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ON THE THETA DIVISOR OF $SU(r, 1)$

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Abstract. Let $SU(r, 1)$ be the moduli space of stable vector bundles, on a smooth curve C of genus $g \geq 2$, with rank $r \geq 3$ and determinant $O_C(p)$, $p \in C$; let $\mathcal{L}$ be the generalized theta divisor on $SU(r, 1)$. In this paper we prove that the map $\phi_{\mathcal{L}}$, defined by $\mathcal{L}$, is a morphism and has degree 1.

§0. Introduction

Let C be a smooth, irreducible, complex, projective curve, of genus $g \geq 2$. Let $SU(r, d)$ denotes the moduli space of semistable vector bundles with rank r and fixed determinant $L \in \mathrm{Pic}^d(C)$. $SU(r, d)$ is an irreducible projective variety of dimension $(r^2 - 1)(g - 1)$, (see [S] and [N-R]), its Picard variety is free cyclic, see [D-N], the ample generator $\mathcal{L}$ is called the generalized theta divisor of $SU(r, d)$. Let $\phi_{\mathcal{L}}\colon SU(r, d) \to |\mathcal{L}|^*$ be the map associated to the theta divisor: if $r = 2$, then $\phi_{\mathcal{L}}$ is an embedding, see [Be1], [L],[B-V1], [vG-I] for d even, [D-R], [Be2] and [B-V2] for d odd. In this paper, we will assume $r \geq 3$ and we will consider $SU(r, 1)$, where $L = O_C(p)$ and p is a given point of C, our first result is the following:

Theorem 0.0.1. *For any curve C of genus $g \geq 2$: $\deg(\phi_{\mathcal{L}}) = 1$, the linear system $|\mathcal{L}|$ on $SU(r, 1)$ is base points free, i.e. the map $\phi_{\mathcal{L}}$ is a morphism.*

As a second result we prove the following:

Theorem 0.0.2. *For any curve C of genus $g \geq 2$, we have $\deg(\phi_{\mathcal{L}}) = 1$.*

The paper is organized as follows. The first section is devoted to proving theorem (0.0.1). In section 2, we study rank r-bundles with $r + 1$

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sections extending the rank 2 case dealed in [B-V2]. Fix a line bundle $l \in \mathrm{Pic}^g(C)$: we can identify the spaces $SU(r,1)$ and $SU(r, O_C(p+rl))$, let $E \in SU(r, O_C(p+rl))$, assume that $h^0(E) = r+1$ and the natural map $w_E\colon \wedge^r H^0(E) \to H^0(\det E)$ is injective, then $\mathrm{Im}\, w_E$ is a $(r+1)$-dimensional subspace of $H^0(O_C(p+rl))$. This allow us to define a map

$$g_l\colon SU(r, O_C(p+rl)) \to G_l(r+1, H^0(O_C(p+rl)),$$

we prove that g_l is a birational map and it is defined by a linear system in $|\mathcal{L}|$. In section 3, we prove theorem (0.0.2). Actually, we perform a non empty open subset $\mathcal{U} \subset SU(r,1)$ such that the restriction $\phi_{\mathcal{L}|\mathcal{U}}$ is an embedding. $\mathcal{U}$ is naturally defined as the set of bundles ξ for which exists $l \in \mathrm{Pic}^g(C)$, s.t. g_l is biregular at the point $E = \xi(l)$. If $r = 2$, in [B-V2] we proved that actually $\mathcal{U} = SU(2,1)$, which allows us to conclude that $\mathcal{L}$ is very ample. If $r \geq 3$, actually $\mathcal{U}$ can be a proper subset of $SU(r,1)$, (see lemma (3.2.1)), this unable us to extend completely the result of rank 2.

Finally, we would like to remember that rank 2 vector bundles with 3 sections were useful also in proving that $\phi_{\mathcal{L}}$ is an embedding at singular points of $SU(2)$, see [I-vG].

0.1. Notations.

We reserve the notation ξ for points of $SU(r,1)$; with some abuse, the same notation will be used for the vector bundle corresponding to ξ. For a vector bundle ξ of degree d and rank r we denote by $\mu\colon= \frac{d}{r}$ the slope of ξ. We say that ξ is semistable iff for every proper subbundle $\eta \subset \xi$ we have $\mu(\eta) \leq \mu(\xi)$, it is stable iff the inequality is strict. Given two vector bundles ξ, η on C, they are said complementary if $\chi(\xi \otimes \eta) = 0$.

