Nihonkai Math. J.
Vol.5 (1994), 33-42

Structure of certain solvable j-algebras

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Abstract. In this paper we study the stability subgroups of certain solvable Lie groups with respect to the coadjoint action in connection with j-algebras. For this aim we generalize Piatetskii-Shapiro's theory on normal (split solvable) j-algebras. We prove the connectedness of the stabilizers for certain solvable j-algebras. In the last section we give an example of j-simple solvable j-algebras which satisfy our assumtion (1.1) and have rank > 1. Such phenomena do not occur for solvable j-algebras of exponential type which were already treated by I.I.Piatetskii-Shapiro and H.Fujiwara.

1. Introduction and Main Results

We find many literatures which treat the j-algebras in connection with the homogeneous Kähler manifolds (e.g.[2],[3],[8],[11],[14]) or the holomorphically induced unitary representations of Lie groups(e.g. [10],[1],[4],[7],[9],[13]). In this paper we study solvable j-algebras satisfying the condition (1.1), which is given in Theorem 1. Our main motivation to study these j-algebras is to generalize R.Penney's theorem on exponential solvable j-algebras [10],Theorem 2. We believe that our structure theorem is useful to achieve this aim.

Definition. Suppose that $\omega : \mathfrak{g} \to \mathbf{R}$ is a linear functional on a finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbf{R}$. Denote its complexification $\omega^{\mathbf{C}} : \mathfrak{g}^{\mathbf{C}} \to \mathbf{C}$ by the same letter ω. Suppose that $\mathfrak{h}$ is a complex Lie subalgebra of $\mathfrak{g}^{\mathbf{C}}$. The algebra $\mathfrak{h}$ is said to be an *algebraic polarization* of $\mathfrak{g}$ at ω if the following conditions are fulfilled:
i) $\omega([Z_1, Z_2]) = 0$ for every $Z_1, Z_2 \epsilon \mathfrak{h}$. ii) If $Z_0 \epsilon \mathfrak{g}^{\mathbf{C}}$ satisfies $\omega([Z_0, Z]) = 0$ for every $Z \epsilon \mathfrak{h}$, then Z_0 is an element of $\mathfrak{h}$. iii)$\mathfrak{h} + \overline{\mathfrak{h}}$ is a Lie subalgebra of $\mathfrak{g}^{\mathbf{C}}$. An algebraic polarization $\mathfrak{h}$ is said to be *totally complex* if the condition iv) $\mathfrak{h} + \overline{\mathfrak{h}} = \mathfrak{g}^{\mathbf{C}}$ is satisfied. An algebraic polarization $\mathfrak{h}$ at ω is said to be *positive* if v)$\sqrt{-1}\omega([Z, \overline{Z}]) \geq 0$ holds for every $Z \epsilon \mathfrak{h}$. Denote by G the connected, simply connected Lie group with Lie algebra $\mathfrak{g}$. Denote by G_ω the stabilizer of ω,i.e., $G_\omega = \{g \epsilon G : \omega(Ad(g)(X)) = \omega(X) \quad \text{for every} X \epsilon \mathfrak{g}\}$ and by $\mathfrak{g}_\omega$ the Lie algebra of G_ω,i.e.,$\mathfrak{g}_\omega = \{X \epsilon \mathfrak{g} : \omega([X,Y])=0 \text{ for every } Y \epsilon \mathfrak{g}\}$.

We remark that an algebraic polarization $\mathfrak{h}$ of $\mathfrak{g}$ at ω is necessarily globally invariant under the adjoint action of the principal connected component $(G_\omega)_0$ of the stabilizer G_ω. In §3, we give an example of algebraic polarizations which are not glabally invariant under the stabilizer. In the case $\mathfrak{g}$ is an exponential solvable Lie algebra, G_ω is always connected for every $\omega\epsilon\widehat{\mathfrak{g}}$, where $\widehat{\mathfrak{g}}$ is the dual space of the real vector space $\mathfrak{g}$. The following question is crucial.

