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LOWER BOUNDS FOR FUNDAMENTAL UNITS OF REAL QUADRATIC FIELDS

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Abstract. Let d be a square-free positive integer and $l(d)$ be the period length of the simple continued fraction expansion of ω_d, where ω_d is integral basis of $\mathbb{Z}[\sqrt{d}]$. Let $\varepsilon_d = (t_d + u_d\sqrt{d})/2$ (> 1) be the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{d})$. In this paper new lower bounds for ε_d, t_d, and u_d are described in terms of $l(d)$. The lower bounds of ε_d are sharper than the known bounds and those of t_d and u_d have been yet unknown. In order to show the strength of the method of the proof, some interesting examples of d are given for which ε_d and Yokoi's d-invariants are determined explicitly in relation to continued fractions of the form $[a_0, \overline{1, \dots, 1, a_{l(d)}}]$.

Introduction

For a positive square-free integer d, let D be the discriminant of the real quadratic field $\mathbb{Q}(\sqrt{d})$ and $l(d)$ be the period length in the simple continued fraction expansion of the algebraic integer $\omega_d = (\sigma_d - 1 + \sqrt{d})/\sigma_d$, where $\sigma_d = 1$ (resp. 2) for $d \not\equiv 1 \pmod 4$ (resp. $d \equiv 1 \pmod 4$). It is well-known that the fundamental unit $\varepsilon_d = (t_d + u_d\sqrt{d})/2$ (> 1) of $\mathbb{Q}(\sqrt{d})$ has lower bounds that increase with $l(d)$. For example, there are $((1 + \sqrt{5})/2)^{l(d)}$ (see, for example [1, p. 240]) and $\sqrt{D}\,(3/2)^{l(d)-2}$ (see [2, p. 98]). For d with sufficiently large $l(d)$, ε_d is much larger than these lower bounds. So we will caluculate a sharper lower bound in terms of $l(d)$ to study the sufficiently large ε_d for d. Furthermore, we will calculate the lower bounds for t_d and u_d in terms of $l(d)$ that have been yet unknown. In order to study Yokoi's d-invariants, which are concerned with the class number one problem for real quadratic fields, we need to investigate t_d and u_d. We have obtained the following theorem.

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THEOREM 1. *Let d be a positive square-free integer with $l(d) \geq 2$ and D be the discriminant of $\mathbb{Q}(\sqrt{d})$. Then we have*

$$\varepsilon_d > \begin{cases} \dfrac{\sqrt{D}}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^{l(d)-1} & \text{if } l(d) \text{ is even,} \\ \dfrac{1}{\sqrt{5}}\left(\sqrt{D}-\dfrac{\sqrt{5}-1}{2}\right)\cdot\left(\dfrac{1+\sqrt{5}}{2}\right)^{l(d)} & \text{if } l(d) \text{ is odd.} \end{cases}$$

Moreover, for t_d and u_d in $\varepsilon_d = (t_d + u_d\sqrt{d})/2$ (> 1), we have

$$t_d > \begin{cases} \dfrac{\sqrt{D}}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^{l(d)-1} & \text{if } l(d) \text{ is even,} \\ \dfrac{1}{\sqrt{5}}(\sqrt{D}-\sqrt{5}+1)\cdot\left(\dfrac{1+\sqrt{5}}{2}\right)^{l(d)} & \text{if } l(d) \text{ is odd} \end{cases}$$

and

$$u_d > \begin{cases} \dfrac{2}{\sigma_d\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^{l(d)-1} & \text{if } l(d) \text{ is even,} \\ \dfrac{2}{\sigma_d\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^{l(d)-2} & \text{if } l(d) \text{ is odd.} \end{cases}$$