We recall that there exists a Poincaré family on $SU(r,1)$, see [N-R], i.e. a vector bundle U on $SU(r,1) \times C$ such that $U_{|\xi \times C} \simeq \xi$, for any $\xi \in SU(r,1)$. Let as usual π_i denote the natural projections of $SU(r,1) \times C$ onto factors. Note that if U is a Poincaré bundle, then for any $A \in \mathrm{Pic}(SU(r,1))$, $U \otimes \pi_1^* A$ is a Poincaré bundle too. Actually there exists a unique Poincaré bundle U on $SU(r,1) \times C$ with the further following property, (see [Ra]):

$$\det U_{|SU(r,1) \times \{x\}} \simeq \mathcal{L},$$

where $\mathcal{L}$ is the theta divisor of $SU(r,1)$. Following [Ra], we will call such a bundle U the universal bundle.

§1. On the base points of the theta divisor

1.0.

Let θ be an effective divisor of degree $g-1$ on C, θ defines a natural isomorphism

$$f_\theta\colon SU(r,1) \to SU(r, r(g-1)-1) \tag{1}$$

sending ξ to $\xi^*(\theta)$. Let $(\xi,\eta) \in SU(r,1) \times SU(r, r(g-1)-1)$ we have

$$\chi(\xi \otimes \eta) = 0, \tag{2}$$

hence the subset

$$\hat\Theta_\xi := \{\eta \in SU(r, r(g-1)-1) \ \ / \ h^0(\xi \otimes \eta) > 0\} \tag{3}$$

is either $SU(r, r(g-1)-1)$ or a theta divisor of $SU(r, r(g-1)-1)$, see [D-N].

Lemma 1.0.1. *Let $U_\xi \subset \hat\Theta_\xi$ be the locus of points η such that each non zero morphism $u\colon \eta^* \to \xi$ is a monomorphism. Then U_ξ is a non empty open subset.*

Proof. Let $\mathcal{F}$ be a family of stable vector bundles on $S \times C$, let $U\colon \mathcal{F}^* \to {\pi_2}^*\xi$ be a non zero morphism of vector bundles. It is enough to show that the locus Δ of points $s \in S$ such that U_s is not a monomorphism is closed. This is immediate because Δ is the projection of the degeneracy locus of U. The non emptyness follows from the exact sequence

$$0 \to \xi(-\theta) \to \xi \to O_\theta \otimes \xi \to 0 \tag{4}$$

where $\eta^* = \xi(-\theta)$. ∎

Lemma 1.0.2. *We have:* $\dim U_\xi \le (r^2-1)(g-1)-1$.

Proof. Let $\eta \in U_\xi$: then there exists an exact sequence as follows

$$0 \to \eta^* \to \xi \to O_D \to 0, \tag{5}$$

where D is a divisor in the linear system $|\det(\xi) \otimes \det(\eta)|$, that is $|r\theta|$. Let's consider the natural rational map

$$V_D\colon \mathrm{Hom}(\xi, O_D) \to SU(r, r(g-1)-1) \tag{6}$$

which associates to any epimorphism $v\colon \xi \to O_D$ the sheaf $(\ker v)^*$. Then any $\eta \in U_\xi$ belongs to the image of the above map V_D. The group of automorphisms $\mathrm{Aut}(O_D)$ naturally acts on $\mathrm{Hom}(\xi, O_D)$; the action is faithful on points v which are epimorphisms, moreover $\mathrm{Aut}(O_D)$ contains the torus $\mathbf{C}^{*\deg D}$. Since $\mathrm{Hom}(\xi, O_D) \simeq \xi^* \otimes O_D$, it follows that the dimension of the image of the previous map at V_D is bounded by

$$(r-1)\deg D = (r^2-1)(g-1) - (r-1)(g-1).$$

On the other hand, if $r \geq 3$ we have $\dim |D| = (r-1)(g-1) - 1$. Let $\mathcal{D} \subset C \times |D|$ be the universal divisor, then U_ξ is contained in the image of the natural map

$$\pi_{2*}((\pi_1{}^*\xi) \otimes O_{\mathcal{D}}) \to SU(r, r(g-1)-1). \tag{7}$$

By the previous count the dimension of this image is at most $(r^2-1)(g-1)-1$. ∎

Let $\xi \in SU(r,1)$, with the above notations, we define

$$\Theta_\xi := f_\theta{}^*\hat{\Theta}_\xi. \tag{8}$$

Proposition 1.0.1. *For any $\xi \in SU(r,1)$, Θ_ξ is actually a theta divisor on $SU(r,1)$.*

Proof. It is enough to prove the assertion for $\hat{\Theta}_\xi$. Note that $\hat{\Theta}_\xi$ is either a theta divisor on $SU(r, r(g-1)-1)$ or it is $SU(r, r(g-1)-1)$. Let's assume that $\hat{\Theta}_\xi = SU(r, r(g-1)-1)$, then by the preceding lemma U_ξ is an open subset of $SU(r, r(g-1)-1)$: but this is impossible because $\dim U_\xi < \dim SU(r, r(g-1)-1)$. This implies the claim. ∎

1.1.