QUESTION. *Suppose that $\mathfrak{h}$ is a totally complex positive algebraic polarization of $\mathfrak{g}$ at ω.*
A)Whether the stabilizer G_ω is connected, or not?
B)Whether $\mathfrak{h}$ is globally invariant under the stabilizer, or not ?

It is known that the answer to the question A) is affirmative for the case $\mathfrak{g}$ is a semi-simple Lie algebra(cf.[2],[8],[14]). In Theorem 1 we give an affirmative answer to the question A) for the system $\{\mathfrak{g},\mathfrak{h},\omega\}$ where $\mathfrak{g}$ is a solvable Lie algebra and $\{\mathfrak{h},\omega\}$ satisfies the condition (1.1) $\omega([\mathfrak{g},\mathfrak{g}]\cap\mathfrak{g}_\omega)=\{0\}$. The most crucial idea to prove this is concentrated on Proposition 1. Roughly speaking, after the proof of Proposition 1, for such systems we may use the arguments analogous to those for normal or exponential solvable j-algebras $\{\mathfrak{g},j,\omega\}$ with $\omega([jX,X])>0$ for $X\epsilon\mathfrak{g}, X\neq 0$ (cf.[11],[4]). But some new phenomena occur for our systems. You shall understand this quickly from Example 2 in §3.

THEOREM 1. *Suppose that G is a connected, simply-connected solvable Lie group with Lie algebra $\mathfrak{g}$ and that $\mathfrak{h}$ is a totally complex positive algebraic polarization at $\omega\epsilon\widehat{\mathfrak{g}}$. If the condition*

$$\omega([\mathfrak{g},\mathfrak{g}]\cap\mathfrak{g}_\omega)=\{0\} \tag{1.1}$$

is satisfied, then the stabilizer G_ω is necessarily connected.

Proposition 1. Suppose that $\mathfrak{g}$ is a solvable Lie algebra over $\mathbf{R}$ with $1\leq dim\mathfrak{g}<+\infty$ and $\mathfrak{h}$ is a totally complex positive algebraic polarization at $\omega\epsilon\widehat{\mathfrak{g}}$ and the condition

$$[\mathfrak{g},\mathfrak{g}]\cap\mathfrak{g}_\omega=\{0\} \tag{1.2}$$

is satisfied. Then every minimal ideal $\mathfrak{m}$ of $\mathfrak{g}$ is *one-dimensional.*

2.Decomposiotion of solvable j-algebras

In this section we prove Proposition 1 and develop a structure theory for solvable j-algebras satisfying the condition (1.1) or (1.2), and prove Theorem 1.

First we consider the meaning of condition (1.1). We assume that $\mathfrak{g}$ is a solvable Lie algebra and $\mathfrak{h}$ is a totally complex positive algebraic polarization at $\omega\epsilon\hat{\mathfrak{g}}$. Set $\mathfrak{g}_\omega=\{X : \omega([X,Y]) = 0 \text{ for every} Y\epsilon\mathfrak{g}\}$. Then the mapping $\widetilde{ad(X)} : Y + \mathfrak{g}_\omega(\epsilon\mathfrak{g}/\mathfrak{g}_\omega) \mapsto [X,Y] + \mathfrak{g}_\omega(\epsilon\mathfrak{g}/\mathfrak{g}_\omega)$ is complex diagonalizable for every $X\epsilon\mathfrak{g}_\omega$ (cf. [1],p.279). So the space $[\mathfrak{g},\mathfrak{g}]\cap\mathfrak{g}_\omega$ is an ideal of $\mathfrak{g}$. Set $\mathfrak{n} = \{X\epsilon[\mathfrak{g},\mathfrak{g}]\cap\mathfrak{g}_\omega : \omega(X) = 0\}$. Then $\mathfrak{n}$ is also an ideal of $\mathfrak{g}$ and satisfies

(2.1) $dim([\mathfrak{g},\mathfrak{g}]\cap\mathfrak{g}_\omega) \leq dim\ \mathfrak{n}+1$.

