\tilde{\phi}$ *is a cross section of* $F(T(M))$ *over* $T(U)$ *and is expressed by the following:*

$$\tilde{\phi}\Big(\sum \dot{x}^i\Big(\frac{\partial}{\partial x^i}\Big)_x\Big) = \Big(\cdots, \phi^i_j(x)\Big(\frac{\partial}{\partial x^i}\Big)_X + \frac{\partial \phi^i_j}{\partial x^k}\dot{x}^k\Big(\frac{\partial}{\partial \dot{x}^i}\Big)_X, \cdots,$$

$$\phi^i_j(x)\Big(\frac{\partial}{\partial \dot{x}^i}\Big)_X, \cdots\Big),$$

where $\{x^1, \cdots, x^n, \dot{x}^1, \cdots, \dot{x}^n\}$ *in the right hand side is the induced local coordinate system on* $T(U)$ *and* $X = \sum \dot{x}^i\Big(\frac{\partial}{\partial x^i}\Big)_x \in T(U)$.

Proof. Let $\pi : F(M) \longrightarrow M$ and $\tilde{\pi} : F(T(M)) \longrightarrow T(M)$ be the projections. Let Φ_U and Ψ_U be the local trivialization of $F(M)$ and $F(T(M))$ over U and $T(U)$ respectively as in Definition 2. 5. By Proposition 2. 6 we have

$$j_M | TF(U) = \Psi_U \circ (1_{T(U)} \times j_n) \circ (T\Phi_U)^{-1}.$$

First, $\tilde{\pi} \circ \tilde{\phi} = \tilde{\pi} \circ j_M \circ T\phi = T\pi \circ T\phi = T(\pi \circ \phi) = T1_U = 1_{T(U)}$, which shows that $\tilde{\phi}$ is a cross section of $F(T(M))$ over $T(U)$. Now, by putting $f(x) = (\phi^i_j(x))$ for $x \in U$, and using Lemma 5. 2 we calculate as follows:

$$\begin{aligned}\tilde{\phi} &= \Psi_U \circ (1_{T(U)} \times j_n) \circ (T\Phi_U)^{-1} \circ T(\Phi_U \circ \Phi_U^{-1} \circ \phi)\\ &= \Psi_U \circ (1_{T(U)} \times j_n) \circ T(1_U \times f)\\ &= \Psi_U \circ (1_{T(U)} \times j_n \circ Tf),\end{aligned}$$

which implies

$$\tilde{\phi}\Big(\sum \dot{x}^i\Big(\frac{\partial}{\partial x^i}\Big)_x\Big) = \Psi_U\left((x, \dot{x}), \begin{pmatrix} (\phi^i_j(x)) & 0 \\ \Big(\sum \frac{\partial \phi^i_j}{\partial x^i}\dot{x}^k\Big) & (\phi^i_j(x)) \end{pmatrix}\right)$$

$$= \Big(\cdots, \phi^i_j(x)\Big(\frac{\partial}{\partial x^i}\Big)_X + \frac{\partial \phi^i_j}{\partial x^k}\dot{x}^k\Big(\frac{\partial}{\partial \dot{x}}\Big)_X, \cdots,$$

$$\phi^i_j(x)\Big(\frac{\partial}{\partial \dot{x}^i}\Big)_X, \cdots\Big).$$

Thus, Proposition 5. 3 is proved.

COROLLARY 5. 4. *Let* P *be an integrable* G*-structure on* M, *then the prolongation* $\tilde{P}$ *of* P *is also integrable.*

Proof. Take any $X_0 \in T_p(M)$. Let U be a coordinate neighborhood of

p in M with the local coordinate system $\{x^1, \cdots, x^n\}$ such that, if we define ϕ by $\phi(x) = \left(\cdots, \left(\frac{\partial}{\partial x^i}\right)_x, \cdots\right)$ for $x \in U$, then ϕ is a cross section of P over U. Let $\{x^1, \cdots, x^n, \dot{x}^1, \cdots, \dot{x}^n\}$ be the induced local coordinate system on $T(U)$. Then, by Proposition 5. 3 $\tilde{\phi} = j_M \circ T\phi$ is a cross section of $F(T(M))$ over $T(U)$ and is expressed by

$$\tilde{\phi}(X) = \left(\cdots, \left(\frac{\partial}{\partial x^i}\right)_X, \cdots, \left(\frac{\partial}{\partial \dot{x}^i}\right)_X, \cdots\right)$$

for $X = \sum \dot{x}^i \left(\frac{\partial}{\partial x^i}\right)_x \in T(U)$. Now we have:

$$\tilde{\phi}(T(U)) = j_M \circ T\phi(T(U)) \subset j_M(T\phi(U)) \subset j_M(T(P)) = \tilde{P},$$

which shows that $\tilde{\phi}$ is actually a cross section of $\tilde{P}$ over $T(U)$. Since $X_0 \in T(M)$ is arbitrary, we have proved Corollary 5. 4.

Conversely, we have the following:

PROPOSITION 5. 5. *Let P be a G-structure on M. If the prolongation $\tilde{P}$ of P is integrable, then P is also integrable.*

Proof. Take a point $p \in M$ and take a coordinate neighborhood U of p with local coordinate system $\{x^1, \cdots, x^n\}$ on U such that there is a local cross section $\phi : U \longrightarrow P$ of P over U. Then by Proposition 5. 3 and the proof of Corollary 5. 4, $\tilde{\phi} = j_M \circ T\phi$ is a cross section of $\tilde{P}$ over $T(U)$.

Now, let $X_0 \in T(U)$ be the zero tangent vector of M at p. Since $\tilde{P}$ is integrable, there can be found a coordinate neighborhood $\tilde{U}$ of X_0 with local coordinate system $\{y^1, \cdots, y^{2n}\}$ such that $\tilde{U} \subset T(U)$ and that if we define $\tilde{\phi}_0$ by

$$\tilde{\phi}_0(X) = \left(\left(\frac{\partial}{\partial y^1}\right)_X, \cdots, \left(\frac{\partial}{\partial y^{2n}}\right)_X\right)$$

$\tilde{\phi}_0$ is a cross section of $\tilde{P}$ over $\tilde{U}$. Since $\tilde{\phi}|\tilde{U}$ and $\tilde{\phi}_0$ are both cross sections of $\tilde{P}$ over $\tilde{U}$, there exists a map $\tilde{g} : \tilde{U} \longrightarrow \tilde{G}$ such that

$$\tilde{\phi}(X) = \tilde{\phi}_0(X) \cdot \tilde{g}(X) \tag{5. 2}$$

for $X \in \tilde{U}$. By virtue of Proposition 1. 12, we can write

$$\tilde{g}(X) = \begin{pmatrix} g(X) & 0 \\ B(X)g(X) & g(X) \end{pmatrix},$$

where $g:\tilde{U}\longrightarrow G$, $B:\tilde{U}\longrightarrow \mathfrak{g}$ are C^∞-maps, $\mathfrak{g}$ being the Lie algebra of G. Since $\{y^1,\cdots,y^{2n}\}$ and $\{x^1,\cdots,x^n, \dot{x}^1,\cdots,\dot{x}^n\}$ are both coordinate systems on $\tilde{U}$ we can express:

$$y^\nu = f^\nu(x^1,\cdots,x^n, \dot{x}^1,\cdots,\dot{x}^n)$$

for $(x,\dot{x})\in\tilde{U}$, where f^ν are differentiable functions for $\nu=1,2,\cdots,2n$. Now, if

$$\phi(x) = \left(\sum \phi_1^i(x)\left(\frac{\partial}{\partial x^i}\right)_x, \cdots, \sum \phi_n^i(x)\left(\frac{\partial}{\partial x^i}\right)_x\right)$$

for $x\in U$, then by Proposition 5. 3, (5. 2) can be written as follows:

$$\tag{5. 3} \begin{aligned} &\sum_i \phi_j^i(x)\left(\frac{\partial}{\partial x^i}\right)_X + \sum_{k,i}\frac{\partial\phi_j^i}{\partial x^k}\dot{x}^k\left(\frac{\partial}{\partial \dot{x}^i}\right)_X \\ &= \sum_i g_j^i(X)\left(\frac{\partial}{\partial y^i}\right)_X + \sum_{i,k} B_k^i(X) g_j^k(X)\left(\frac{\partial}{\partial y^{n+i}}\right)_X, \end{aligned}$$

where $g(X)=(g_j^i(X))$ and $B(X)=(B_j^i(X))$ for $X\in\tilde{U}$.