Fix $\theta \in C^{(g-1)}$, the previous remarks allow us to define a map

$$\phi_\theta\colon SU(r,1) \to |\mathcal{L}| \simeq \mathbf{P}^n \tag{9}$$

just sending ξ to the divisor Θ_ξ. From the previous proposition follows immediately that ϕ_θ is a morphism.

Lemma 1.1.1. *We have: $\phi_\theta{}^* O_{\mathbf{P}^n}(1) = \mathcal{L}$.*

Proof. Let $\xi_0 \in SU(r,1)$ be a point wich is not a point of ramificaton of ϕ_θ, set $\eta := f_\theta(\xi_0) = {\xi_0}^*(\theta)$. Let H be the hyperplane of $\mathbf{P}^n$ consisting of divisors passing through ξ_0. We have:

$$ {\phi_\theta}^* H = \{\xi \in SU(r,1) : \eta \in \hat{\Theta}_\xi\} = \{\xi \in SU(r,1) : h^0(\xi \otimes \eta) > 0\}; $$

and set theoretically this is a divisor in the linear system $|\mathcal{L}|$. Actually, $\xi_0 \in {\phi_\theta}^* H$ and since it is not a point of ramification , we have ${\phi_\theta}^* H = \mathcal{L}$. ∎

As an immediate consequence we have

PROPOSITION 1.1.1. *The map associated to the theta divisor $\phi_{\mathcal{L}}$: $SU(r,1) \to |\mathcal{L}|^*$ is a morphism.*

§2. Bundles with $r+1$ sections

2.1. Definition

Let $(\xi, l) \in SU(r,1) \times \mathrm{Pic}^g(C)$. We say that (ξ, l) satisfies condition $(*)$ if the following three properties hold:

(i) $h^0(\xi(l)) = r+1$,

(ii) $\xi(l)$ is globally generated,

(iii) the determinant map $w_{\xi,l} : \wedge^r H^0(\xi(l)) \to H^0(\det \xi(l))$ is injective.

We will set

$$ X_l := \{\xi \in SU(r,1)/(\xi, l) \quad \text{satisfies } (*)\}. \tag{10} $$

2.2. Remark

Assume that a pair $(\xi, l) \in SU(r,1) \times \mathrm{Pic}^g(C)$ satisfies properties (i) and (ii), then it satisfies (iii) too. First of all, note that since $h^0(\xi(l)) = r+1$, every vector of $\wedge^r H^0(\xi(l))$ is indecomposable. So assume that $v \neq 0$ is in the kernel of the map $w_{\xi,l}$, then $v = s_1 \wedge s_2 \wedge ... \wedge s_r$, with $s_i \in H^0(\xi(l))$, $i = 1...r$. Then the sections $s_1, ...s_r$ would generate a subbundle $\eta \subset \xi(l)$ with the following properties: $rk\eta = s \leq r-1$, $h^0(\eta) \geq r$, and η is globally generated too. This implies $rk\eta = r-1$, $h^0(\eta) = r$ and the following commutative diagram

$$ \begin{tikzcd} 0 \arrow[r] & (\det \eta)^{-1} \arrow[r] \arrow[d] & H^0(\eta) \otimes O_C \arrow[r, "e"] \arrow[d] & \eta \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & O_C(p+rl)^{-1} \arrow[r] & H^0(\xi(l)) \otimes O_C \arrow[r] & \xi(l) \arrow[r] & 0, \end{tikzcd} \tag{11} $$

which implies $\deg \eta \geq 1 + rg$, which contradicts the stability of $\xi(l)$.

We will show later that actually for any $l \in \mathrm{Pic}^g(C)$, X_l is a non empty open subset of $SU(r,1)$.

2.3. Definition

Let $l \in \mathrm{Pic}^g(C)$, we will consider the Grassmannian

$$G_l = G(r+1, H^0(O_C(p+rl))) \tag{12}$$

of $(r+1)$ dimensional subspaces of $H^0(O_C(p+rl))$. If (ξ, l) satisfies assumptions (i) and (iii) then the image of the determinant map $w_{\xi,l}$ is a $(r+1)$ dimensional subspace of $H^0(O_C(p+rl))$, let's denote it by

$$W := \mathrm{Im}\, w_{\xi,l}.$$

This defines a map

$$g_l \colon SU(r,1) \to G_l(r+1, H^0(O_C(p+rl))) \tag{13}$$

by sending ξ to the point of the Grassmannian corresponding to the subspace $W \hookrightarrow H^0(O_C(p+rl))$.