In this paper, the simple continued fraction with period l is generally denoted by $[a_0, \overline{a_1, \ldots, a_l}]$, and $[x]$ means the greatest integer less than or equal to x. Let $\{F_i\}$ be the Fibonacci numbers determined by $F_0 = 0$, $F_1 = 1$, $F_{i+1} = F_i + F_{i-1}$ ($i \geq 1$). The Fibonacci numbers play an important role in showing Theorem 1, because we use the inequality $\varepsilon_d \geq ([\omega_d]+\omega_d)F_{l(d)} + F_{l(d)-1}$ (resp. $([\omega_d]-1+\omega_d)F_{l(d)} + F_{l(d)-1}$) in the case that $d \not\equiv 1 \pmod 4$ (resp. $d \equiv 1 \pmod 4$). We are interested in whether the equality is possible. We got an affirmative answer and we have discovered real quadratic fields with $m_d \neq 0$. Here m_d is one of Yokoi's d-invariant, and it is defined by $m_d = [u_d^2/t_d]$. Another is defined by $n_d = [t_d/u_d^2]$. We know that there exist only finitely many d satisfying both class number one and $n_d \neq 0$ (i.e. $m_d = 0$) (see [7, p. 188]). So it is very important to investigate the case that $m_d \neq 0$ (i.e. $n_d = 0$). Now these results are stated as follows:

THEOREM 2. *Let d be a positive square-free integer and l be a positive integer satisfying $l \geq 2$. Assume that*

$$d = (2F_l + 1)^2 + 8F_{l-1} + 4.$$

Then $d \equiv 1 \pmod 4$, $\omega_d = [F_l + 1, \overline{1, \ldots, 1, 2F_l + 1}]$, and $l(d) = l$ hold.

Moreover, we have

$$\varepsilon_d = \frac{1}{2}(2F_l^2 + F_l + 2F_{l-1} + F_l\sqrt{d}) \ (= ([\omega_d] - 1 + \omega_d)F_l + F_{l-1}),$$

$$\begin{cases} t_d = 2F_l^2 + F_l + 2F_{l-1}, \\ u_d = F_l, \end{cases}$$

and

$$n_d = \begin{cases} 5 & \text{if } l = 2, \\ 3 & \text{if } l = 3, \\ 2 & \text{if } l \geq 5. \end{cases}$$

THEOREM 3. *Let d be a positive square-free integer and l be a positive integer satisfying that $l \geq 2$ and $l \equiv 1, 2,$ or $4 \pmod 6$. Assume that*

$$d = ((F_l + 1)/2)^2 + F_{l-1} + 1.$$

Then $d \not\equiv 1 \pmod 4$, $\omega_d = [(F_l + 1)/2, \overline{1, \ldots, 1, F_l + 1}]$, and $l(d) = l$ hold.

Moreover, we have

$$\varepsilon_d = \frac{1}{2}(F_l^2 + F_l + 2F_{l-1} + 2F_l\sqrt{d}) \ (= ([\omega_d] + \omega_d)F_l + F_{l-1}),$$

$$\begin{cases} t_d = F_l^2 + F_l + 2F_{l-1}, \\ u_d = 2F_l, \end{cases}$$

and

$$m_d = \begin{cases} 1 & \text{if } l = 2, \\ 2 & \text{if } l = 4, \\ 3 & \text{if } l \geq 7. \end{cases}$$

Remark 1. Our aim is to show that, for any l, there exists d satisfying $l(d) = l$ and $\omega_d = [a_0, \overline{1, \ldots, 1, a_{l(d)}}]$. The case that $l = 6n+1$ was treated in [4], but our proof is simpler than theirs. Their aim is to consider Eisenstein's problem.

Preliminary

In order to prove our theorems, we need several lemmas.

LEMMA 1. *For a square-free positive integer d, we suppose $\omega_d = [a_0, \overline{a_1, \dots, a_l}]$. Moreover let q_i be the integers determined by $q_0 = 0$, $q_1 = 1$, $q_{i+1} = a_i q_i + q_{i-1}$ $(i \geq 1)$. Then the fundamental unit $\varepsilon_d = (t_d + u_d\sqrt{d})/2$ (> 1) of $\mathbb{Q}(\sqrt{d})$ is given by the following formula:*

If $d \not\equiv 1 \pmod 4$, then

$$\varepsilon_d = (a_0 + \sqrt{d})q_{l(d)} + q_{l(d)-1}, \qquad \begin{cases} t_d = 2(a_0 q_{l(d)} + q_{l(d)-1}), \\ u_d = 2q_{l(d)}. \end{cases}$$

If $d \equiv 1 \pmod 4$, then

$$\varepsilon_d = \left(\frac{2a_0 - 1 + \sqrt{d}}{2}\right)q_{l(d)} + q_{l(d)-1}, \qquad \begin{cases} t_d = (2a_0 - 1)q_{l(d)} + 2q_{l(d)-1}, \\ u_d = q_{l(d)}. \end{cases}$$

Proof is omitted (see proof of Lemma 1 in [3]).