Since $\left(\frac{\partial}{\partial x^i}\right)_X = \sum\frac{\partial f^\nu}{\partial x^i}\left(\frac{\partial}{\partial y^\nu}\right)_X$ and $\left(\frac{\partial}{\partial \dot{x}^i}\right)_X = \sum\frac{\partial f^\nu}{\partial \dot{x}^i}\left(\frac{\partial}{\partial y^\nu}\right)_X$, (5. 3) can be written as follows:

$$\tag{5. 4} \begin{aligned} &\sum_{i=1}^n\sum_{\nu=1}^{2n}\phi_j^i(x)\frac{\partial f^\nu}{\partial x^i}\left(\frac{\partial}{\partial y^\nu}\right)_X + \sum_{\nu=1}^{2n}\sum_{i,k=1}^n\frac{\partial\phi_j^i}{\partial x^k}\dot{x}^k\frac{\partial f^\nu}{\partial \dot{x}^i}\left(\frac{\partial}{\partial y^\nu}\right)_X \\ &= \sum_{i=1}^n g_j^i(X)\left(\frac{\partial}{\partial y^i}\right)_X + \sum_{i,k=1}^n B_k^i(X) g_j^k\left(\frac{\partial}{\partial y^{n+i}}\right)_X. \end{aligned}$$

Comparing the coefficients of $\left(\frac{\partial}{\partial y^k}\right)_X$ for $k\leqq n$ in (5. 4), we have:

$$\tag{5. 5} \phi_j^i(x)\frac{\partial f^s}{\partial x^i} + \frac{\partial\phi_j^i}{\partial x^k}\dot{x}^k\frac{\partial f^s}{\partial \dot{x}^i} = g_j^s(X)$$

for $j,s=1,2,\cdots,n$. Now, define maps $\bar{f}_s:U'\longrightarrow R$ $(s=1,2,\cdots,n)$ and $\bar{g}:U'\longrightarrow G$ by

$$\bar{f}_s(x) = f^s(x,0) \text{ and } (\bar{g}(x)^{-1})_j^s = g_j^s(x,0)$$

for $x\in U'=\pi(U)$. Putting $\dot{x}^k=0$ $(k=1,2,\cdots,n)$ in (5. 5) we get

$$\tag{5. 6} \sum\phi_j^i(x)\frac{\partial \bar{f}_s}{\partial x^i} = (\bar{g}(x)^{-1})_j^s$$

for $x \in U'$. Now define $\bar{x}^i(x) = \bar{f}_i(x)$ for $x \in U'$, then there exists a neighborhood U_0 of p such that $\{\bar{x}^1, \cdots, \bar{x}^n\}$ is a coordinate system on U_0, because. $\det\left(\frac{\partial \bar{f}_i}{\partial x^j}\right) \neq 0$ by virtue of (5. 6). We shall prove

$$\bar{\phi}(x) = \phi(x) \cdot \bar{g}(x) \tag{5. 7}$$

for $x \in U_0$, where we have defined $\bar{\phi}$ by

$$\bar{\phi}(x) = \left(\left(\frac{\partial}{\partial \bar{x}^1}\right)_x, \cdots, \left(\frac{\partial}{\partial \bar{x}^n}\right)_x\right) \tag{5. 8}$$

for $x \in U_0$. In fact, since $\left(\frac{\partial}{\partial x^i}\right)_x = \sum \frac{\partial \bar{f}_s}{\partial x^i}\left(\frac{\partial}{\partial \bar{x}^s}\right)_x$, we have

$$\begin{aligned}
\phi(x) \cdot \bar{g}(x) &= \left(\cdots, \sum \phi^i_j(x)\left(\frac{\partial}{\partial x^i}\right)_x \bar{g}^j_l(x), \cdots\right) \\
&= \left(\cdots, \sum \phi^i_j(x)\bar{g}^j_l(x)\frac{\partial \bar{f}_s}{\partial x^i}\left(\frac{\partial}{\partial \bar{x}^s}\right)_x, \cdots\right) \\
&= \left(\cdots, \sum (\bar{g}(x)^{-1})^s_j \bar{g}^j_l(x)\left(\frac{\partial}{\partial \bar{x}^s}\right)_x, \cdots\right) \\
&= \left(\left(\frac{\partial}{\partial \bar{x}^1}\right)_x, \cdots, \left(\frac{\partial}{\partial \bar{x}^n}\right)_x\right) = \bar{\phi}(x).
\end{aligned}$$

Since $\bar{g}(x) \in G$ for $x \in U_0$, the map $\bar{\phi} : U_0 \longrightarrow F(M)$ is a cross section of P. Thus, for any point $p \in M$, there exists a coordinate neighborhood U_0 of p with coordinate system $\{\bar{x}^1, \cdots, \bar{x}^n\}$ such that the map $\bar{\phi}$ defined by (5. 8) is a cross section of P over U_0. Thus Proposition 5. 5 is proved.

Combining Corollary 5. 4 and Proposition 5. 5 we obtain the following

THEOREM 5. 6. *Let P be a G-structure on a manifold M. Then, P is integrable if and only if the prolongation $\tilde{P}$ is integrable.*

§ 6. Prolongations of some classical G-structures.

(I) $G = GL(n, C)$.

We take a linear automorphism $J : R^{2n} \longrightarrow R^{2n}$ such that $J^2 = -1$, and denote by $GL(n, C)$ the group of all $a \in GL(2n, R)$ such that $a \circ J = J \circ a$. More precisely, $GL(n, C)$ will be denoted by $GL(n, C; J)$.

LEMMA 6. 1. *The tangential map TJ of J is a linear automorphism of the vector space $T(R^{2n})$ such that $(TJ)^2 = -1$.*

Proof. By virtue of Proposition 1. 3, TJ is a linear endomorphism of $T(R^{2n})$. Moreover,

$$(TJ)^2 = TJ \circ TJ = T(J^2) = T(-1) = -1 .$$

Thus Lemma 6. 1 is proved.

PROPOSITION 6. 2. *Let* $G = GL(n, C; J)$, *then* $\tilde{G} \subset GL((2n, C; TJ)$.

Proof. Take an element $\tilde{a} \in \tilde{G}$. We can take an element $A \in T(G)$ such that $\tilde{a} = j_{2n}(A)$. Then, by the notations in Definition 1. 8, we can write $A = [a, B]$, where $a \in G$ and $B \in \mathfrak{g}$, $\mathfrak{g}$ being the Lie algebra of G. Hence, by Proposition 1. 12 we have the following expressions:

$$\tilde{a} = j_{2n}([a, B]) = \begin{pmatrix} a & 0 \\ Ba & a \end{pmatrix},$$

$$TJ = j_{2n}([J, 0]) = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix},$$

Since $B \in \mathfrak{g}$, we have the equality $BJ = JB$. Therefore, we get

$$\tilde{a} \circ TJ = \begin{pmatrix} a & 0 \\ Ba & a \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} = \begin{pmatrix} aJ & 0 \\ BaJ & aJ \end{pmatrix}$$
$$= \begin{pmatrix} Ja & 0 \\ JBa & Ja \end{pmatrix} = TJ \circ \tilde{a},$$

where we have used the equality $BJ = JB$ in the third equality. Thus $\tilde{a} \in GL(2n, C; TJ)$ and Proposition 6. 2 is proved.

THEOREM 6. 3. (1) *If a manifold* M *has an almost complex structure, then* $T(M)$ *has a canonical almost complex structure.*

(2) *If a manifold* M *is a complex manifold, then* $T(M)$ *has a canonical complex structure.*

Proof. (1) As is well known, M has an almost complex structure if and only if M has a $GL(n, C; J)$-structure. If M has a G-structure then $T(M)$ has a canonical $\tilde{G}$-structure $\tilde{P}$. Applying this assertion for $G = GL(n, C; J)$, we see that the $\tilde{G}$-structure $\tilde{P}$ induces canonically a $GL(2n, C; TJ)$-structure $\tilde{P}'$ by virtue of Proposition 6. 2., which means that $T(M)$ has a canonical almost complex structure. Thus the assertion (1) is proved.

(2) It is well known that a $GL(n, C; J)$-structure is integrable if and only if the associated almost complex structure is a complex structure. Therefore, if M is a complex manifold, M has a canonical integrable $GL(n, C; J)$-structure P. Then, by Corollary 5. 4 the prolongation $\tilde{P}$ of P is also integrable. Therefore the canonical $GL(2n, C; TJ)$-structure $\tilde{P}'$ is again integrable. Thus $T(M)$ has a canonical complex structure and the assertion (2) is proved.

(II) $G = S_p(m)$.

Let $f : R^{2m} \times R^{2m} \longrightarrow R$ be a skew-symmetric non-degenerate bilinear form on R^{2m}. We denote by $S_p(m)$ the group of all $a \in GL(2m, R)$ such that

$$f(ax, ay) = f(x, y)$$

for all $x, y \in R^{2m}$. More precisely, we write $S_p(m) = S_p(m, f)$. We shall denote by π the projection of $T(R) = R \times R$ onto R defined by

$$\pi\Big(c\Big(\frac{d}{dt}\Big)_s\Big) = c$$

for $c, s \in R$, where t is the natural coordinate in R.