Note that there is a canonical isomorphism $\wedge^r H^0(\xi(l)) \simeq H^0(\xi(l))^*$, which induces an inclusion

$$w'_\xi \colon H^0(\xi(l))^* \hookrightarrow H^0(O_C(p+rl)), \tag{14}$$

whose image is again W. Assume now that $\xi(l)$ is globally generated too, then we have an exact sequence

$$0 \to O_C(p+rl)^{-1} \to H^0(\xi(l)) \otimes O_C \to \xi(l) \to 0, \tag{15}$$

and its dual

$$0 \to \xi(l)^* \to H^0(\xi(l))^* \otimes O_C \to O_C(p+rl) \to 0; \tag{16}$$

passing to cohomology we have

$$0 \to H^0(\xi(l)^*) \to H^0(\xi(l))^* \xrightarrow{\pi} H^0(O_C(p+rl)) \to ...., \tag{17}$$

since $\xi(l)$ is stable, then $H^0(\xi(l)^*) = 0$, so we can conclude that π is injective. We claim that $\mathrm{Im}\, \pi = W$, so that we can identify the maps π and w'_ξ.

2.4.

Let $W \in G_l(r+1, H^0(O_C(p+rl))$, assume that $|W|$ is base point free. Then we can consider the evaluation map $e: W \otimes O_C \to O_C(p+rl)$, which is surjective, so its kernel is a rank r vector bundle, let's define

$$E_W := (\operatorname{Ker} e)^*. \tag{18}$$

We have $\det E_W = O_C(p+rl)$, moreover we have the following exact sequence

$$0 \to O_C(p+rl)^{-1} \to W^* \otimes O_C \to E_W \to 0, \tag{19}$$

so we can conclude that E_W is generated by $(r+1)$ global sections spanning the subspace $\operatorname{Im}(W^* \hookrightarrow H^0(E_W))$. Passing to cohomology, we have

$$(20)\ 0 \to W^* \to H^0(E_W) \to H^1(O_C(p+rl)^{-1}) \to W^* \otimes H^1(O_C) \to ..;$$

note that $h^0(E_W) = r+1$ if and only $H^0(E_W) \simeq W^*$, that is the following multiplication map is an isomorphism

$$\mu_W: W \otimes H^0(\omega_C) \to H^0(\omega_C \otimes O_C(p+rl)). \tag{21}$$

We have the following results:

Lemma 2.4.1. *Let E be a rank r vector bundle with $h^0(E) = \chi(E) = r+1$, which is globally generated , then E is stable.*

Proof. By Riemann Roch theorem we have $\deg(E) = 1 + rg$, and $\mu(E) = g + \frac{1}{r}$. Assume there exists a destabilizying subbundle $F \subset E$ with $rk(F) = s \leq r-1$ and $\mu(F) \geq g + \frac{1}{r}$. This implies $\deg(F) \geq 1 + sg$ and $\chi(F) \geq s+1$. Since E is generated by $r+1$ global sections spanning $H^0(E)$, then $h^0(F) = s+1$ and F is globally generated too. So we have a commutative diagramm

$$\begin{tikzcd}
0 \arrow[r] & (\det F)^{-1} \arrow[r] \arrow[d] & H^0(F) \otimes O_C \arrow[r, "e"] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \det E^{-1} \arrow[r] & H^0(E) \otimes O_C \arrow[r, "e"] & E \arrow[r] & 0,
\end{tikzcd} \tag{22}$$

from the inclusion $(\det F)^{-1} \hookrightarrow \det E^{-1}$ we have $sg+1 \geq rg+1$, which is impossible. This concludes the proof.

Assume that for a subspace W the map μ_W is an isomorphism, then by preceding lemma E_W is stable, so that $E_W(-l) = \xi \in SU(r,1)$. Moreover, (ξ, l) satisfies conditions (i),(ii) so that the map g_l is defined at the point ξ and we actually have $g_l(\xi) = W$. We would remark that the exact sequence

$$0 \to \xi(l)^* \to W \otimes O_C \to O_C(p + rl) \to 0 \tag{23}$$

is just the pull-back of the Euler sequence

$$0 \to \Omega_{\mathbf{P}^{r-1}}(1) \to W \otimes O_{\mathbf{P}^{r-1}} \to O_{\mathbf{P}^{r-1}}(1) \to 0 \tag{24}$$

under the morphism $f\colon C \to \mathbf{P}^{r-1} = \mathbf{P}(W^*)$ defined by $|W|$. Hence it turns out that

$$\xi(l) \simeq f^* T_{\mathbf{P}^{r-1}}(-1). \tag{25}$$

Let's define the following subsets of G_l:

$$B_l := \{W \in G_l \colon |W| \ \ has \quad base \ \ points \ \}\tag{26}$$

and D_l as the set of W such that the multiplication map

$$\mu_W \colon W \otimes H^0(\omega_C) \to H^0(\omega_C \otimes O_C(p + rl)) \tag{27}$$

is not surjective. Note that $\forall l$, we have $B_l \subset D_l$. Moreover, we have the following fact:

LEMMA 2.4.2. *For any $l \in \mathrm{Pic}^g(C)$, D_l is a Cartier divisor on G_l.*

Proof. For more details see also [B], th.(0.0.1).