LEMMA 2. *For a positive square-free integer d, denote by D the discriminant of $\mathbb{Q}(\sqrt{d})$. Then we have*

$$\varepsilon_d > (\sqrt{D} - 1)F_{l(d)} + F_{l(d)-1}.$$

Moreover, for t_d and u_d in $\varepsilon_d = (t_d + u_d\sqrt{d})/2$ (> 1), we have

$$t_d > (\sqrt{D} - 2)F_{l(d)} + 2F_{l(d)-1} \quad \text{and} \quad u_d \geq \left(\frac{2}{\sigma_d}\right)F_{l(d)}.$$

Proof. In the case that $d \not\equiv 1 \pmod 4$, since $a_0 > \sqrt{d} - 1$ and $q_i \geq F_i$ for any integer i (≥ 1), from Lemma 1 we get

$$\begin{aligned} \varepsilon_d &\geq (a_0 + \sqrt{d})F_{l(d)} + F_{l(d)-1} \\ &> (2\sqrt{d} - 1)F_{l(d)} + F_{l(d)-1} \\ &= (\sqrt{D} - 1)F_{l(d)} + F_{l(d)-1}. \end{aligned}$$

For t_d and u_d, in the case that $d \not\equiv 1 \pmod 4$, we have

$$t_d > 2\{(\sqrt{d} - 1)F_{l(d)} + F_{l(d)-1}\} = (\sqrt{D} - 2)F_{l(d)} + 2F_{l(d)-1}$$

and

$$u_d \geq 2F_{l(d)}.$$

In the case that $d \equiv 1 \pmod 4$, we get the results in the same way. □

We get the following lemma by straightforward calculations.

LEMMA 3. *For* $i \geq 1$,

$$F_i > \begin{cases} \dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^{i-1} & \text{if } i \text{ is even,} \\ \dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^{i} & \text{if } i \text{ is odd.} \end{cases}$$

Proof of Theorem 1

Proof of Theorem 1. We put $\alpha = (1+\sqrt{5})/2$. First we shall show the lower bound of ε_d. From Lemma 2, we know $\varepsilon_d > (\sqrt{D}-1)F_{l(d)} + F_{l(d)-1}$.

In the case that $l(d)$ (≥ 3) is odd, we have, from Lemma 3,

$$\begin{aligned}(\sqrt{D}-1)F_{l(d)} + F_{l(d)-1} &> (\sqrt{D}-1)\frac{\alpha^{l(d)}}{\sqrt{5}} + \frac{1}{\sqrt{5}}\alpha^{l(d)-2} \\ &= \frac{\alpha^{l(d)}}{\sqrt{5}}\left\{\sqrt{D} - \frac{\sqrt{5}-1}{2}\right\}.\end{aligned}$$

This proves the odd case.

In the case that $l(d)$ (≥ 2) is even, we have

$$\begin{aligned}(\sqrt{D}-1)F_{l(d)} + F_{l(d)-1} &> (\sqrt{D}-1)\frac{\alpha^{l(d)-1}}{\sqrt{5}} + \frac{\alpha^{l(d)-1}}{\sqrt{5}} \\ &= \frac{\sqrt{D}}{\sqrt{5}}\alpha^{l(d)-1}.\end{aligned}$$

Next we shall show the lower bounds of t_d and u_d. From Lemma 2, we know

$$t_d > (\sqrt{D}-2)F_{l(d)} + 2F_{l(d)-1}.$$

In the case that $l(d)$ (≥ 3) is odd, we have

$$\begin{aligned}(\sqrt{D}-2)F_{l(d)} + 2F_{l(d)-1} &> (\sqrt{D}-2)\frac{\alpha^{l(d)}}{\sqrt{5}} + 2\frac{\alpha^{l(d)-2}}{\sqrt{5}} \\ &= \frac{\alpha^{l(d)}}{\sqrt{5}}(\sqrt{D}-\sqrt{5}+1).\end{aligned}$$