LEMMA 6. 4. *If f is a skew-symmetric non-degenerate bilinear form on R^{2m}, then $\tilde{f} = \pi \circ Tf$ is also a skew-symmetric non-degenerate bilinear form on $R^{4m} = T(R^{2m})$.*

Proof. We define $\tau : R^{2m} \times R^{2m} \longrightarrow R^{2m} \times R^{2m}$ by $\tau(u, v) = (v, u)$ for $u, v \in R^{2m}$. Then, we have $f \circ \tau = -f$ and hence

$$Tf \circ T\tau = T(f \circ \tau) = T(-f) = T(-1) \circ Tf = (-1) \circ Tf,$$

where "$\circ$" in the last term means the scalar multiplication in $T(R^{2m})$. Therefore, we have

$$(\pi \circ Tf) \circ (T\tau) = -\pi \circ Tf.$$

Thus, $\tilde{f}$ is skew-symmetric on $T(R^{2m})$. We take a skew-symmetric matrix $(a^i_j) \in GL(2m, R)$ such that

$$f(x, y) = \sum_{i,j=1}^{n} a^i_j x^i y^j$$

for $x = (x^1, \cdots, x^n)$ and $y = (y^1, \cdots, y^n)$ with $n = 2m$. Then it follows that

$$Tf\left(\left(\frac{\partial}{\partial x^i}\right)_{(x,y)}\right) = \frac{\partial f}{\partial x^i}\left(\frac{d}{dt}\right)_{f(x,y)} = \sum_j a^i_j y^j \left(\frac{d}{dt}\right)_{f(x,y)},$$

$$Tf\left(\left(\frac{\partial}{\partial y^j}\right)_{(x,y)}\right) = \frac{\partial f}{\partial y^j}\left(\frac{d}{dt}\right)_{f(x,y)} = \sum_i a^i_j x^i \left(\frac{d}{dt}\right)_{f(x,y)}.$$

Therefore, we have for any $b_i, c_j \in R$:

$$\begin{aligned} & Tf\left(\sum b_i \left(\frac{\partial}{\partial x^i}\right)_x, \ \sum c_j \left(\frac{\partial}{\partial y^j}\right)_y\right) \\ &= Tf\left(\sum b_i \left(\frac{\partial}{\partial x^i}\right)_{(x,y)} + \sum c_j \left(\frac{\partial}{\partial y^j}\right)\right)_{(x,y)} \\ &= \sum b_i \, Tf\left(\left(\frac{\partial}{\partial x^i}\right)_{(x,y)}\right) + \sum c_j \, Tf\left(\left(\frac{\partial}{\partial y^j}\right)_{(x,y)}\right) \\ &= \sum b_i a^i_j y^j \left(\frac{d}{dt}\right)_{f(x,y)} + \sum c_j a^i_j x^i \left(\frac{d}{dt}\right)_{f(x,y)} \\ & \sum (b_i a^i_j y^j + c_j a^i_j x^i)\left(\frac{d}{dt}\right)_{f(x,y)}. \end{aligned}$$

Hence, we obtain

$$\tilde{f}\left(\sum b_i \left(\frac{\partial}{\partial x^i}\right)_x, \ \sum c_j \left(\frac{\partial}{\partial y^j}\right)_y\right) = \sum (b_i a^i_j y^j + c_j a^i_j x^i). \tag{6. 1}$$

Now, we calculate as follows:

$$\begin{aligned} & \tilde{f}\left(c \circ \sum b_i \left(\frac{\partial}{\partial x^i}\right)_x, \ \sum c_j \left(\frac{\partial}{\partial y^j}\right)_y\right)_y\Big) \\ &= \tilde{f}\left(\sum c b_i \left(\frac{\partial}{\partial x^i}\right)_{cx} \cdot \sum c_j \left(\frac{\partial}{\partial y^j}\right)_y\right) \\ &= \sum (c b_i a^i_j y^j + c_j a^i_j c x^i) \\ &= c \cdot f\left(\sum b_i \left(\frac{\partial}{\partial x^i}\right)_x, \ \sum c_j \left(\frac{\partial}{\partial y^j}\right)_y\right) \end{aligned}$$

for any $c \in R$. Similarly, we obtain

$$\begin{aligned} & \tilde{f}\left(\sum b_i \left(\frac{\partial}{\partial x^i}\right)_x \oplus \sum b'_i \left(\frac{\partial}{\partial x^i}\right)_{x'}, \ \sum c_j \left(\frac{\partial}{\partial y^j}\right)_y\right) \\ &= \tilde{f}\left(\sum b_i \left(\frac{\partial}{\partial x^i}\right)_x, \ \sum c_j \left(\frac{\partial}{\partial y^j}\right)_y\right) \\ & \quad + \tilde{f}\left(\sum b'_i \left(\frac{\partial}{\partial x^i}\right)_{x'}, \ \sum c_j \left(\frac{\partial}{\partial y^j}\right)_y\right), \end{aligned}$$

which means that $\tilde{f}$ is bilinear on $T(R^{2m})$. If we identify a tangent vector $\sum b_i\left(\frac{\partial}{\partial x^i}\right)_x \in T(R^n)$ with the vector $(x^1, \cdots, x^n, b_1, \cdots, b_n) \in R^{2n}$, then we have the following expression of $\tilde{f}$:

$$
\tag{6. 2}
\begin{aligned}
&\tilde{f}((x^i, b_i), (y^i, c_i)) \\
&= (x^1, \cdots, x^n, b_1, \cdots, b_n)\begin{pmatrix} 0 & (a^i_j) \\ (a^i_j) & 0 \end{pmatrix}\begin{pmatrix} y^1 \\ \vdots \\ y^n \\ c_1 \\ \vdots \\ c_n \end{pmatrix}
\end{aligned}
$$

which implies in particular, that $\tilde{f}$ is non-degenerate. Thus, Lemma 6. 4 is proved.

PROPOSITION 6. 5. *Let $G = S_p(m, f)$, then $\tilde{G} \subset S_p(2m, \tilde{f})$.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G. Then, if we denote by $A = (a^i_j)$ the matrix in the proof of Lemma 6. 4, we have the equality $B\cdot A + A\cdot {}^tB = 0$ for any $B \in \mathfrak{g}$, ${}^tB$ being the transpose of B. Now, take any $\tilde{a} \in \tilde{G}$, then $\tilde{a} = j_{2m}(X)$ for some $X \in T(G)$. By Proposition 1. 12 we can write $X = [a, B]$ with $a \in G$ and $B \in \mathfrak{g}$. Then, we have the following expression:

$$
\tilde{a} = \begin{pmatrix} a & 0 \\ Ba & a \end{pmatrix}.
$$

Now, we can calculate as follows:

$$
\begin{aligned}
\tilde{a}\cdot\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}\cdot {}^t\tilde{a} &= \begin{pmatrix} a & 0 \\ Ba & a \end{pmatrix}\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}\begin{pmatrix} {}^ta & {}^ta\,{}^tB \\ 0 & {}^ta \end{pmatrix} \\
&= \begin{pmatrix} 0 & 2A \\ aA & BaA \end{pmatrix}\cdot\begin{pmatrix} {}^ta & {}^ta\,{}^tB \\ 0 & {}^ta \end{pmatrix} \\
&= \begin{pmatrix} 0 & aA{}^ta \\ aA\,{}^ta & aA\,{}^ta\,{}^tB + BaA\,{}^ta \end{pmatrix} \\
&= \begin{pmatrix} 0 & A \\ A & A\cdot{}^tB + B\cdot A \end{pmatrix} = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix},
\end{aligned}
$$

which means that $\tilde{a} \in S_p(2m, \tilde{f})$, since $\tilde{f}$ is expressed by $\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$ as in (6. 2). Thus Proposition 6. 5 is proved.

THEOREM 6. 6. (1) *If a manifold M has an almost symplectic structure, then $T(M)$ has a canonical almost symplectic structure.*

(2) *If M has symplectic structure, then $T(M)$ has a canonical symplectic structure.*

Proof. (1) A manifold M has an almost symplectic structure if and only if M has a $S_p(m)$-structure. Hence, by Proposition 6. 5, $T(M)$ has a canonical almost symplectic structure if M has an almost symplectic structure.

Now, it is well known that an almost symplectic structure is a symplectic structure if and only if the associated $S_p(m)$-structure is integrable. Therefore, we see that the assertion (2) follows from Corollary 5. 4.

(III) $G = GL(V, W)$.