There exists a homomorphism between vector bundles $\mu\colon \mathcal{G} \to \mathcal{F}$ such that at the point $W \in G_l$ is actually the multiplication map

$$\mu_W \colon W \otimes H^0(\omega_C) \to H^0(\omega_C \otimes O_C(p + rl)); \tag{28}$$

so that D_l is actually the degeneracy locus of μ. From Thom-Porteous's formula it is either a Cartier divisor or $D_l = G_l$. Actually we show that there exists $W \notin D_l$.

Claim: let $r \geq 1$, for any line bundle L of degree $rg+1$, base points free and non special, there exists a subspace $W \subset H^0(L)$ s.t. μ_W is surjective.

We will prove the claim by recurrence on r. Note that if $r = 1$, and L is a non special base points free line bundle of degree $g + 1$, the assertion follows from the base points free pencil trick, see [A-C-G-H]. Assume that the claim is true for degree $1+(r-1)g$. Let L be any line bundle of degree $1+rg$: choose $x_1,...,x_g \in C$ with the following properties:

i) $x_1+...+x_g$ is non special,

ii) $L(-x_1-...-x_g)$ is base points free and non special.

By the induction hypothesys there exists an r dimensional subspace $\bar{W} \subset H^0(L(-x_1-...-x_g))$ for which $\mu_{\bar{W}}$ is surjective. Choose a global section $s \in H^0(L)$, such that $s(x_i) \neq 0$, for $i=1,...g$, and define the subspace

$$W := \langle \bar{W}, s \rangle.$$

By i), we can find indipendent global sections ω_1, ω_2,... ,ω_g such that $\omega_i(x_j) \neq 0$ if and only if $j=i$; let $f_i = \mu_W(s \otimes \omega_i)$, then it is easy to see that $f_1,...,f_g$ are indipendent global sections of $H^0(L \otimes \omega_C)$. This implies the following commutative diagramm

$$\begin{array}{lccc} & \bar{W} \otimes H^0(\omega_C) & \xrightarrow{\mu_{\bar{W}}} & H^0(L(-x_1-..-x_g)\otimes\omega_C)) \\ & \downarrow & & \downarrow \\ (29) & W \otimes H^0(\omega_C) & \xrightarrow{\mu_W} & H^0(L\otimes\omega_C) \\ & \downarrow & & \downarrow \\ & < s > \otimes H^0(\omega_C) & \xrightarrow{\mu_s} & < f_1,..,f_g > \end{array}$$

Since both $\mu_{\bar{W}}$ and μ_s are surjective, we can conclude that μ_W is surjective too.

As an immediate consequence of the lemma we have that X_l is a non empty open subset of $SU(r,1)$: in fact if W is a point in $G_l - D_l$, then by the previous arguments $E_W(-l) = \xi \in X_l$. Moreover the map $h_l\colon G_l - D_l \to X_l$ sending W to $E_W(-l)$ is actually the inverse map of g_l.

2.5.

Let's consider the Pluecker embedding of the grassmannian G_l:

$$p_l\colon G_l(r+1, H^0(O_C(p+rl)) \hookrightarrow \mathbf{P}^N = \mathbf{P}(\wedge^{r+1}H^0(O_C(p+rl)) \tag{30}$$

and look at the composition map

$$p_l \cdot g_l\colon SU(r,1) \to \mathbf{P}^N, \tag{31}$$

we have the following result:

Proposition 2.5.1. *Let* $l \in \mathrm{Pic}^g(C)$,

(1) $g_l: SU(r,1) \to G_l(r+1, H^0(O_C(p+rl))$ *is a birational map, the restriction* $g_{l|X_l}: X_l \to G_l - D_l$ *is biregular;*

(2) *the rational map* $p_l \cdot g_l: SU(r,1) \to \mathbf{P}^N$ *is defined by* $N+1$ *indipendent global sections of* $H^0(\mathcal{L})$, *where* $\mathcal{L}$ *is the generalized theta divisor on* $SU(r,1)$.