Every element $Z\epsilon[\mathfrak{g},\mathfrak{g}]\cap\mathfrak{g}_\omega$ satisfies $[Z,\mathfrak{g}] \subseteq \mathfrak{n}$. Set $\dot{\mathfrak{g}} = \mathfrak{g}/\mathfrak{n}$, $\dot{\mathfrak{h}} = \mathfrak{h}/\mathfrak{n}^{\mathbf{C}}$. Define a linear functional $\dot{\omega}$ on $\dot{\mathfrak{g}}$ by $\dot{\omega}(X+\mathfrak{n})=\omega(X)$ for $X\epsilon\mathfrak{g}$. Then the questions A) and B) in §1 for $\{\mathfrak{g},\mathfrak{h},\omega\}$ are reduced to those for $\{\dot{\mathfrak{g}}, \dot{\mathfrak{h}}, \dot{\omega}\}$. If the former satisfies the condition (1.1), then the latter satisfies (1.2). On account of this we restrict our attension to solvable j-algebras satisfying the condition (1.2).

Second we prove Proposition 1.

Proof of Proposition 1 I).Suppose that $\mathfrak{m}$ is a minimal ideal of $\mathfrak{g}$. By the assumption $\mathfrak{h}+\bar{\mathfrak{h}} = \mathfrak{g}^{\mathbf{C}}$, for every $Y\epsilon\mathfrak{g}$ there exists an element X for which $X-\sqrt{-1}Y\epsilon\mathfrak{h}$. Define a linear subspace $\mathfrak{a}$ of $\mathfrak{g}$ by $\mathfrak{a} =\{X : X-\sqrt{-1}Y\epsilon\mathfrak{h}$ for some $Y\epsilon\mathfrak{m}\}$.

$\mathfrak{m}\cap[\mathfrak{g},\mathfrak{g}] \neq \mathfrak{m}$,then by the minimality of $\mathfrak{m}$, $\mathfrak{m}\cap[\mathfrak{g},\mathfrak{g}] = \{0\}$ and hence $[\mathfrak{m},\mathfrak{g}] \subseteq \mathfrak{m}\cap[\mathfrak{g},\mathfrak{g}] = \{0\}$, $dim\ \mathfrak{m} = 1$. Therefore we may assume that $\mathfrak{m} = [\mathfrak{m},\mathfrak{g}]$. II). Under this assumption we shall show the following : α)The strict inequality (2.2) $0 < \omega([Y,X]) = -\omega([X,Y])$ holds for $Y\epsilon\mathfrak{m}$,$Y \neq 0$ and $X\epsilon\ \mathfrak{a}$ for which $X-\sqrt{-1}Y\epsilon\mathfrak{h}$ and β)The equation (2.3) $[X_1,Y_2]+[Y_1,X_2] = 0$ holds for $Y_1,Y_2\epsilon\mathfrak{m}$ and $X_1,X_2\epsilon\mathfrak{a}$ for which $X_1-\sqrt{-1}Y_1\epsilon\mathfrak{h}$, $X_2-\sqrt{-1}Y_2\epsilon\mathfrak{h}$. First we show α). By the positivity of $\mathfrak{h}$, we have $2\omega([Y,X]) = \sqrt{-1}\omega([X-\sqrt{-1}Y, X+\sqrt{-1}Y]) \geq 0$. If the equality holds in this, then $X-\sqrt{-1}Y$ is an element of $\mathfrak{g}_\omega^{\mathbf{C}} = \mathfrak{h}\cap\bar{\mathfrak{h}}$ and hence $Y\epsilon\mathfrak{m}\cap\mathfrak{g}_\omega \subseteq [\mathfrak{g},\mathfrak{g}]\cap\mathfrak{g}_\omega = \{0\}$ which contradicts the assumption on Y. Therefore the strict inequality (2.2) holds. Second we prove β). Since $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}^{\mathbf{C}}$ and $\mathfrak{m}$ is abelian, $[X_1,X_2]-\sqrt{-1}\{[X_1,Y_2]+[Y_1,X_2]\}$ belongs to $\mathfrak{h}$. Since $[X_1,Y_2]+[Y_1,X_2]$ belongs to the minimal ideal $\mathfrak{m}$ of $\mathfrak{g}$, the equation $\omega([[X_1,X_2],[X_1,Y_2]+[Y_1,X_2]]) = \omega(0) = 0$ holds.