Let $V = R^n$ and W be a vector subspace of V. We denote by $GL(V, W)$ the group of all $a \in GL(V)$ such that $a(W) = W$. The following lemma is easily verified:

LEMMA 6. 7. *Let $G = GL(V, W)$, then $\tilde{G} \subset GL(T(V), T(W))$.*

PROPOSITION 6. 8. (1) *If a manifold M has a k-dimensional differential system, (i.e. a differentiable assignement $M \ni x \longrightarrow D(x) \subset T_x(M)$ of vector subspaces $D(x)$ with $\dim D(x) = k$ for $x \in M$) then $T(M)$ has a canonical $2k$-dimensional differential system.*

(2) *If a k-dimensional differential systems on M is completely integrable, then the canonical $2k$-dimensional differential system on $T(M)$ is also completely integrable.*

Proof. (1) A manifold M has a k-dimensional differential system if and only if M has a $GL(V, W)$-structure with $\dim V = \dim M$, $\dim W = k$. If M has a $GL(V, W)$-structure, then $T(M)$ has a canonical $GL(T(V), T(W))$-structure by virtue of Lemma 6. 7. Hence $T(M)$ has a canonical $2k$-dimensional differential system, since $\dim T(W) = 2k$.

(2) It is well known that a differential system on M is completely integrable if and only if the associated $GL(V, W)$-structure on M is integrable. Therefore, the assertion (2) follows from Corollary 5. 4.

(IV) $G=0(k, n-k)$

Let f be a symmetric non-degenerate bilinear form on R^n of signature $(k, n-k)$ and let $\pi: T(R) \longrightarrow R$ be the same projection as in (II) and let $\tilde{f}$ be the map $\tilde{f}=\pi \circ Tf: T(R^n) \times T(R^n) \longrightarrow R$. We denote by $0(k, n-k)$ the group of all $a \in GL(n)$ such that $f(ax, ay)=f(x, y)$ for $x, y \in R^n$.

LEMMA 6. 9. *The notations being as above, $\tilde{f}$ is a symmetric non-degenerate bilinear form on $T(R^n)$ of signature (n, n)*

Proof. By the natural basis of $T(R^n)$ induced by the natural basis of R^n, $\tilde{f}$ is expressed by the matrix $\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$ as in (6. 1), if f is expressed by the matrix A. From this fact, Lemma 6. 9 follows.

The following proposition can be proved in the same way as the proof of Proposition 6. 5.

PROPOSITION 6. 10. *Let $G=0(k, n-k)$, then $\tilde{G} \subset 0(n, n)$.*

THEOREM 6. 11. *If M has a quasi-Riemannian metric, then $T(M)$ has a canonical quasi-Riemannian metric.*

Proof. A quasi-Riemannian metric on M is nothing but a $0(k, n-k)$-structure on M. Therefore, the prolongation of this $0(k, n-k)$-structure induces a canonical $0(n, n)$-structure on $T(M)$ by virtue of Proposition 6. 10, and hence $T(M)$ has a canonical quasi-Riemannian metric of signature (n, n).

(V) $G=SL(n, R)$.

As usual, $SL(n, R)$ denotes the group of all $a \in GL(n)$ with $\det(a)=1$.

LEMMA 6. 12. *Let $G=SL(n, R)$, then $\tilde{G} \subset SL(2n, R)$.*

Proof. Take $\tilde{a} \in \tilde{G}$, then $\tilde{a}=j_n(X)=\begin{pmatrix} a & 0 \\ Ba & a \end{pmatrix}$, where $X=[a, B]$ with $a \in G$ and $B \in \mathfrak{g}$, $\mathfrak{g}$ being the Lie algebra of G. Since $\det(a)=1$, we have $\det(\tilde{a})=1$. Thus Lemma 6. 12 is proved.

The following proposition is easily verified:

PROPOSITION 6. 13. *If a manifold M has a $SL(n, R)$-structure, then $T(M)$ has a canonical $SL(2n, R)$-structure.*

(VI) $G = U(n) \times 1 \subset GL(2n+1, R)$.

As usual, $U(n)$ denotes the unitary group of degree n. Then $U(n) \times 1 \subset GL(n, C) \times 1 \subset GL(2n+1, R)$ by the usual injection.

LEMMA 6. 14. *Let* $G = U(n) \times 1 \subset GL(2n+1, R)$, *then we have* $\tilde{G} \subset GL(2n+1, C)$.

Proof. Let $J : R^{2n} \longrightarrow R^{2n}$ be the linear isomorphism such that $GL(n, C) = GL(n, C; J)$. We denote by $J_0 : T(R) \longrightarrow T(R)$ the linear isomorphism of $T(R)$ defined by $J_0\left(a\left(\frac{d}{dt}\right)_s\right) = s\left(\frac{d}{dt}\right)_{-a}$ for $a, s \in R$, where t is the natural coordinate of R. Let $\tilde{J} : T(R^{2n+1}) \longrightarrow T(R^{2n+1})$ be the linear map defined by $\tilde{J} = TJ \times J_0$. Then it is readily seen that $\tilde{G} \subset GL(2n+1, C; J)$. Thus Lemma 6. 14 is proved.

PROPOSITION 6. 15. *If M has an almost contact structure (cf.* [3], [6] *for the definition), then $T(M)$ has a canonical almost complex structure.*

Proof. An almost contact structure on M induces a $U(n) \times 1$-structure P on M. Then, by Lemma 6. 14, the prolongation $\tilde{P}$ of P induces a canonical $GL(2n+1, C)$-structure on $T(M)$, that is, an almost complex structure on $T(M)$. Thus Proposition 6. 15 is proved.

§7. Relations between the prolongations of *G*-structures and the prolongations of tensor fields.

(I) We shall prove that our prolongation of $GL(m, C)$-structure given in Theorem 6. 3 is exactly the complete lift of the associated almost complex structure given by Yano and Kobayashi [9].

Let $P(M, \pi, GL(m, C))$ be a $G(m, C)$-structure on a manifold M. We take a coordinate neighborhood U in M with a local coordinate system $\{x^1, \cdots, x^n\}$ with $n = 2m$. Let ϕ be a cross section of P over U. Then the map $\tilde{\phi} = j_M \circ T\phi$ is also a cross section of the prolongation $\tilde{P}$ of P by Proposition 5. 3 and the proof of Corollary 5. 4. Let J be the linear isomorphism of R^{2m} such that $GL(m, C) = GL(m, C; J)$ and let $\psi(x) : T_x(M) \longrightarrow T_x(M)$ be the map defined by

$$\psi(x) = \phi(x) \circ J \circ \phi(x)^{-1}$$

for $x \in U$. Then ψ is the (globally defined) almost complex structure associated with the $GL(m, C)$-structure P. If we take $\left\{\left(\frac{\partial}{\partial x^1}\right)_x, \cdots, \left(\frac{\partial}{\partial x^n}\right)_x\right\}$ as a basis of the tangent space $T_x(M)$ we can write $\phi(x)$ and $\psi(x)$ as follows:

$$(7.1)\qquad \begin{cases} \phi(x) = \left(\cdots, \sum \phi_j^i(x)\left(\frac{\partial}{\partial x^i}\right)_x, \cdots\right), \\ \psi(x) = (\phi_j^i(x)) \circ J \circ (\phi_j^i(x))^{-1}. \end{cases}$$

We define $\tilde{\psi}(X) : T_X(T(M)) \longrightarrow T_X(TM))$ by

$$\tilde{\psi}(X) = \tilde{\phi}(X) \circ TJ \circ \tilde{\phi}(X)^{-1}$$

for $X \in T(U)$. Then, since $(TJ)^2 = -1$, we have $(\tilde{\psi}(X))^2 = -1$. Thus $\tilde{\psi}$ is an almost complex structure on $T(M)$. In fact, we see easily that $\tilde{\psi}$ is the canonical almost complex structure on $T(M)$ given in Theorem 6. 3. We take $\left\{\left(\frac{\partial}{\partial x^1}\right)_X, \cdots, \left(\frac{\partial}{\partial x^n}\right)_X, \left(\frac{\partial}{\partial \dot{x}^1}\right)_X, \cdots, \left(\frac{\partial}{\partial \dot{x}^n}\right)_X\right\}$ as the basis of $T_X(TM))$, where $\{x^1, \cdots, x^n, \dot{x}^1, \cdots, \dot{x}^n\}$ is the induced local coordinate system on $T(U)$. Then, using Proposition 5. 3, the map $\tilde{\psi}(X)$ can be expressed as follows:

$$(7.2)\qquad \tilde{\psi}(X) = \begin{pmatrix} (\phi_j^i(x)) & 0 \\ \left(\frac{\partial \phi_j^i}{\partial x^k}\dot{x}^k\right) & (\phi_j^i(x)) \end{pmatrix} \cdot TJ \cdot \begin{pmatrix} (\phi_j^i(x)) & 0 \\ \left(\frac{\partial \phi_j^i}{\partial x^k}\dot{x}^k\right) & (\phi_j^i(x)) \end{pmatrix}^{-1}$$

for $X = (x, \dot{x}) = \sum \dot{x}^k\left(\frac{\partial}{\partial x^k}\right)_x \in T(U)$, where we have $TJ = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$ by virtue of Proposition 1. 12.

