HIGHER DERIVATIONS AND CENTRAL SIMPLE ALGEBRAS

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(Dedicated to the memory of Tadasi Nakayama)

Introduction. Let K be a commutative ring, A a K-algebra, and B a K-subalgebra of A. The object of this paper is to prove some results on higher derivations (in the sense of Jacobson [4]) of B into A. In § 1 we introduce a notion of equivalence among higher derivations. With this notion of equivalence, we prove in § 2 (Theorem 1) that the equivalence classes of higher K-derivations of B into A are in one-one correspondence with the isomorphism classes of certain filtered $B\otimes_K A^\circ$-modules, where A° denotes the opposite algebra of A. In § 3 we give a cohomological criterion for the extendability of a higher derivation of a commutative ring to a crossed product. We use this result in § 4 to show (Theorem 2) that if A is central simple over K and B is semi-simple, then any higher derivation of B into A which maps K into K can be extended to a higher derivation of A. This result is a generalization of a theorem of Jacobson-Hochschild ([2], Theorem 6) on extendability of derivations.

§ 1 Generalities on higher derivations.

Let B be a subring of a ring A. We recall that a *higher derivation of rank* n of B into A is a sequence of additive maps $\delta=(d_0=1, d_1, \cdots, d_n)$ of B into A such that

$$d_i(bb') = \sum_{0\leqslant j\leqslant i} d_j(b)d_{i-j}(b'),$$

$b, b'\in B$, $0\leqslant i\leqslant n$. If A is an algebra over a commutative ring K and B a K-subalgebra of A, then δ is called a *higher K-derivation* if the maps d_i are K-linear, i.e. if the maps d_i vanish on K for $i\geqslant 1$. The following statement is easily checked:

(1. 1) If $(d_0=1, d_1, \cdots, d_{n-1}, d_n)$ and $(d_0=1, d_1, \cdots, d_{n-1}, d'_n)$

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are higher derivations of B into A, then $d_n - d'_n$ is a derivation.

For any ring Λ, let $T_n(\Lambda)$ be the ring $\Lambda[X]/(X^{n+1})$. We shall denote the image of X in $T_n(\Lambda)$ by x. Let $\eta_\Lambda : T_n(\Lambda) \to \Lambda$ be the ring epimorphism defined by $\eta_\Lambda(\lambda_0 + \lambda_1 x + \cdots + \lambda_n x^n) = \lambda_0$. Since $\ker \eta_\Lambda$ is nilpotent, $1+\ker \eta_\Lambda$ is a subgroup of the group of units of $T_n(\Lambda)$. We shall denote this subgroup by $U_n(\Lambda)$.

With A and B as above, if $\delta : B \to A$ is a higher derivation, then the map $\alpha_\delta : B \to T_n(A)$ given by $\alpha_\delta(b) = \sum_{0 \leqslant i \leqslant n} d_i(b)x^i$ is a *section* of η_A on B, i.e., α_δ is a ring homomorphism such that $\eta_A \circ \alpha_\delta = \text{identity}$. Conversely, let α be a section of η_A on B. If $\alpha(b) = \sum_{0 \leqslant i \leqslant n} d_i(b)x^i$, then $(d_0 = 1, d_1, \cdots, d_n)$ is a higher derivation of B into A.

If $\delta, \delta' : B \to A$ are two higher derivations, we say that they are *equivalent*, if there exists an element $u \in U_n(A)$ such that $\alpha_{\delta'} = \text{int } u \circ \alpha_\delta$, where int u denotes the inner automorphism of $T_n(A)$ given by u. Clearly, this is an equivalence relation. More explicitly, δ and δ' are equivalent if and only if there exist elements $u_0 = 1, u_1, \cdots, u_n \in A$ such that

$$\sum_{0 \leqslant j \leqslant i} u_j d_{i-j}(b) = \sum_{0 \leqslant j \leqslant i} d'_{i-j}(b) u_j, \qquad (*)$$

for $b \in B$ and $0 \leqslant i \leqslant n$. A higher derivation is called *inner* if it is equivalent to the higher derivation $(d_0 = 1, d_1, \cdots, d_n)$, where $d_i = 0$ for $i \geqslant 1$.