Then, by virtue of the assertion α), we have the equation (2.3).

III). By using α), β), we show that $dim\ \mathfrak{m} = 1$. We use a reduction to absurdity. By S.Lie's theorem, $\mathfrak{m}$ is abelian and $dim\ \mathfrak{m}$ is 1 or 2. We assume $dim\ \mathfrak{m} = 2$. Then there exists a basis $\{R_1, R_2\}$ of $\mathfrak{m}$ for which the following equations hold for every $T \epsilon \mathfrak{g}$:

$$(2.4) \qquad \begin{cases} [T, R_1] = \psi_1(T)R_1 + \psi_2(T)R_2 \\ [T, R_2] = -\psi_2(T)R_1 + \psi_1(T)R_2. \end{cases}$$

In the above ψ_1, ψ_2 are linear functionals on $\mathfrak{g}$. We set $\phi_1 = \psi_1|_{\mathfrak{a}}\ \phi_2 = \psi_2|_{\mathfrak{a}}$ First we assume that ϕ_1 and ϕ_2 are linearly dependent. We choose a linear functional $\phi \neq 0$ on $\mathfrak{a}$ so that $\phi_1 = \lambda\phi, \phi_2 = \mu\phi$ for some $\lambda, \mu \epsilon \mathbf{R}$. We choose real numbers a, b so that $(\lambda a - \mu b)R_1 + (\mu a + \lambda b)R_2 \epsilon (Ker\ \omega) \cap (\mathbf{R}R_1 + \mathbf{R}R_2)$ and $(a, b) \neq (0, 0)$. Then, by (2.4) we have the equation (2.5) $\omega([T, aR_1 + bR_2]) = \phi(T)\omega(\{\lambda a - \mu b\}R_1 + \{\mu a + \lambda b\}R_2) = 0$ for every $T \epsilon \mathfrak{a}$, which contradicts the inequality (2.2). Second we assume that ϕ_1 and ϕ_2 are lineary independent. Then there exist elements Q_1, Q_2 of $\mathfrak{a}$ for which $[Q_1, R_1] = R_2, [Q_1, R_2] = -R_1, [Q_2, R_1] = R_1, [Q_2, R_2] = R_2$. Every element P of $\mathfrak{a}$ is uniquely written as $P = mQ_1 + wQ_2 + \widetilde{P}$ where $m, w \epsilon \mathbf{R}$ and $[\widetilde{P}, R_1] = [\widetilde{P}, R_2] = 0$. We choose $P_1, P_2 \epsilon \mathfrak{a}$ so that $P_1 - \sqrt{-1}R_1 \epsilon \mathfrak{h}$, $P_2 - \sqrt{-1}R_2 \epsilon \mathfrak{h}$. Then there exist real numbers s, t, u, v for which $P_1 = sQ_1 + tQ_2 + \widetilde{P_1}, P_2 = uQ_1 + vQ_2 + \widetilde{P_2}$. Then by (2.3) we have the equation $[P_1, R_2] + [R_1, P_2] = -(s + v)R_1 + (t - u)R_2 = 0$. Hence $v = -s, t = u$. Therefore we have $\sqrt{-1}\omega([P_1 - \sqrt{-1}R_1, P_1 + \sqrt{-1}R_1]) = -2\omega([P_1, R_1]) = -2\{t\omega(R_1) + s\omega(R_2)\}$ and $\sqrt{-1}\omega([P_2 - \sqrt{-1}R_2, P_2 + \sqrt{-1}R_2]) = -2\omega([P_2, R_2]) = -2\{v\omega(R_2) - u\omega(R_1)\} = -2\{-t\omega(R_1) - s\omega(R_2)\}$. Since R_1 and R_2 are non-zero elements of $\mathfrak{m}$, the inequality (2.2) implies the following two strict inequalities (2.6) $0 < -2\omega([P_1, R_1]) = -2t\omega(R_1) - 2s\omega(R_2)$, and (2.7) $0 < -2\omega([P_2, R_2]) = 2t\omega(R_1) + 2s\omega(R_2)$. Obviously (2.6) and (2.7) are not compatible. Thus we proved the proposition.