Now, we shall prove the following

PROPOSITION 7. 1. *The notations being as above, if* $\psi(x) = (\psi_j^i(x))$ *then* $\tilde{\psi}(X)$ *is given by the following:*

$$\tilde{\psi}(X) = \begin{pmatrix} (\psi_j^i(x)) & 0 \\ \left(\sum \frac{\partial \psi_j^i}{\partial x^k}\dot{x}^k\right) & (\psi_j^i(x)) \end{pmatrix}$$

for $X = (x, \dot{x})$.

Proof. By virtue of (7. 1) and (7. 2), it is sufficient to show the following

$$(7.\ 3)\qquad \begin{pmatrix} (\psi^i_j) & 0 \\ \left(\dfrac{\partial\psi^i_j}{\partial x^k}\dot{x}^k\right) & (\psi^i_j) \end{pmatrix}\cdot\begin{pmatrix} (\phi^i_j) & 0 \\ \left(\dfrac{\partial\phi^i_j}{\partial x^k}\dot{x}^k\right) & (\phi^i_j) \end{pmatrix}$$

$$= \begin{pmatrix} (\phi^i_j) & 0 \\ \left(\dfrac{\partial\phi^i_j}{\partial x^k}\dot{x}^k\right) & (\phi^i_j) \end{pmatrix}\cdot\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}.$$

The left hand side of (7. 3) is equal to

$$(7.\ 4)\qquad \begin{pmatrix} (\psi^i_j)(\phi^i_j) & 0 \\ \left(\dfrac{\partial\psi^i_j}{\partial x^k}\dot{x}^k\right)\cdot(\phi^i_j)+(\psi^i_j)\left(\dfrac{\partial\phi^i_j}{\partial x^k}\dot{x}^k\right) & (\psi^i_j)(\phi^i_j) \end{pmatrix},$$

while the right hand side of (7. 3) is equal to

$$(7.\ 5)\qquad \begin{pmatrix} (\phi^i_j)J & 0 \\ \left(\dfrac{\partial\phi^i_j}{\partial x^k}\dot{x}^k\right)J & (\phi^i_j)J \end{pmatrix}.$$

Now, from (7. 1) we get the following

$$(7.\ 1)'\qquad (\psi^i_j(x))\,(\phi^i_j(x)) = (\phi^i_j(x))\cdot J.$$

Differentiating the both hand sides of (7. 1)′ with respect to x^k, multiplying $\dot{x}^k$ and summing up for $k=1,2,\cdots,n$, we obtain

$$(7.\ 6)\qquad \left(\frac{\partial\psi^i_j}{\partial x^k}\dot{x}^k\right)(\phi^i_j)+(\psi^i_j)\left(\frac{\partial\phi^i_j}{\partial x^k}\dot{x}^k\right)=\left(\frac{\partial\phi^i_j}{\partial x^k}\dot{x}^k\right)\cdot J.$$

By (7. 1)′ and (7. 6) we see that (7. 4) is equal to (7. 5) and hence (7. 3) holds. Thus Proposition 7. 1 is proved.

THEOREM 7. 2. *The canonical almost complex structure $\tilde{\phi}$ on $T(M)$ induced by a $GL(m, C)$-structure P on M is just the complete lift ϕ^C of the associated almost complex structure ϕ with P.*

Proof. By the formula of the complete lift of a $(1,1)$-tensor field on M given in [9] p. 204, we see that $\phi^C=\tilde{\phi}$ by virtue of Proposition 7. 1.

(II) Take $G=0(k, n-k)$ and a G-structure $P(M,\pi,G)$ on a manifold M. Then the prolongation $\tilde{P}$ of P induces an $0(n,n)$-structure $Q=Q(T(M), \tilde{\pi}, 0(n,n))$ as proved in Proposition 6. 11. We take a symmetric non-degenerate bilinear form f on R^n such that G is the group of all $a \in GL(n)$ satisfying $f(a\cdot u, a\cdot v) = f(u,v)$ for $u, v \in R^n$. Then $0(n,n)$ is the group of all $\tilde{a} \in GL(2n)$ such that $\tilde{f}(\tilde{a}\cdot\tilde{u},\ \tilde{a}\cdot\tilde{v}) = \tilde{f}(\tilde{u},\tilde{v})$ for $\tilde{u},\tilde{v} \in T(R^n)$, where $\tilde{f} = \pi\circ Tf$ as in §6 (IV).

Let U be a coordinate neighborhood in M with a local coordinate system $\{x^1, \cdots, x^n\}$ such that there is a cross section ϕ of P over U. For any $x \in U$ we denote by g_x the symmetric non-degenerate bilinear form on $T_x(M)$ defined by

$$g_x(X, Y) = f(\phi(x)^{-1}X,\ \phi(x)^{-1}Y), \tag{7. 7}$$

then g is the associated quasi-Riemannian metric on M with the G-structure P. We now define, for any $X \in T(U)$, the symmetric non-degenerate bilinear form $\tilde{g}_X$ on $T_X(T(M))$ by

$$\tilde{g}_X(\tilde{X},\tilde{Y}) = \tilde{f}(\tilde{\phi}(X)^{-1}\tilde{X}, \tilde{\phi}(X)^{-1}\tilde{Y}) \tag{7. 8}$$

for $\tilde{X},\tilde{Y} \in T_X(T(M))$. We can easily see that $\tilde{g}$ is the (globally defined) canonical quasi-Riemannian metric on $T(M)$ induced by the prolongation $\tilde{P}$ of P given in Theorem 6. 11. We shall now study the relations between g and $\tilde{g}$. Let $(\phi^i_j(x))$ be the matrix defined by $\phi(x)\cdot e_j = \sum \phi^i_j(x)\left(\frac{\partial}{\partial x^i}\right)_x$ for $x \in U$, where $e_j = (\delta^1_j, \cdots, \delta^n_j) \in R^n$. Let $\{x^1, \cdots, x^n, \dot{x}^1, \cdots, \dot{x}^n\}$ be the induced local coordinate system on $T(U)$. We now define the functions $g_{ij}(x)$ on U and $\tilde{g}_{\nu\mu}(X)$ on $T(U)$ for $i, j = 1, 2, \cdots, n$ and for $\nu, \mu = 1, 2, \cdots, 2n$ as follows

$$\left\{\begin{aligned} g_{ij}(x) &= g_x\left(\left(\frac{\partial}{\partial x^i}\right)_x, \left(\frac{\partial}{\partial x^j}\right)_x\right), \\ \tilde{g}_{\nu\mu}(X) &= \tilde{g}_X\left(\left(\frac{\partial}{\partial x^\nu}\right)_X, \left(\frac{\partial}{\partial x^\mu}\right)_X\right), \end{aligned}\right. \tag{7. 9}$$

for $x \in U$ and $X \in T(U)$, where we have put $x^{n+i} = \dot{x}^i$ for $i = 1, 2, \cdots, n$. We define $A = (a^j_k) \in GL(n, R)$ by $f(e_j, e_k) = a^j_k$ for $j, k = 1, 2, \cdots, n$. Since $g_x(\phi(x)e_j,\ \phi(x)e_k) = f(e_j, e_k) = a^j_k$, (7. 9) implies

$$\sum_{i,l=1}^{n} \phi^i_j(x)\phi^l_k(x)g_{i,l}(x) = a^j_k \tag{7. 10}$$

for $j, k = 1, 2, \cdots, n$. By virtue of (6. 1) we have

$$\tilde{f}\left(\sum b_i\left(\frac{\partial}{\partial u^i}\right)_u,\ \sum c_j\left(\frac{\partial}{\partial u^j}\right)_v\right) = \sum (a^i_j b_i v^j + c_j a^i_j u^i) \tag{7. 11}$$

for $b_i, c_j \in R$.

On the other hand, by Proposition 5.3 we have

$$(7.12)\qquad \tilde{\phi}(X)\Big(\sum b_i\Big(\frac{\partial}{\partial u^i}\Big)_u\Big) = \sum_{i,j} u_j\Big(\phi^i_j(x)\Big(\frac{\partial}{\partial x^i}\Big)_X + \frac{\partial\phi^i_j}{\partial x^k}\dot{x}^k\Big(\frac{\partial}{\partial \dot{x}^i}\Big)_X\Big)$$
$$+ \sum b_j\phi^i_j(x)\Big(\frac{\partial}{\partial \dot{x}^i}\Big)_X$$

for $X = \sum \dot{x}^i\Big(\frac{\partial}{\partial x^i}\Big)_x \in T(U)$. Since $\tilde{g}_X(\tilde{\phi}(X)\cdot\tilde{u},\ \tilde{\phi}(X)\tilde{v}) = \tilde{f}(\tilde{u},\tilde{v})$ for $\tilde{u},\tilde{v}\in T(R^n)$, the formulas (7.9) and (7.11) imply

$$(7.13)\qquad \tilde{g}_X\Big(\tilde{\phi}(X)\cdot\sum b_i\Big(\frac{\partial}{\partial u^i}\Big)_u,\ \tilde{\phi}(X)\cdot\sum c_i\Big(\frac{\partial}{\partial u^i}\Big)_v\Big)$$
$$= \sum a^i_j(b_iv^j + c_ju^i).$$