Proof. (1) Let $l \in \mathrm{Pic}^g(C)$, note that we can identify the two moduli spaces $SU(r,1)$ and $SU(r, O_C(p+rl))$ via the natural isomorphism sending $\xi \to \xi(l)$. Let $\mathcal{U}_l$ be the universal bundle on $SU(r, O_C(p+rl)) \times C$, let as usual π_i, with $i = 1,2$, denote the natural projections. We recall that

$$\det \mathcal{U}_{l|SU(r,O_C(p+rl))\times x} \simeq \mathcal{L}, \tag{32}$$

moreover $\det \mathcal{U}_{l|\xi(l)\times C} \simeq O_C(p+rl)$, so that we can conclude that

$$\det \mathcal{U}_l \simeq {\pi_2}^* O_C(p+rl) \otimes {\pi_1}^* \mathcal{L}. \tag{33}$$

We will consider, on $SU(r, O_C(p+rl))$, the torsion free sheaf $\pi_{1*}\mathcal{U}_l$, whose fibre at the point ξ is $H^0(\xi(l))$. Let'consider the following open subset of $SU(r, O_C(p+rl))$

$$V_l := \{\xi(l) : h^0(\xi(l)) = r+1\}, \tag{34}$$

then $\pi_{1*}\mathcal{U}_{l|V_l}$ is a vector bundle of rank $r+1$. There is a natural map between sheaves on $SU(r, O_C(p+rl)) \times C$, see [H],

$$E: {\pi_1}^*(\pi_{1*}\mathcal{U}_l) \to \mathcal{U}_l, \tag{35}$$

let's consider the map $\wedge^r E$

$$\wedge^r E: {\pi_1}^*(\wedge^r \pi_{1*}\mathcal{U}_l) \to \wedge^r \mathcal{U}_l = \det \mathcal{U}_l, \tag{36}$$

and tensor this map with ${\pi_1}^*\mathcal{L}^{-1}$, so we have

$$ {\pi_1}^*(\wedge^r(\pi_{1*}\mathcal{U}_l) \otimes \mathcal{L}^{-1})) \to {\pi_2}^* O_C(p+rl). \tag{37}$$

Finally let's push down this map on $SU(r, O_C(p+rl))$, by using the projecting formula and recalling that $\pi_{1*}O_{SU(r,O_C(p+rl))\times C} \simeq O_{SU(r,O_C(p+rl))}$, we will have the following map

$$G: \wedge^r(\pi_{1*}\mathcal{U}_l) \otimes \mathcal{L}^{-1} \to \pi_{1*}{\pi_2}^* O_C(p+rl). \tag{38}$$

Note that $\pi_{1*}\pi_2{}^*O_C(p+rl)$ is the trivial bundle on $SU(r,O_C(p+rl))$ with fibre $H^0(O_C(p+rl))$, moreover at the point $\xi(l)$ G is actually the determinant map

$$w_\xi\colon \wedge^r H^0(\xi(l)) \to H^0(O_C(p+rl)). \tag{39}$$

If $g_l(\xi)$ is defined, then $(\operatorname{Im} G)_{\xi(l)} = g_l(\xi)$ and this shows that g_l is a rational map. Moreover, let $U_l \subset SU(r,O_C(p+rl))$ the set of points $\xi(l)$ satisfying properties (i) and (iii), then $X_l \subset U_l$ and the restriction $G_{|U_l}$ is an injection of vector bundle, and $\operatorname{codim} U_l \geq 2$.

Since $\dim SU(r,1) = \dim G_l = (r^2-1)(g-1)$, and moreover both $SU(r,1)$ and G_l are smooth and irreducible, then by Zariski's main theorem it is enough to show that $g_{l|X_l}$ is injective, but this follows from the preceding section.

(2) Since $\wedge^r(\pi_{1*}\mathcal{U}_l) \simeq \pi_{1*}\mathcal{U}_l{}^* \otimes \det(\pi_{1*}\mathcal{U}_l)$, $G_{|U_l}$ gives the following injection

$$(\pi_{1*}\mathcal{U}_l)^* \otimes \det(\pi_{1*}\mathcal{U}_l) \otimes \mathcal{L}^{-1} \hookrightarrow H^0(O_C(p+rl)) \otimes O_{SU(r,1)}, \tag{40}$$

which is actually the pull back of the universal subbundle $\mathcal{W}$ on G_l, via the map $g_{l|U_l}$. Since the Pluecker map p_l of G_l is defined by the line bundle $\det W^*$, we can conclude that

$$(p_l \cdot g_l)^*(O_{\mathbf{P}^N(1)}) \simeq \det(g_l{}^* W^*). \tag{41}$$

We will prove that actually $g_l{}^*W^* \simeq \pi_{1*}\mathcal{U}_l$ and $\det \pi_{1*}\mathcal{U}_l = \mathcal{L}$.