In the case that $l(d)$ (≥ 2) is even, we have

$$(\sqrt{D}-2)F_{l(d)}+2F_{l(d)-1} > (\sqrt{D}-2)\frac{\alpha^{l(d)-1}}{\sqrt{5}}+2\frac{\alpha^{l(d)-1}}{\sqrt{5}} = \frac{\sqrt{D}}{\sqrt{5}}\alpha^{l(d)-1}.$$

From Lemma 2 and Lemma 3, we can get the lower bound of u_d in a similar way as in the proof of t_d. Theorem has been completely proved. ∎

From Theorem 1, we get the following corollary for the period $l(d)$ and Yokoi's d-invariant m_d:

Corollary. *If there exist a positive integer M and a positive square-free integer d such that $d > 13$ and*

$$l(d) \geq \frac{\log(M+1)+\log(\sqrt{5d})-\log 2}{\log\left(\dfrac{1+\sqrt{5}}{2}\right)}+1,$$

then $m_d > M$.

Proof. From the assumption, we have

$$\frac{\sqrt{D}}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{l(d)-1} \geq (M+1)d.$$

Since it holds that

$$\frac{1}{\sqrt{5}}\left(\sqrt{D}-\frac{\sqrt{5}-1}{2}\right)\cdot\left(\frac{1+\sqrt{5}}{2}\right)^{i} > \frac{\sqrt{D}}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{i-1}$$

for any positive integer i, we get, from Theorem 1,

$$\varepsilon_d > (M+1)d.$$

Moreover, $m_d = [\varepsilon_d/d]$ if $d > 13$ from Theorem 1.1 in [8], therefore we have

$$m_d > \frac{\varepsilon_d}{d}-1 > M.$$

∎

Remark 2. We describe the comparison between the lower bound in Theorem 1 and the two well-known lower bounds given in the introduction of this paper. From the proof of Corollary, our lower bound for d with odd $l(d)$ is greater than $(\sqrt{D}/\sqrt{5})\cdot((1+\sqrt{5})/2)^{l(d)-1}$. Moreover, if $D>13$, then we have the following:

$$\frac{\sqrt{D}}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{l(d)-1}-\left(\frac{1+\sqrt{5}}{2}\right)^{l(d)}>0.22\left(\frac{1+\sqrt{5}}{2}\right)^{l(d)-1}.$$

Furthermore, if $l(d)\geq 7$, then

$$\frac{\sqrt{D}}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{l(d)-1}-\sqrt{D}\left(\frac{3}{2}\right)^{l(d)-2}>0.038\sqrt{D}\left(\frac{1+\sqrt{5}}{2}\right)^{l(d)-2}.$$

Hence, our lower bounds are sharper than theirs.

Proof of Theorems 2 and 3

Proof of Theorems 2 *and* 3. First we shall prove the first half of Theorem 3. Suppose that $l\equiv 1,2,$ or $4 \pmod 6$. Since it holds that

$$F_{l+6}\equiv F_l \pmod 4 \qquad (l\geq 0),$$

we have $d\not\equiv 1 \pmod 4$. We put

$$\omega_R=(F_l+1)/2+[(F_l+1)/2,\overline{\underbrace{1,\dots,1}_{l-1},F_l+1}].$$

Then we have

$$\omega_R=F_l+1+\frac{1}{1}{+}\cdots{+}\frac{1}{1}{+}\frac{1}{\omega_R}.$$

By a straightforward induction argument, we obtain that

$$\omega_R=F_l+1+\frac{F_{l-1}\omega_R+F_{l-2}}{F_l\omega_R+F_{l-1}}.$$

Here, using $F_l=F_{l-1}+F_{l-2}$ $(l\geq 2)$, we have

$$\omega_R^2-(F_l+1)\omega_R-(F_{l-1}+1)=0.$$

Since $\omega_R>0$, it holds that

$$\omega_R=\frac{F_l+1}{2}+\sqrt{d}.$$

Hence, we obtain that $\omega_d = [(F_l+1)/2, \overline{1,\ldots,1,F_l+1}]$ and $l(d)=l$, and we can determine ε_d, t_d and u_d from Lemma 1. In the case of Theorem 2, we put

$$\omega_R = F_l + [F_l+1, \overline{\underbrace{1,\ldots,1}_{l-1}, 2F_l+1}].$$

Since we have $\omega_R = F_l + \omega_d$ in the same way, we can determine ε_d, t_d and u_d.