Substituting $b_i=c_i=0$, $u^i=\delta^j_i$ and $v^i=\delta^k_i$ in (7.13) for fixed $j,k=1,2,\cdots,n$, we have:

$$\tilde{g}_X\Big(\phi^i_j(x)\Big(\frac{\partial}{\partial x^i}\Big)_X + \frac{\partial\phi^i_j}{\partial x^l}\dot{x}^l\Big(\frac{\partial}{\partial \dot{x}^i}\Big)_X,\ \phi^i_k(x)\Big(\frac{\partial}{\partial x^i}\Big)_X + \frac{\partial\phi^i_k}{\partial x^l}\dot{x}^l\Big(\frac{\partial}{\partial \dot{x}^l}\Big)_X\Big)$$
$$=0$$

and hence we obtain, by (7.9), the following

$$(7.14)\qquad \phi^i_j(x)\phi^s_k(x)\tilde{g}_{i,s}(X) + \phi^i_j(x)\frac{\partial\phi^s_k}{\partial x^l}\dot{x}^l\tilde{g}_{i,s+n}(X)$$
$$+\frac{\partial\phi^s_j}{\partial x^l}\dot{x}^l\phi^i_k(x)\,\tilde{g}_{s+n,i}(X) + \frac{\partial\phi^i_j}{\partial x^p}\dot{x}^p\,\frac{\partial\phi^s_k}{\partial x^l}\dot{x}^l\,\tilde{g}_{i+n,s+n}(X)$$
$$=0.$$

Substituing $b_i=0$, $v=0$, $c_i=\delta^k_i$ and $u^i=\delta^j_i$ in (7.13) we have

$$\tilde{g}_X\Big(\phi^i_j(x)\Big(\frac{\partial}{\partial x^i}\Big)_X + \frac{\partial\phi^i_j}{\partial x^l}\dot{x}^l\Big(\frac{\partial}{\partial \dot{x}^i}\Big)_X,\ \phi^i_k(x)\Big(\frac{\partial}{\partial \dot{x}^i}\Big)_X\Big) = a^j_k$$

and hence we obtain the following

$$(7.15)\qquad \phi^i_j(x)\phi^s_k(x)\tilde{g}_{i,s+n}(X) + \frac{\partial\phi^i_j}{\partial x^l}\dot{x}\phi^s_k(x)\tilde{g}_{i+n,s+n}(X) = a^j_k.$$

Similarly, substituting $u=v=0$, $b_i=\delta^k_i$ and $c_i=\delta^j_i$ in (7.13), we have $\tilde{g}_X\Big(\phi^i_j(x)\Big(\frac{\partial}{\partial \dot{x}^i}\Big)_X,\ \phi^i_k(x)\Big(\frac{\partial}{\partial \dot{x}^i}\Big)_X\Big)=0$ and we obtain

$$(7.16)\qquad \phi^i_j(x)\phi^s_k(x)\tilde{g}_{i+n,s+n}(X) = 0.$$

Since $\det(\phi^i_j(x)) \neq 0$, we see that $\tilde{g}_{i+n,s+n}(X) = 0$, which we insert in (7. 15) and we obtain the following

$$\phi^i_j(x)\phi^s_k(x)\tilde{g}_{i,s+n}(X) = a^j_k. \tag{7. 17}$$

Then (7. 10) and (7. 17) imply $\tilde{g}_{i,s+n}(X) = g_{i,s}(x)$. Therefore we get the following equality by (7. 14)

$$\phi^i_j\phi^s_k\tilde{g}_{i,s}(X) + \phi^i_j \frac{\partial \phi^s_k}{\partial x^l} \dot{x}^l g_{i,s}(x) + \frac{\partial \phi^s_j}{\partial x^l} \dot{x}^l \phi^i_k g_{s,i}(x) = 0,$$

and hence we obtain

$$\phi^i_j\phi^s_k\tilde{g}_{i,s}(X) + \left(\phi^s_j \frac{\partial \phi^i_k}{\partial x^l} \dot{x}^l + \frac{\partial \phi^s_j}{\partial x^l} \dot{x}^l \cdot \phi^i_k\right) g_{i,s} = 0. \tag{7. 18}$$

On the other hand, by differentiating the both hand sides of (7. 10) with respect to x^k, multiplying $\dot{x}^k$ and summing up for $k = 1, 2, \cdots, n$, we obtain:

$$\left(\frac{\partial \phi^i_j}{\partial x^p} \dot{x}^p \phi^s_k + \phi^i_j \frac{\partial \phi^s_k}{\partial x^p} \dot{x}^p\right) g_{i,s} + \phi^i_j \phi^s_k \frac{\partial g_{is}}{\partial x^p} \dot{x}^p = 0. \tag{7. 19}$$

Hence, (7. 18) and (7. 19) imply $\phi^i_j\phi^s_k\tilde{g}_{i,s} = \phi^i_j\phi^s_k \dfrac{\partial g_{i,s}}{\partial x^p} \dot{x}^p$, and thus we obtain

$$\tilde{g}_{i,s} = \frac{\partial g_{i,s}}{\partial x^p} \dot{x}^p. \tag{7. 20}$$

Finally we obtain the expression of $\tilde{g}$ with respect to the induced local coordinate system as follows

$$\tilde{g} \longleftrightarrow \begin{pmatrix} \left(\dfrac{\partial g_{ij}}{\partial x^k} \dot{x}^k\right) & (g_{ij}) \\ (g_{ij}) & 0 \end{pmatrix},$$

which is exactly the complete lift of the pseudo-Riemannian metric g as written in [9] p. 203. Thus we have proved the following:

THEOREM 7. 3. *The canonical pseudo-Riemannian metric on $T(M)$ induced by the prolongation of $0(k, n-k)$-structure P on a manifold M is the complete lift of the pseudo-Riemannian metric g associated with P.*

In the same way as in the case of $0(k, n-k)$-structure we can prove the analogous fact on a symplectic structure in the following proposition whose proof will be omitted.

PROPOSITION 7. 4. *The symplectic form on $T(M)$ associated with the prolongation of a $S_p(m)$-structure on a manifold M is the complete lift of the symplectic form associated with the structure P.*

(III) Let $P(M,\pi,0(n))$ be an $0(n)$-structure on M, $0(n)$ being the orthogonal group of degree n. Let Γ be a connection on the principal fibre bundle P. (We denote by the same letter Γ the connection on $F(M)$ induced natually by the connection Γ on P, (cf. [4] for the general theory of connections). Then Γ is nothing but a cross section of the fibre bundle $T(P)/0(n)$ over $T(M)$ (cf. Remark 3. 7). Then the bundle homomorphism $j_M : T(F(M)) \longrightarrow F(T(M))$ induces a cross section Γ_0 of $\tilde{P}/j_n(0(n))$ over $T(M)$. Hence the cross section Γ_0 reduces the structure group $\tilde{G}$ of $\tilde{P}$ to $j_n(0(n)) = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| a \in 0(n)\right\}$, and we obtain a $j_n(0(n))$-subbundle $\tilde{P}_0$ of $\tilde{P}$. Now, since $j_n(0(n))$ is included in $0(2n)$, $\tilde{P}_0$ induces canonically an $0(2n)$-structure $\tilde{P}_1$, such that $\tilde{P}_0$ is a subbundle of $\tilde{P}_1$. We denote by g (resp. $\tilde{g}$) the Riemmannian metric on M (resp. on $T(M)$) associated with the $0(n)$-structure P (resp. $0(2n)$-structure $\tilde{P}_1$). We shall study the relations between g, Γ and $\tilde{g}$.

LEMMA 7. 5. *Let ϕ be a cross section of P over an open set U in M, then we can find a cross section $\tilde{\phi}_\Gamma$ of $\tilde{P}$ over $T(U)$ such that $p \circ \tilde{\phi}_\Gamma = \Gamma_0$ on $T(U)$, where $p : \tilde{P} \longrightarrow \tilde{P}/j_n(0(n))$ is the natural projection.*

Proof. Take any tangent vector $X \in T_x(U)$, then there is a unique horizontal tangent vector $\tilde{X}$ of P at $\phi(x)$ with respect to the connection Γ such that $(T\pi)\cdot\tilde{X} = X$, π being the projection of P onto M. We denote this $\tilde{X}$ by $\phi_\Gamma(X)$. Then ϕ_Γ is a map of $T(U)$ into $T(P)$ such that $(T\pi)\circ\phi_\Gamma = 1_{T(U)}$. Now, put $\tilde{\phi}_\Gamma = j_M \circ \phi_\Gamma$, then it is easy to see that the map $\tilde{\phi}_\Gamma$ satisfies the required conditions.