Let's consider again the natural map of sheaves

$$E\colon \pi_1{}^*(\pi_{1*}\mathcal{U}_l) \to \mathcal{U}_l, \tag{42}$$

the restriction at $\xi(l) \times C$ is actually the evaluation map: assume that $\xi(l) \in X_l$, then $E_{|\xi(l)\times C}$ is surjective and $(\ker E)_{|\xi(l)\times C} \simeq O_C(p+rl)^{-1}$. Let's consider the set $V \subset SU(r,O_C(p+rl)) \times C$ of pairs $(\xi(l),x)$ with $\xi(l) \in X_l$: we have

$$(\operatorname{Ker} E)_{|V} = \pi_2{}^*O_C(p+rl)^{-1} \otimes \pi_1{}^*B, \tag{43}$$

with $B \in \operatorname{Pic}(SU(r,O_C(p+rl)))$. Look at the following exact sequences on V:

$$0 \to \ker E_{|V} \to \pi_1{}^*(\pi_{1*}\mathcal{U}_l)_{|V} \to \mathcal{U}_{l|V} \to 0, \tag{44}$$

$$0 \to \mathcal{U}^*_{l|V} \to \pi_1{}^*(\pi_{1*}\mathcal{U}_l)^*_{|V} \to (\operatorname{Ker} E)^*_{|V} \to 0, \tag{45}$$

by pushing down to $SU(r, O_C(p+rl))$ we obtain an injective map Π:

$$\Pi\colon (\pi_{1*}\mathcal{U}_l)^*_{|V} \to \pi_{1*}(\operatorname{Ker} E)^*_{|V}, \tag{46}$$

where $\pi_{1*}(\operatorname{Ker} E)^*{}_{|V} = \pi_{1*}\pi_2{}^* O_C(p+rl) \otimes B$. Note that by construction, Π turns out to be the restriction to V of the above map G, so we can conclude that actually $B = O_{SU(r, O_C(p+rl))}$ and $\det \pi_{1*}\mathcal{U}_l = \mathcal{L}$, and this concludes the proof. □

As an immediate consequence we have an alternative proof of the following well known result, see [N]:

Proposition 2.5.2. *$SU(r,1)$ is a rational variety.*

§3. The main result

Let $\phi_{\mathcal{L}}\colon SU(r,1) \to |\mathcal{L}|^*$ be the map associated to the theta divisor. By prop. (2.5.1) there exist s_0, ... , s_N, indipendent global sections of $H^0(\mathcal{L})$ which define the rational map $p_l \cdot g_l$. Let V be the subspace spanned by them, we have a natural inclusion $V \hookrightarrow H^0(\mathcal{L})$, which induces a linear projection

$$\pi_l\colon |\mathcal{L}|^* \to \mathbf{P}(V^*) = \mathbf{P}^N \tag{47}$$

such that $g_l = \pi_l \cdot \phi_{\mathcal{L}}$. This allows us to prove that for any curve C of genus $g \geq 2$, the map $\phi_{\mathcal{L}}\colon SU(r,1) \to \mathbf{P}^n$ has degree one.

3.1. Proof of theorem (0.0.2)

Actually, we will perform a non empty open subset $\mathcal{U}$ of $SU(r,1)$, such that the restriction of $\phi_{\mathcal{L}}$ to $\mathcal{U}$ is actually injective, moreover we will prove that the tangent map $d(\phi_{\mathcal{L}})_\xi$ at a point ξ of $\mathcal{U}$ is injective too.

Consider in $SU(r,1) \times \operatorname{Pic}^g(C)$ the set X containing pairs (ξ, l) satisfying property $(*)$. We will denote by

$$\mathcal{U} := p_1(X), \tag{48}$$

then $\mathcal{U}$ is a non empty open subset of $SU(r,1)$. First of all note that if $\xi \in \mathcal{U}$ the following set

$$\{l \in \operatorname{Pic}^g(C) \quad (\xi, l) \text{satisfies } (*)\} \tag{49}$$

is a non empty open subset of $\operatorname{Pic}^g(C)$. Now let ξ_1 and ξ_2 be any two points of $\mathcal{U}$: then there exists l such that $(\xi_i, l) \in U$, for $i = 1, 2$. For such an l, let $g_l \colon X_1 \to G_l$ be the rational map defined in (2.3), then by pr. (2.5.1) the restriction $g_{l|X_l}$ is biregular and both ξ_1 and ξ_2 are in X_l. Now assume that $\phi_{\mathcal{L}}(\xi_1) = \phi_{\mathcal{L}}(\xi_2)$. Since $g_l = \pi_l \cdot \phi_{\mathcal{L}}$, then we have $g_l(\xi_1) = g_l(\xi_2)$. But $g_{l|X_l}$ is injective, so we can conclude that $\xi_1 \simeq \xi_2$.