§ 2. Higher derivations and filtered modules

Let K be a commutative ring, A a K-algebra, and B a K-subalgebra of A. For any positive integer n, we denote by $\bar{A}(n)$, the graded $B \otimes_K A^\circ$-module $\sum_{0 \leqslant i \leqslant n} \bar{A}_i$, where $\bar{A}_i$ is the $B \otimes_K A^\circ$-module A. Let $\bar{e}_i$ denote the element 1 of $\bar{A}_i$. Let $\bar{\theta}$ denote the graded endomorphism of degree -1 of $\bar{A}(n)$ defined by $\bar{\theta}_i(\bar{e}_i) = \bar{e}_{i-1}$ for $i > 0$, and $\bar{\theta}_0 = 0$.

We consider the class $\mathscr{C}$ of triples (M, ψ, θ), where M is a $B \otimes_K A^\circ$-module with a filtration $0 \subset M_0 \subset M_1 \subset \cdots \subset M_n = M$, θ a $B \otimes_K A^\circ$-endomorphism of degree -1 of M and $\psi : E^\circ(M) \to \bar{A}(n)$ an isomorphism of graded $B \otimes_K A^\circ$-modules, where $E^\circ(M)$ denotes the associated graded module of M, such that the diagram

$$\begin{array}{ccc} E^\circ(M) & \xrightarrow{E^\circ(\theta)} & E^\circ(M) \\ \psi\downarrow & & \downarrow\psi \\ \bar{A}(n) & \xrightarrow[\bar{\theta}]{} & \bar{A}(n) \end{array}$$

is commutative. With the natural filtration on $\bar{A}(n)$, the triple $(\bar{A}(n), 1_{\bar{A}(n)}, \bar{\theta})$ is clearly a member of $\mathscr{C}$. We define a *morphism* $(M, \psi, \theta) \to (M', \psi', \theta')$ in $\mathscr{C}$ to be a map of filtered $B \otimes_K A^\circ$-modules $M \to M'$ which is compatible with ψ, ψ' and θ, θ'.

Thus $\mathscr{C}$ becomes a category. Clearly, every morphism in $\mathscr{C}$ is an isomorphism.

Let $\delta = (d_0 = 1, d_1, \cdots, d_n)$ be a higher K-derivation of rank n of B into A. On the free right A-module $A_\delta = \sum_{0 \leqslant i \leqslant n} e_i A$, with basis (e_i), we define a left B-module structure by setting $b(e_i a) = (\sum_{0 \leqslant j \leqslant i} e_j d_{i-j} b) a$ for $0 \leqslant i \leqslant n$, $b \in B$, $a \in A$. This makes A_δ a $B \otimes_K A^\circ$-module. We define a filtration $0 \subset (A_\delta)_0 \subset (A_\delta)_1 \subset \cdots \subset (A_\delta)_n = A_\delta$ by taking $(A_\delta)_i$ to be the $B \otimes_K A^\circ$-submodule of A_δ generated by $e_0, \cdots, e_i$. We also define a $B \otimes_K A^\circ$-endomorphism θ_δ of degree -1 of the filtered module A_δ by setting $\theta_\delta(e_0) = 0$ and $\theta_\delta(e_i) = e_{i-1}$ for $i \geqslant 1$. The map $(A_\delta)_i \to \bar{A}_i$ which sends $\sum_{0 \leqslant j \leqslant i} e_j a_j$ to $\bar{e}_i a_i$ is $B \otimes_K A^\circ$-linear. This map is an isomorphism for $i = 0$ and has $(A_\delta)_{i-1}$ as its kernel for $i \geqslant 1$. We thus get an isomorphism

$$\psi_\delta : E^\circ(A_\delta) \to \bar{A}(n)$$

of graded $B \otimes_K A^\circ$-modules. Clearly, $(A_\delta, \psi_\delta, \theta_\delta)$ is an object of $\mathscr{C}$.