Next we shall show the remaining part of Theorem 3. Since $1/F_l$ is monotone decreasing in l, it holds that

$$3.4 < 4\left(1+\frac{3}{F_l}\right)^{-1} \leq 4\left(1+\frac{1}{F_l}+\frac{2F_{l-1}}{{F_l}^2}\right)^{-1} < 4$$

for $l \geq 7$. Hence, we have $m_d = 3$ for $l \geq 7$ from definition of m_d. And we can get m_d in the case that $l = 2$ or 4 from straightforward calculations.

Lastly we shall show the remaining part of Theorem 2. We have $n_d = 2$ for $l \geq 5$ from the definition of n_d, because

$$2 \leq n_d \leq 2 + \frac{1}{F_l} + \frac{2F_{l-1}}{F_l^2} < 2 + \frac{3}{F_l} < 3$$

for $l \geq 5$. And we can get n_d in the case that $l = 2$ or 3 from straightforward calculations. Theorems have been proved. □

Table: Square-free positive integers d with $2 < l(d) \leq 15$ represented by the Fibonacci numbers:

d	$l(d)$ (mod 6)	$l(d)$	h_d	$F_{l(d)}$	ω_d
3	2	2	1	1	$[1, \overline{1, 2}]$
21	2	2	1	1	$[2, \overline{1, 3}]$
37	3	3	1	2	$[3, \overline{1, 1, 5}]$
7	4	4	1	3	$[2, \overline{1, 1, 1, 4}]$
69	4	4	1	3	$[4, \overline{1, 1, 1, 7}]$
149	5	5	1	5	$[6, \overline{1, 1, 1, 1, 11}]$
58	1	7	2	13	$[7, \overline{1, 1, 1, 1, 1, 1, 14}]$
797	1	7	1	13	$[14, \overline{1, 1, 1, 1, 1, 1, 27}]$
4933	3	9	3	34	$[35, \overline{1, 1, 1, 1, 1, 1, 1, 1, 69}]$
32485	5	11	8	89	$[90, \overline{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 179}]$
84237	0	12	6	144	$[145, \overline{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 289}]$

Table: Square-free positive integers d with $2 < l(d) \leq 15$ represented by the Fibonacci numbers:

d	$l(d)$ (mod 6)	$l(d)$	h_d	$F_{l(d)}$	ω_d
13834	1	13	22	233	$[117, \overline{1,1,1,1,1,1,1,1,1,1,1,1,234}]$
219245	1	13	12	233	$[234, \overline{1,1,1,1,1,1,1,1,1,1,1,1,467}]$
1493861	3	15	20	610	$[611, \overline{1,1,1,1,1,1,1,1,1,1,1,1,1,1,1221}]$

Finally, we have the above table for d concerning Theorem 2 and Theorem 3. Here h_d is the class number of $\mathbb{Q}(\sqrt{d})$.

References

[1] R. A. Mollin, Quadratics, CRC Press, Boca Rato, FL., 1996.

[2] R. Sasaki, *A characterization of certain real quadratic fields*, Proc. Japan Acad., **62**, Ser. A (1986), no. 3, 97–100.

[3] K. Tomita, *Explicit representation of fundamental units of some quadratic fields*, Proc. Japan Acad., **71**, Ser. A (1995), no. 2, 41–43.

[4] K. S. Williams and N. Buck, *Comparison of the lengths of the continued fractions of* $\sqrt{D}$ *and* $\frac{1}{2}(1+\sqrt{D})$, Proc. Amer. Math. Soc., **120** (1994), no. 4, 995–1002.

[5] H. Yokoi, *The fundamental unit and class number one problem of real quadratic fields with prime discriminant*, Nagoya Math. J., **120** (1990), 51–59.

[6] H. Yokoi, *a note on class number one problem for real quadratic fields*, Proc. Japan Acad., **69**, Ser. A (1993), 22–26.

[7] H. Yokoi, *The fundamental unit and bounds for class numbers of real quadratic fields*, Nagoya Math. J., **124** (1991), 181–197.

[8] H. Yokoi, *New invariants and class number problem in real quadratic fields*, Nagoya Math. J., **132** (1993), 175–197.

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