LEMMA 7. 6. *Let ω be the map of $T(P)$ into $T_e(G)$ such that $h(\tilde{X}) = \tilde{X}\cdot\omega(\tilde{X})$ is the horizontal part of any $\tilde{X} \in T(P)$ with respect to the concetion Γ. (We remark that ω is essentially the connection form of Γ, cf* [4]). *Define the map $\tilde{\omega} : \tilde{P} \longrightarrow T_e(G)$ by*

$$\tilde{\omega}(X') = j_n(\omega(j_M^{-1}(X')))$$

for $X' \in \tilde{P}$. *Then, we have*

$$\tilde{\phi}_\Gamma(X) = \tilde{\phi}(X)\cdot\tilde{\omega}(\tilde{\phi}(X)) \tag{7. 21}$$

for $X \in T(U)$, *where* $\tilde{\phi} = j_M \circ T\phi$ *as in Proposition* 5. 3.

Proof. First, we shall prove that $\phi_\Gamma(X) = T\phi(X)\cdot\omega(T\phi(X))$ for $X \in T(U)$. In fact, if $X \in T_x(U)$ then $T\phi(X)\cdot\omega(T\phi(X))$ is a horizontal tangent vector at $\phi(x)$ and $T\pi(T\phi(X)\cdot\omega(T\phi(X)))=T\pi T\phi(X)=X$, and hence $\phi_\Gamma(X)=T\phi(X)\cdot\omega(T\phi(X))$ holds. Now we calculate as follows:

$$\begin{aligned}\tilde{\phi}_\Gamma(X) &= j_M\circ\phi_\Gamma(X) = j_M(T\phi(X)\cdot\omega(T\phi(X)))\\ &= j_M(T\phi(X))\cdot j_n(\omega T\phi(X)) = \tilde{\phi}(X)\cdot j_n\omega j_M^{-1}j_M T\phi(X)\\ &= \tilde{\phi}(X)\cdot(\tilde{\omega}\circ\tilde{\phi}(X)).\end{aligned}$$

Thus Lemma 7. 6 is proved.

LEMMA 7. 7. *Let* $\phi : U \longrightarrow P$ *be a cross section of* P *over a coordinate neighborhood* U *in* M *with local coordinate system* $\{x^1, \cdots, x^n\}$. *Let* $(\phi^i_j(x)) \in GL(n)$ *be the expression of* ϕ *with respect to the basis* $\left\{\left(\frac{\partial}{\partial x^1}\right)_x, \cdots, \left(\frac{\partial}{\partial x^n}\right)_x\right\}$ *of* $T_x(M)$, *i.e.* $\phi(x)\cdot e_j = \sum \phi^i_j(x)\left(\frac{\partial}{\partial x^i}\right)_x$. *Then we have the following*

$$\omega\circ T\phi\left(\left(\frac{\partial}{\partial x^k}\right)_x\right) = -\frac{\partial\phi^i_j}{\partial x^k}\rho^l_i(x)\left(\frac{\partial}{\partial y_j}\right)_e + \omega\left(\left(\frac{\partial}{\partial x^k}\right)_{\phi(x)}\right), \tag{7. 22}$$

where $(\rho^i_j(x)) = (\phi^i_j(x))^{-1}$ *and* $\{\cdots, y^i_j, \cdots\}$ *is the natural coordinates in* $GL(n, R)$.

Proof. Let $\{x^1, \cdots, x^n, y^i_j\}$ be the induced local coordinate system on $F(U)$. Then we have

$$T\phi\left(\left(\frac{\partial}{\partial x^k}\right)_x\right) = \frac{\partial\phi^i_j}{\partial x^k}\left(\frac{\partial}{\partial y^i_j}\right)_{\phi(x)} + \left(\frac{\partial}{\partial x^k}\right)_{\phi(x)}. \tag{7. 23}$$

Now, it is easily seen that

$$\omega\left(\left(\frac{\partial}{\partial y^i_j}\right)_{\tilde{a}}\right) = -TL_{a^{-1}}\left(\left(\frac{\partial}{\partial y^i_j}\right)_a\right) \tag{7. 24}$$

for $\tilde{a} = \left(\cdots, \sum a^i_j\left(\frac{\partial}{\partial x^i}\right)_x, \cdots\right) \in F(U)$ and $a = (a^i_j)$, where $L_a : GL(n) \longrightarrow GL(n)$ is the left translation of $GL(n)$ with respect to the element $a \in GL(n)$, i.e. $L_a(y) = a\cdot y$ for $y \in GL(n)$. Using (7. 23) and (7. 24), we obtain (7. 22).

Thus Lemma 7. 7 is proved.

DEFINITION 7. 8. We define the function $\Gamma^i_{kj} : U \longrightarrow R$ for $i, j, k = 1, 2, \cdots, n$ by

$$-\omega\Big(\Big(\frac{\partial}{\partial x^k}\Big)_{\tilde{x}}\Big) = \sum \Gamma^i_{kj}(x)\Big(\frac{\partial}{\partial y^i_j}\Big)_e, \tag{7. 25}$$

where $\tilde{x} = \Big(\cdots, \Big(\frac{\partial}{\partial x^i}\Big)_x, \cdots\Big) \in F(U)$.

The following Lemma is well known as a property of the form ω.

LEMMA 7. 9.

$$\omega\Big(\Big(\frac{\partial}{\partial x^k}\Big)_{\phi(x)}\Big) = ad((\phi^i_j(x))^{-1})\omega\Big(\Big(\frac{\partial}{\partial x^k}\Big)_{\tilde{x}}\Big).$$

Now, we define the Riemannian metrics g (and $\tilde{g}$) on M (resp. on $T(M)$) by the following:

$$g_x(X, Y) = (\phi(x)^{-1}X, \phi(x)^{-1}Y)_{R^n} \text{ for } X, Y \in T_x(M), \tag{7. 26}$$

$$\tilde{g}_X(\tilde{X}, \tilde{Y}) = (\tilde{\phi}_\Gamma(X)^{-1}\cdot\tilde{X}, \tilde{\phi}_\Gamma(X)^{-1}\cdot\tilde{Y})_{R^{2n}} \text{ for } \tilde{X}, \tilde{Y} \in T_X(T(M)), \tag{7. 27}$$

where $(\ ,\ )_{R^n}$ denotes the usual inner product in R^n. These metrics g and $\tilde{g}$ are independent on the choice of the local cross section ϕ. In fact g is the Riemannian metric associated with the $O(n)$-structure P and $\tilde{g}$ is the Riemannian metric canonically induced by g and the connection Γ on P. We define the functions $g_{ij} : U \longrightarrow R$ and $\tilde{g}_{\nu\mu} : T(U) \longrightarrow R$ for $i, j = 1, 2, \cdots, n$ and $\nu, \mu = 1, 2, \cdots, 2n$ in the same way as (7. 9). By using Lemma 7. 6 and Proposition 5. 3, we can calculate, for $X = \sum \dot{x}^k\Big(\frac{\partial}{\partial x^k}\Big)_x$, as follows:

$$\tilde{\phi}_\Gamma(X)\Big(\sum c_i\Big(\frac{\partial}{\partial u^i}\Big)_u\Big) = \tilde{\phi}(X)\begin{pmatrix} E & 0 \\ \omega T\phi(X) & E \end{pmatrix}\cdot\Big(\sum c_i\Big(\frac{\partial}{\partial u^i}\Big)_u\Big)$$

$$= \Big(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial \dot{x}^1}, \cdots, \frac{\partial}{\partial \dot{x}^n}\Big)\begin{pmatrix} (\phi^i_j) & 0 \\ \Big(\frac{\partial \phi^i_j}{\partial x^k}\dot{x}^k\Big) & (\phi^i_j) \end{pmatrix}\cdot\begin{pmatrix} E & 0 \\ \omega T\phi(X) & E \end{pmatrix}\begin{pmatrix} u^1 \\ \vdots \\ u^n \\ c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

$$= \left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial \dot{x}^n}\right) \begin{pmatrix} \phi & 0 \\ \dfrac{\partial \phi}{\partial x} \dot{x} + \phi \cdot \omega T \phi(X) & \phi \end{pmatrix} \begin{pmatrix} u^1 \\ \vdots \\ u^n \\ c_1 \\ \vdots \\ c_n \end{pmatrix}$$

where we have omitted the indices in the matrix of the last term. We shall put

$$\psi^i_j = \frac{\partial \phi^i_j}{\partial x^k} \dot{x}^k + \phi^i_l (\omega T \phi(X))^l_j \, . \tag{7. 28}$$

Thus we have proved the following

LEMMA 7. 10.

$$\tilde{\phi}_{\Gamma}(X) \left(\sum c_i \left(\frac{\partial}{\partial u^i}\right)_u\right) = \sum u^j \left(\phi^i_j \left(\frac{\partial}{\partial x^i}\right)_X + \psi^i_j \left(\frac{\partial}{\partial \dot{x}^i}\right)_X\right) + \sum c_i \phi^i_j \left(\frac{\partial}{\partial \dot{x}^i}\right)_X \tag{7. 29}$$

for $X = \sum \dot{x}^i \left(\frac{\partial}{\partial x^i}\right)_x$.