Now let $\delta = (d_0 = 1, d_1, \cdots, d_n)$ and $\delta' = (d'_0 = 1, d'_1, \cdots, d'_n)$ be two equivalent higher K-derivations of B into A. There exist elements $u_0 = 1, u_1, \cdots, u_n \in A$ satisfying the condition $(*)$ of § 1. The isomorphism $A_\delta \to A_{\delta'}$ of right A-modules which sends e_i to $\sum_{0 \leqslant j \leqslant i} e'_j u_{i-j}$ is easily verified to be left B-linear and actually gives an isomorphism in $\mathscr{C}$ of $(A_\delta, \psi_\delta, \theta_\delta)$ onto $(A_{\delta'}, \psi_{\delta'}, \theta_{\delta'})$. Thus, equivalent higher K-derivations of B into A give rise to isomorphic objects in $\mathscr{C}$.

Consider now any object $(M, \psi, \theta) \in \mathscr{C}$. We then have for $1 \leqslant i \leqslant n$, the following commutative diagrams with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{i-2} & \longrightarrow & M_{i-1} & \xrightarrow{\psi_{i-1}} & \bar{A}_{i-1} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle\theta_{i-1}} & & \uparrow{\scriptstyle\theta_i} & & \uparrow{\scriptstyle\bar{\theta}_i} & & \\ 0 & \longrightarrow & M_{i-1} & \longrightarrow & M_i & \xrightarrow[\psi_i]{} & \bar{A} & \longrightarrow & 0 \end{array}$$

where $M_{-1}=0$. Let $s_n : \bar{A}_n \to M_n$ be a right A-linear map such that $\psi_n \circ s_n =$ identity. The map s_n induces right A-linear maps $s_i (0 \leqslant i < n)$ such that $\theta_i \circ s_i = s_{i-1} \circ \bar{\theta}_i$ and we have $\psi_i \circ s_i =$ identity. If $s_i(\bar{e}_i) = m_i$, we have $M_i = m_0 A + m_1 A + \cdots + m_i A$. Since for any $b \in B$, $\psi_i(bm_i - m_i b) = 0$, it follows that $bm_i - m_i b \in M_{i-1}$. Let $bm_n - m_n b = \sum_{0 \leqslant i \leqslant n-1} m_i d_{n-i} b$. Applying $\theta_{i+1} \circ \cdots \circ \theta_n$, we get

$$bm_i - m_i b = \sum_{0 \leqslant j \leqslant i-1} m_j d_{i-j} b,$$

since $\theta_i(m_j) = m_{j-1}$ for $1 \leqslant j \leqslant i$ and $\theta_i(m_0) = 0$. Now (setting $d_0 = 1$)

$$\begin{aligned} \sum_{0 \leqslant k \leqslant n-1} m_{n-k} d_k(bb') &= bb'm_n - m_n bb' \\ &= b(b'm_n - m_n b') + (bm_n - m_n b)b' \\ &= \sum_{0 \leqslant i \leqslant n-1} bm_i d_{n-i} b' + \Big(\sum_{0 \leqslant i \leqslant n-1} m_i d_{n-i} b\Big) b' \\ &= \sum_{0 \leqslant i \leqslant n-1} \Big(\sum_{0 \leqslant j \leqslant i} m_j d_{i-j} b\Big) d_{n-i} b' \\ &\qquad + \sum_{0 \leqslant i \leqslant n-1} m_i (d_{n-i} b) b'. \end{aligned}$$

Comparing the coefficients of m_{n-k} on both sides, we get

$$d_k(bb') = \sum_{0 \leqslant i \leqslant k} d_i(b)\, d_{k-i}(b'), \qquad 1 \leqslant k \leqslant n,$$

i.e. $\delta = (d_0 = 1, d_1, \cdots, d_n)$ is a higher derivation of rank n of B into A.

The right A-linear map $f : A_\delta \to M$ defined by $f(e_i) = m_i$ is clearly B-linear, and is in fact an isomorphism in $\mathscr{C}$.