Third, to an arbitrary solvable j-algebra $\mathfrak{g}$ satisfying (1.2), we give its decomposition $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}'$ like as the decomposition (9) of [11],p.55. We take a minimal (1-dimensinal) ideal $\mathfrak{r}_1 = \mathbf{R}R_1$ of $\mathfrak{g}$ contained in $[\mathfrak{g}, \mathfrak{g}]$ for which $\omega(R_1) > 0$. Multiplying R_1 by a suitable positive number, we may assume that there exists an element A_1 of $\mathfrak{a}$ for which $A_1 + \sqrt{-1}R_1 \epsilon \mathfrak{h}$ and $[A_1, R_1] = R_1$. We set $\mathfrak{u} = \{X \epsilon \mathfrak{g} : \omega([X, R_1]) = \omega([X, A_1]) = 0\}$

$=\{X\epsilon\mathfrak{g} : [X,R_1]=0,$ there exists $Y\epsilon\mathfrak{g}$ for which $X-\sqrt{-1}Y\epsilon\mathfrak{h}, [Y,R_1]=0\}$.
Then as a vector space $\mathfrak{g}$ is the direct sum of $\mathfrak{r}_1, \mathfrak{u}$ and $\mathbf{R}A_1$. By arguments analogous to those in [11]p.53-55 (cf.[4]), we may prove the following :a) $\mathfrak{u}$ is globally invariant under $ad_{\mathfrak{g}}(A_1)$. b) $ad_{\mathfrak{u}}(A_1)$ is complex diagonalizable. c)Every eigenvalue λ of $(ad_{\mathfrak{u}}(A_1))^{\mathbf{C}}$ satisfies $\mathfrak{Re}\,(\lambda)=\frac{1}{2}$ or $\mathfrak{Re}\,(\lambda)=0$. For every $\lambda\epsilon\mathbf{C}$, we set $V(\lambda)=\{Z\epsilon\mathfrak{u}^{\mathbf{C}}: [A_1,Z]=\lambda Z\}$, $W(\lambda)=\{Z+\overline{Z} : Z\epsilon V(\lambda)\}$. We set $\mathfrak{z}_1=\sum\{W(\lambda) : \mathfrak{Re}\,(\lambda)=\frac{1}{2}\}$, $\mathfrak{g}'=\{W(\lambda) : \mathfrak{Re}\,(\lambda)=0\}$, and $\mathfrak{g}_1=\mathbf{R}A_1+\mathfrak{r}_1+\mathfrak{z}_1$. The following holds:d) $\mathfrak{g}_1$ and $\mathfrak{g}'$ are Lie subalgebras of $\mathfrak{g}$.
e) $[R_1,\mathfrak{g}']=\{0\}$. f) $\mathfrak{g}'\supseteq W(0)\supseteq\mathfrak{g}_\omega$. g) $\mathfrak{h}'=\mathfrak{h}\cap\{\mathfrak{g}'\}^{\mathbf{C}}$ is a totally complex algebraic polarization of $\mathfrak{g}'$ at $\omega'=\omega|_{\mathfrak{g}'}$. h) $(\mathfrak{g}')_{\omega'}=\mathfrak{g}_\omega$. i) $(\mathfrak{g}')_{\omega'}\cap[\mathfrak{g}',\mathfrak{g}']=\{0\}$. j) $[A_1,\mathfrak{h}']\subseteq\mathfrak{h}'$.
In contrast to the case $\mathfrak{g}$ is an exponential j-algebra, the algebra $\mathfrak{g}_1$ *is not necessarily an ideal* of $\mathfrak{g}$ (cf.Example 2 in §3).
If $\mathfrak{g}'$ is abelian,then $\mathfrak{g}'=\mathfrak{g}_\omega$. We assume that $\mathfrak{g}'$ is not abelian. Then,by Proposition 1 there exist a 1-dimensional ideal $\mathfrak{r}_2=\mathbf{R}R_2$ of $\mathfrak{g}'$ and an element A_2 of $\mathfrak{g}'$ for which $[A_2,R_2]=R_2, A_2+\sqrt{-1}R_2\epsilon\mathfrak{h}$ and $\omega(R_2)>0$. In the case $\mathfrak{g}$ is an exponential j-algebra it is trivial by the definition of $\mathfrak{g}'$ that the equation $[A_1,R_2]=0$ holds. But this equation is not trivial for our systems. Fourth we shall prove this equation in a generalized fashion.