Assume now that $d(\phi_{\mathcal{L}})_\xi(v) = 0$ for a point $\xi \in \mathcal{U}$ and a tangent vector $v \in T_{SU(r,1),\xi}$. Let $l \in \mathrm{Pic}^g(C)$ such that $(\xi, l) \in U$: then consider the rational map g_l, the linear projection π_l is defined at $\phi_{\mathcal{L}}(\xi)$, so we have

$$(d\pi_l)_{\phi_{\mathcal{L}}(\xi)} \cdot (d\phi_{\mathcal{L}})_\xi = (dg_l)_\xi. \tag{50}$$

Since $\xi \in X_l$ and $g_{l|X_l}$ is biregular, then $(dg_l)_\xi(v) = 0$, hence $v = 0$, and $(d\phi_{\mathcal{L}})_\xi$ is injective. This concludes the proof.

3.2.

For $r \geq 3$, $\mathcal{U}$ may be a proper subset of $SU(r,1)$, that is there exist bundles ξ such that for any $l \in \mathrm{Pic}^g(C)$ we have $\xi(l) \notin X_l$.

Let E be a semistable bundle on C of rank r, for any $l \in \mathrm{Pic}^g(C)$ we have $h^0(E(l)) \geq max(0, \chi(E(l))$; actually there exists an open subset $U \subset \mathrm{Pic}^g(C)$ such that for $l \in U$ this value is constant, following Raynaud, let's denote it by $h^0(E(l_{gen}))$, (see [R]). If $r \leq 2$ or $r = 3$ and the curve is general, then Raynaud proved that for any bundle we have $h^0(E(l_{gen})) = max(0, \chi(E(l))$; for $r \geq 4$ he showed the existence of bundles which do not satisfy this property, we will call such bundles Raynaud bundles, see [R].

Let $\eta \in SU(r)$: for any non zero morphism $\lambda \in \mathrm{Hom}(\eta, C_p)$ the sheaf $\ker\lambda$ is actually a vector bundle on C with $\det\ker\lambda = O_C(-p)$:

$$0 \to \ker\lambda \to \eta \xrightarrow{\lambda} C_p \to 0. \tag{51}$$

We claim that if η is stable then $\ker\lambda$ is stable too. In fact, if $\alpha \subset \ker\lambda \subset \eta$ is a destabilizying subbundle of $\ker\lambda$, then $\mu(\alpha) = \frac{d}{s} \geq \frac{-1}{r}$, with $s \leq r-1$: this implies $d \geq 0$ and contradicts the stability of η. Let's define

$$\xi := \ker\lambda^*, \tag{52}$$

we can conclude that $\xi \in SU(r, O_C(p))$, and fits into the exact sequence

$$0 \to \eta^* \to \xi \xrightarrow{v} C_p \to 0. \tag{53}$$

In the above notations, we can prove the following fact:

Lemma 3.2.1. *If $\eta^* \in SU(r)$ is a stable Raynaud bundle, then for any $l \in \mathrm{Pic}^g(C)$, we have $\xi(l) \notin X_l$.*

Proof. Consider the exact sequence (53) and tensor with $l \in \mathrm{Pic}^g(C)$,

$$0 \to \eta^*(l) \to \xi(l) \xrightarrow{v_l} C_p \to 0, \tag{54}$$

passing to cohomology, we can consider the following commutative diagramm

$$\begin{tikzcd} 0 \arrow[r] & H^0(\eta^*(l)) \arrow[r] \arrow[d] & H^0(\xi(l)) \arrow[r, "\bar{v_l}"] \arrow[d, "e_p"] & C \arrow[d] & \\ & (\eta^*(l))_p \arrow[r] & \xi(l)_p \arrow[r, "\bar{v_{l,p}}"] & C_p \arrow[r] & 0 \end{tikzcd} \tag{55}$$

Since η^* is a Raynaud bundle, then $h^0(\eta^*(l)) \geq r+1$ for any $l \in \mathrm{Pic}^g(C)$, this implies that either $h^0(\xi(l)) \geq r+2$ for any $l \in \mathrm{Pic}^g(C)$, or $h^0(\xi(l)) = r+1$ for l generic, and moreover $\bar{v_l}$ is the zero map. In this case, $\mathrm{Im}\, e_p \subset \mathrm{Ker}(\bar{v_{p,l}})$ for any l, which implies that $\xi(l)$ is not globally generated at p for any l. So we can conclude that $\xi \notin X_l$ for any $l \in \mathrm{Pic}^g(C)$, and $\mathcal{U}$ is a proper subset of $SU(r, O_C(p))$. ∎

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