Let now $s'_n : \bar{A}_n \to M_n$ be another right A-linear map such that $\psi_n \circ s'_n$ $=$ identity and let $s'_i : \bar{A}_i \to M_i$ be such that $\theta_i \circ s'_i = s'_{i-1} \circ \bar{\theta}_i$ for $0 \leqslant i \leqslant n$. Let $s'_i(\bar{e}_i) = m'_i$. Since $\psi_n(m'_n - m_n) = 0$, we have $m'_n - m_n \in M_{n-1}$. We thus have elements $u_0 = 1, u_1, \cdots, u_n \in A$ such that

$$m'_n = \sum_{0 \leqslant i \leqslant n} m_{n-i} u_i. \qquad (*)_n$$

Applying $\theta_{k+1} \circ \cdots \circ \theta_n$, we get

$$m'_k = \sum_{0 \leqslant i \leqslant k} m_{k-i} u_i \,. \qquad (*)_k$$

Let $\delta' = (d'_0 = 1, d'_1, \cdots, d'_n)$ be the higher K-derivation corresponding to s'_n. Then, for any $b \in B$,

$$bm'_k - m'_k b = \sum_{1 \leqslant i \leqslant k} m'_{k-i} d'_i b \,.$$

From $(*)_n$ we have,

$$\begin{aligned} \sum_{1 \leqslant i \leqslant n} m'_{n-i} d'_i b &= bm'_n - m'_n b \\ &= \sum_{0 \leqslant i \leqslant n} bm_{n-i} u_i - \sum_{0 \leqslant i \leqslant n} m_{n-i} u_i b \\ &= \sum_{0 \leqslant i \leqslant n} \Big(\sum_{0 \leqslant j \leqslant n-i} m_j d_{n-i-j} b \Big) u_i - \sum_{0 \leqslant i \leqslant n} m_{n-i} u_i b \,. \end{aligned}$$

Substituting for m'_{n-i} from $(*)_{n-i}$ in the above equation, and comparing the coefficients of m_{n-k}, we get

$$\sum_{0 \leqslant i \leqslant k} u_i d'_{k-i} b = \sum_{0 \leqslant i \leqslant k} d_{k-i} b u_i \,,$$

for $0 \leqslant k \leqslant n$. Thus δ' is equivalent to δ. It follows now that for a given isomorphism class in $\mathscr{C}$, there exists a higher K-derivation δ of B into A, unique up to equivalence, such that $(A_\delta, \psi_\delta, \theta_\delta)$ belongs to that class.

Thus we have the following

THEOREM 1. *Let A be a K-algebra, B a K-subalgebra, and let $\mathscr{C}$ denote the category of triples (M, ψ, θ) constructed above. The map $\delta / (A_\delta, \psi_\delta, \theta_\delta)$ of the set of higher K-derivations $\delta : B \to A$ into* obj $\mathscr{C}$ *induces a bijection of the set of equivalence classes of these higher derivations onto the set of isomorphism classes of* obj $\mathscr{C}$. *Under this bijection, the equivalence class of inner higher derivations corresponds to the isomorphism class of $(\bar{A}(n), 1_{\bar{A}(n)}, \bar{\theta})$.*

§ 3. Extension of higher derivations to crossed products.

Let L be a commutative ring and let $\delta : L \to L$ be a higher derivation of rank n. Let L^* denote the group of units of L. We then have a homomorphism $\delta^* : L^* \to U_n(L)$ of groups, defined by

$$\delta^*(\lambda) = \sum_{0 \leqslant i \leqslant n} \lambda^{-1} d_i \lambda x^i \,, \qquad \lambda \in L^* \,.$$

Now, let G be a finite group of automorphisms of L. Let G operate on $T_n(L)$ by setting $s\sum \lambda_i x^i = \sum s(\lambda_i)x^i$, $s \in G$, $\lambda_i \in L$. Clearly $U_n(L)$ is stable under the action of G. If δ is a higher G-*derivation* (i.e., if $d_i \circ s = s \circ d_i$ for all $s \in G$ and $0 \leqslant i \leqslant n$), then δ^* is a G-homomorphism. Thus δ^* induces a homomorphism $H^2(\delta^*) : H^2(G, L^*) \to H^2(G, U_n(L))$. Let $f : G \times G \to L^*$ be a 2-cocycle. We recall that the crossed product (L, G, f) is defined to be the free left L-module with a basis $(e_s)_{s\in G}$ together with a multiplication given by $(\lambda e_s)(\mu e_t) = \lambda s(\mu) f(s, t) e_{st}$, $\lambda, \mu \in L$, $s, t \in G$.