Using (7. 26) ~ (7. 28), we obtain the following equalities in the same way as in the proof of (7. 14) ~ (7. 16):

$$\phi^i_j \phi^s_k \tilde{g}_{i,s} + \phi^i_j \psi^s_k \tilde{g}_{i,s+n} + \psi^s_j \phi^i_k \tilde{g}_{s+n,i} + \psi^i_j \psi^s_k \tilde{g}_{i+n,s+n} = \delta^k_j \, , \tag{7. 30}$$

$$\phi^i_j \phi^s_k \tilde{g}_{i,s+n} + \psi^i_j \phi^s_k \tilde{g}_{i+n,s+n} = 0 \, , \tag{7. 31}$$

$$\phi^i_j \phi^s_k \tilde{g}_{i+n,s+n} = \delta^k_j \, , \tag{7. 32}$$

for $j, k = 1, 2, \cdots, n$, the summation notations with respect to the repeated indices being omitted.

On the other hand, since $g_x(\phi(x)e_i, \phi(x)e_j) = (e_i, e_j)_{R^n} = \delta^j_i$, we have the following

$$\phi^k_j \phi^l_j g_{k,l}(x) = \delta^j_i \, . \tag{7. 33}$$

Now, (7. 32) and (7. 33) imply

$$\tilde{g}_{i+n,j+n}(X) = g_{i,j}(x) \, . \tag{7. 34}$$

By (7. 28), (7. 22) and (7. 25), we see that

$$\psi^i_j = \frac{\partial \phi^i_j}{\partial x^k}\dot{x}^k + \phi^i_l\left(-\frac{\partial \phi^s_j}{\partial x^k}\dot{x}^k\rho^l_s - \rho^l_p\Gamma^p_{ks}\phi^s_j\dot{x}^k\right)$$

$$= -\phi^i_l\rho^l_p\Gamma^p_{ks}\phi^s_j\dot{x}^k = -\Gamma^i_{ks}\phi^s_j\dot{x}^k.$$

Substituting this in (7. 31) and using (7. 34) we get:

$$\phi^i_j\phi^s_k\tilde{g}_{i,s+n} = \Gamma^i_{ql}\phi^l_j\dot{x}^q\phi^s_k g_{i,s}$$

which implies

$$\tilde{g}_{i,j+n} = \Gamma^l_{ki}\dot{x}^k g_{l,j}. \tag{7. 35}$$

Combining (7. 30), (7. 31), (7. 33) and (7. 35), we have

$$\begin{aligned}\phi^i_j\phi^s_k\tilde{g}_{i,s} &+ \phi^q_j(-\Gamma^s_{lp}\phi^p_k\dot{x}^l)\cdot\Gamma^i_{mq}\dot{x}^m g_{i,s}\\ &+ (-\Gamma^s_{lp}\phi^p_j\dot{x}^l)\phi^m_k\cdot\Gamma^i_{qm}\dot{x}^q g_{i,s}\\ &+ (-\Gamma^i_{mq}\phi^q_j\dot{x}^m)\cdot(-\Gamma^s_{lp}\phi^p_k\dot{x}^l)g_{i,s}\\ &= \phi^i_j\phi^s_k g_{i,s},\end{aligned}$$

where the third and fourth terms of the left hand side cancel each other and hence we obtain finally:

$$\phi^i_j\phi^s_k\tilde{g}_{i,s} = \phi^i_j\phi^s_k g_{i,s} + \phi^q_j\phi^p_k\Gamma^s_{lp}\Gamma^i_{mq}\dot{x}^l\dot{x}^m g_{i,s},$$

or

$$\tilde{g}_{i,j} = g_{i,j} + \Gamma^k_{pi}\Gamma^l_{qj}\dot{x}^p\dot{x}^q g_{k,l}. \tag{7. 36}$$

THEOREM 7. 11. *If the connection Γ on P is the Riemannian connection induced by the Riemannian metric g, then the induced metric $\tilde{g}$ on $T(M)$ is exactly the same Riemannian metric studied by Sasaki* [5].

In fact, Sasaki ([5] p. 342, (3. 1) ~ (3. 3)) introduced his Riemannian metric g on $T(M)$ by the formulas (7. 34) ~ (7. 36).

§ 8. Final remarks.

Let $P(M,\pi,G)$ be a G-structure on a manifold M, $\tilde{P}(T(M),\tilde{\pi},\tilde{G})$ the prolongation of P to $T(M)$ and Γ be a connection on the principal fibre bundle P. Then, as stated in Remark 3. 7, the structure group $\tilde{G}$ of $\tilde{P}$ is reduced to the subgroup $G_0 = \left\{\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix}\middle| a \in G\right\}$. We denote by $\tilde{P}_0$ the associated G_0-subbundle of $\tilde{P}$. Let f be a diffeomorphism of M onto itself. Suppose f is an automorphism of the G-structure P. Then by Corollary 4. 4, the map Tf is an automorphism of the prolongation $\tilde{P}$. We can prove the following proposition whose proof will be omitted.

PROPOSITION 8. 1. *The tangential map Tf of f is an automorphism of the G_0-structure $\tilde{P}_0$ if and only if f is a Γ-transformation, i.e. f preserves the connection Γ.*

DEFINITION 8. 2. Let $\mathfrak{h}$ be a subspace of the vector space Hom (V,V) of all linear endomorphisms of a vector space V. We define the map $\partial_{\mathfrak{h}}:$ Hom $(V,\mathfrak{h}) \longrightarrow$ Hom $(V \wedge V, V)$ by the following:

$$(\partial_{\mathfrak{h}}\ S)(u,v) = S(u)(v) - S(v)(u),$$

for $u,v \in V$, $S \in$ Hom $(V,\mathfrak{h})$. We denote by $\mathfrak{h}^{(1)}$ the kernel of $\partial_{\mathfrak{h}}$.

Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V) =$ Hom (V,V). We shall call $\mathfrak{g}$ to be of type 1 if $\mathfrak{g}^{(1)} = (0)$.

PROPOSITION 8. 3. *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$ and $\mathfrak{g}_0$ be the Lie subalgebra of $\mathfrak{gl}(V \oplus V)$ consisting of all $A \times A$ for $A \in \mathfrak{g}$. Then $\mathfrak{g}_0$ is of type* 1.

Proof. Take an element $\tilde{S} \in$ Ker $\partial_{\mathfrak{g}_0}$. Then we have

$$\tilde{S}(u \oplus v)(w \oplus x) = \tilde{S}(w \oplus x)(u \oplus v) \tag{8. 1}$$

for any $u,v,w,x \in V$. We define $\tilde{S}_i \in$ Hom $(V,\mathfrak{g}_0)$ by $\tilde{S}_1(u) = \tilde{S}(u \oplus 0)$, $\tilde{S}_2(v) = \tilde{S}(0 \oplus v)$ for $u,v \in V$. Then, (8. 1) can be written as follows:

$$\begin{aligned}\tilde{S}_1(u)(w \oplus t) + \tilde{S}_2(v)(w \oplus t) \\ = \tilde{S}_1(w)(v \oplus v) + \tilde{S}_2(t)(u \oplus v),\end{aligned} \tag{8. 2}$$

for $u,v,w,x \in V$. Now, we define $S_i \in$ Hom $(V,\mathfrak{g})$ by $\tilde{S}_i(u) = S_i(u) \times S_i(u)$ for $u \in V$. Then, from (8. 2) we obtain the following two equations:

$$S_1(u)w + S_2(v)w = S_1(w)u + S_2(t)u \tag{8. 3}$$

$$S_1(u)t + S_2(v)t = S_1(w)v + S_2(t)v \tag{8. 4}$$

for any $u,v,w,x \in V$. If we put $v = t = 0$ in (8. 3), we get $S_1(u)w = S_1(w)u$, which we subtract from (8. 3). Then we get

$$S_2(v)w = S_2(t)u \tag{8. 5}$$

for any $t,u,v,w \in V$. If we put $w = 0$ in (8. 5), we have $S_2(t)u = 0$ for $t,u \in V$ and hence $S_2 = 0$. Therefore, we get $S_1 = 0$ from (8. 4) and hence $\tilde{S} = 0$, which proves Proposition 8. 3.

REMARK 8. 4. There is, in general, no canonical G_0-structure on $T(M)$, even if M has G-structure. In fact, if $T(M)$ has a canioncal G_0-structure for $G = GL(n, R)$, $n = \dim M$, then the group of all diffeomorphisms of M onto itself would be a Lie group of finite dimension, since the Lie algebra of G_0 is of type 1 by Proposition 8. 3. This will be absurd.

REMARK 8. 5. We shall discuss the relation between the tangent connection $T(\Gamma)$ of an affine connection Γ [4] and the complete lift Γ^C of Γ [9] in a forthcoming paper.

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