PROPOSITION 1. *A higher G-derivation $\delta : L \to L$ can be extended to a higher derivation of the crossed product $A = (L, G, f)$ if $H^2(\delta^*)(\bar{f}) = 0$, where $\bar{f}$ denotes the class of f. Conversely, if L is an integral domain and δ admits of an extension to A, then $H^2(\delta^*)(\bar{f}) = 0$.*

Proof. Let $H^2(\delta^*)(\bar{f}) = 0$. This means that there exists a map $h : G \to U_n(L)$ such that

$$\delta^* f(s, t) h(st) = h(s) s h(t), \qquad s, t \in G.$$

Let $h(s) = \sum h_i(s) x^i$. We define additive maps $\bar{d}_i : A \to A$ by setting

$$\bar{d}_i(\lambda e_s) = \sum_{0 \leqslant j \leqslant i} d_j(\lambda) h_{i-j}(s) e_s, \ \lambda \in L, \ s \in G.$$

It is straightforward to check that $(\bar{d}_0 = 1, \bar{d}_1, \cdots, \bar{d}_n)$ is a higher derivation of A which extends δ.

Suppose now that L is an integral domain and that $\bar{\delta} = (\bar{d}_0 = 1, \bar{d}_1, \cdots, \bar{d}_n)$ is an extension of δ to A. We first show that for any $i (0 \leqslant i \leqslant n)$, we have $\bar{d}_i(e_s) = h_i(s) e_s$ for some map $h_i : G \to L$. For, let this be assumed proved for $0 \leqslant j < i$ and let $\bar{d}_i(e_s) = \sum_{t \in G} h_i(s, t) e_t$; $h_i(s, t) \in L$. For any $\lambda \in L$, we have

$$\begin{aligned} \bar{d}_i(e_s \lambda) &= \sum_{0 \leqslant j \leqslant i} (\bar{d}_{i-j} e_s)(d_j \lambda) \\ &= \sum_{t \in G} (h_i(s, t) e_t) \lambda + \sum_{1 \leqslant j \leqslant i} (h_{i-j}(s) e_s) d_j(\lambda). \end{aligned}$$

On the other hand,

$$\begin{aligned} \bar{d}_i(e_s \lambda) = \bar{d}_i(s(\lambda) e_s) &= \sum_{0 \leqslant j \leqslant i} d_j s(\lambda) \bar{d}_{i-j} e_s \\ &= \sum_{1 \leqslant j \leqslant i} d_j s(\lambda) h_{i-j}(s) e_s + s(\lambda) \sum_{t \in G} h_i(s, t) e_t. \end{aligned}$$

Comparing the coefficients of e_t for $t \neq s$, we get

$$h_i(s,t)t(\lambda) = h_i(s,t)s(\lambda),$$

for all $\lambda \in L$. Since $t(\lambda) \neq s(\lambda)$ for some λ, it follows that $h_i(s,t) = 0$ for $s \neq t$. Thus we have functions $h_i : G \to L$ such that $\bar{d}_i(e_s) = h_i(s)e_s$, $0 \leqslant i \leqslant n$.

Now

$$\begin{aligned}\bar{d}_i(e_se_t) &= \sum_{0\leqslant j\leqslant i} (\bar{d}_je_s)(\bar{d}_{i-j}e_t)\\ &= \sum_{0\leqslant j\leqslant i} h_j(s)sh_{i-j}(t)f(s,t)e_{st}.\end{aligned}$$

On the other hand

$$\begin{aligned}\bar{d}_i(e_se_t) &= \bar{d}_i(f(s,t)e_{st})\\ &= \sum_{0\leqslant j\leqslant i} d_jf(s,t)h_{i-j}(st)e_{st}.\end{aligned}$$

Thus, we have, for every i,

$$\sum_{0\leqslant j\leqslant i} d_jf(s,t)h_{i-j}(st) = \sum_{0\leqslant j\leqslant i} h_j(s)sh_{i-j}(t).$$

If $h : G \to U_n(L)$ is defined by $h(s) = \sum_{0\leqslant i\leqslant n} h_i(s)x^i$, then the above equations can be written as

$$\delta^*f(s,t)h(st) = h(s)sh(t),$$

which shows that $H^2(\delta^*)(\bar{f}) = 0$.

COROLLARY. *If $H^2(G,L) = 0$, then any higher G-derivation of L can be extended to any crossed product of G and L.*

The above corollary is an immediate consequence of the above proposition and the following

LEMMA. *If $H^2(G,L) = 0$, then $H^2(G,U_n(L)) = 0$ for every n.*

Proof. We define a G-homomorphism $L \to U_n(L)$ by mapping λ into $1+\lambda x^n$. This is an isomorphism for $n=1$ and so $H^2(G,U_1(L)) = 0$. For $n > 1$ we have an exact sequence of G-modules

$$0 \to L \to U_n(L) \to U_{n-1}(L) \to 1,$$

where the map $U_n(L) \to U_{n-1}(L)$ sends $\sum_{0\leqslant i\leqslant n} \lambda_i x^i$ to $\sum_{0\leqslant i\leqslant n-1} \lambda_i x^i$. We then have an exact sequence

$$H^2(G, L) \to H^2(G, U_n(L)) \to H^2(G, U_{n-1}(L)) .$$

It follows by induction on n that $H^2(G, U_n(L)) = 0$.

§ 4. Higher derivations and central simple algebras

The aim of this section is to establish the following

THEOREM 2. *Let A be a finite dimensional central simple K-algebra and let B be a semi-simple subalgebra of A. Then any higher derivation of B into A, which maps K into itself, can be extended to a higher derivation of A.*

Before proving the theorem, we prove a few lemmas.

LEMMA 1. *Let A be a ring, B a subring of A, and let $\delta, \delta' : B \to A$ be two equivalent higher derivations of rank n. If δ admits of an extension to A then δ' can also be extended to A such that these extensions are equivalent.*

Proof. Let $u \in U_n(A)$ be such that $\alpha_{\delta'} = \text{int } u \circ \alpha_\delta$. If $\bar{\delta}$ is an extension of δ to A, then $\text{int } u \circ \alpha_{\bar{\delta}} : A \to T_n(A)$ is a section of $\eta_A : T_n(A) \to A$ on A. This section gives the required extension of δ' to A.

LEMMA 2. *Let A be a K-algebra and let B be a K-subalgebra of A such that every K-derivation of B into A is inner. Let $\delta, \delta' : B \to A$ be higher derivations of rank n mapping K into itself such that $\delta/K = \delta'/K$. Then δ and δ' are equivalent.*

Proof. The case $n = 1$ follows from the hypothesis that the K-derivations of B into A are inner.

Let now $n > 1$ and assume by induction that $\delta_1 = (d_0 = 1, d_1, \cdots, d_{n-1})$ and $\delta_1' = (d_0' = 1, d_1', \cdots, d_{n-1}')$ are equivalent. Let $u = 1 + u_1 x + \cdots + u_{n-1} x^{n-1} \in U_{n-1}(A)$ be such that $\alpha_{\delta_1'} = \text{int } u \circ \alpha_{\delta_1}$. Consider the element $v = 1 + u_1 x + \cdots + u_{n-1} x^{n-1} \in U_n(A)$. The homomorphism $\text{int } v \circ \alpha_\delta : B \to T_n(A)$ gives a higher derivation $\delta'' = (d_0'' = 1, d_1'', \cdots, d_n'')$ equivalent to δ such that $d_i'' = d_i'$ for $0 \leqslant i \leqslant n-1$. Further $d_n''/K = d_n'/K$. Thus $d_n'' - d_n'$ is a K-derivation of B into A. Therefore there exists a $u_n \in A$ such that $d_n''(b) - d_n'(b) = u_n b - b u_n$. It is easily verified that $\alpha_{\delta''} = \text{int } (1 + u_n x^n) \circ \alpha_{\delta'}$. Thus δ'' and δ' are equivalent, which proves the lemma.

LEMMA 3. *Let K be a field and L/K a finite separable extension. Then any higher derivation of K into itself can be uniquely extended to a higher derivation of L.*

Proof. Let $L=K(\lambda)$ and let f be the minimal polynomial of λ so that we have an isomorphism $K[X]/(f)\rightarrow L$ under which X goes to λ.

Let $\delta=(d_0=1, d_1, \cdots, d_n)$ be a higher derivation of K. We remark that δ can be extended to a higher derivation $\delta'=(d'_0=1, d'_1, \cdots, d'_n)$ of $K[X]$ by prescribing arbitrary values for $d'_1X, \cdots, d'_nX$.

Suppose, by induction, that $(d_0=1, d_1, \cdots, d_{n-1})$ has been extended to a higher derivation $(d'_0=1, d'_1, \cdots, d'_{n-1})$ of $K[X]$ such that the ideal generated by $f(X)$ is stable under each d'_i. Suppose further, that the induced higher derivation $(\bar{d}_0=1, \bar{d}_1, \cdots, \bar{d}_{n-1})$ of L is unique as an extension of $(d_0=1, d_1, \cdots, d_{n-1})$.

Let g be any element of $K[X]$. Let $(d'_0=1, d'_1, \cdots, d'_n)$ be the higher derivation of $K[X]$ for which $d'_nX=g$. It is easily seen that

$$d'_nf=f'g+q\,,$$

where f' is the usual derivative of f and q is a polynomial which depends only on $d'_1X, \cdots, d'_{n-1}X$. Since $f'(\lambda)\neq 0$, there exists a polynomial $f_1\in K[X]$ such that $f_1f'\equiv 1 \pmod{f}$. If we choose $g=-f_1q$, then the ideal (f) is stable under d'_n, and the induced map $\bar{d}_n: L\rightarrow L$ satisfies $\bar{d}_n(\lambda)=-q(\lambda)/f'(\lambda)$. Thus we have a higher derivation $(\bar{d}_0=1, \bar{d}_1, \cdots, \bar{d}_n)$ of L which extends δ and is clearly unique.

Proof of Theorem 2. We first assume that the theorem is true with $B=K$ and prove it for the general case. Let δ be a higher derivation of B into A which maps K into itself and let $\bar{\delta}$ be an extension of δ/K. The restrictions of δ and $\bar{\delta}/B$ to K are the same. Since any K-derivation of B into A is inner ([3], Theorem 7), it follows from lemmas 1 and 2, that δ can be extended to A.

We now prove the theorem in the case $B=K$. Let δ be a higher derivation of K. We first show that it is enough to extend δ to some central simple K-algebra A_1 similar to A. In fact, let $\bar{\delta}$ be an extension of δ to A_1. If D denotes the division algebra of A_1, we have $A_1=M_m(D)$ for some integer m. Let δ_1 be the entrywise extension of δ to $M_m(K)$. Since δ_1 and $\bar{\delta}/M_m(K)$ coincide on K and since any K-derivation of $M_m(K)$ into A_1 is inner, it follows by lemmas 1 and 2, that δ_1 can be extended to a higher derivation $\bar{\delta}_1$ of A_1. Since $M_m(K)$ is stable under $\bar{\delta}_1$, and D is the commutant of $M_m(K)$ in A_1, D is also stable under $\bar{\delta}_1$. Thus,

$\bar{\delta}_1/D$ is an extension of δ, and this can be further extended to A, since A is a matrix ring over D.

We can therefore assume that A is a crossed product (L, G, f) for some Galois extension L/K, where G is the Galois group of L/K ([1], Theorem 1, p. 66). By lemma 3 we have a unique extension $\bar{\delta} = (\bar{d}_0 = 1, \bar{d}_1, \cdots, \bar{d}_n)$ of δ to L. If $s \in G$, then $s\bar{\delta}s^{-1} = (s\bar{d}_0s^{-1} = 1, s\bar{d}_1s^{-1}, \cdots, s\bar{d}_ns^{-1})$ is also a higher derivation of L extending δ, so that we have $s\bar{d}_is^{-1} = d_i$ for $0 \leqslant i \leqslant n$. In other words, $\bar{\delta}$ is a G-derivation. Since $H^2(G, L) = 0$, it follows from the corollary to proposition 1 of § 3, that $\bar{\delta}$ can be extended to A. This completes the proof of the theorem.

References

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