Proposition 2.Let $\{\mathfrak{g},\mathfrak{h},\omega\}$ be a system satisfying the condition (1.2) and let $\mathfrak{g}'$,$\mathfrak{a}$, $\mathfrak{b}$ be Lie subalgebras of $\mathfrak{g}$ such that $\mathfrak{a}$, $\mathfrak{b}$ are abelian and let A_0, R_0 be in $\mathfrak{g}'$. Assume the following conditions: i) $\mathbf{R}R_0$ is a 1-dimensional ideal of $\mathfrak{g}'$. ii) $A_0+\sqrt{-1}R_0\epsilon\mathfrak{h}$. iii) $[\mathfrak{a},\mathfrak{g}']\subseteq\mathfrak{g}'$. iv) $[A_0,R_0]=R_0$. v) $(ad_{\mathfrak{g}'}(A))^{\mathbf{C}}$ is diagonalizable and its eigenvalues are contained in $\sqrt{-1}\mathbf{R}$ for every $A\epsilon\mathfrak{a}$. vi) $[\mathfrak{a},\mathfrak{h}\cap(\mathfrak{g}')^{\mathbf{C}}]\subseteq\mathfrak{h}\cap(\mathfrak{g}')^{\mathbf{C}}$. vii) $[\mathfrak{b},\mathfrak{g}']=\{0\}$. viii)For every $A\epsilon\mathfrak{a}$ there exists $R\epsilon\mathfrak{b}$ for which $A+\sqrt{-1}R\epsilon\mathfrak{h}$.
Then $[A_0,\mathfrak{a}]=[R_0,\mathfrak{a}]=\{0\}$, that is, every element of *the subalgebra* $\mathbf{R}A_0+\mathbf{R}R_0$ *is fixed under the action of* $Ad(\exp[\mathfrak{a}])$.

Proof: First we take an arbitrary element A of $\mathfrak{a}$ and prove the equation $[A,R_0]=0$. We consider the 1-parameter group $\{\alpha_t : t\epsilon\mathbf{R}\}$ of automorphisms of $\mathfrak{g}'$ defined by $\alpha_t=\exp\{t\ ad_{\mathfrak{g}'}(A)\}$. By the condition iv) we have the relation

$$(2.8)\qquad [\alpha_t(A_0),R_0]\longrightarrow R_0 \text{ as } t\to 0 .$$

We set $\mathfrak{m}_t = \alpha_t(\mathbf{R}R_0)$.By the condition v), $\alpha_t^{\mathbf{C}}$ is diagonalizable and its eigenvalues are contained in $\{\mu\epsilon\mathbf{C} : |\mu| = 1\}$. So the conditon "$\alpha_t(R_0) = R_0$ for every $t\epsilon\mathbf{R}$" is equivalent to "$\mathfrak{m}_t = \mathfrak{m}_0$ for every $t\epsilon\mathbf{R}$". We assume that this condition does not hold. Then, for every positive number t_1 there exists a positive number t, $0 < t < t_1$, such that $\mathfrak{m}_t \neq \mathfrak{m}_0$. We take such t. Since $\mathfrak{m}_t, \mathfrak{m}_0$ are different 1-dimensional ideals of $\mathfrak{g}'$, we have the relations, $[\,[A_0, \alpha_t(A_0)], \mathfrak{m}_t + \mathfrak{m}_0] = \{0\}$, $[R_0, \alpha_t(R_0)] = 0$. Therefore $[\,[A_0, \alpha_t(A_0)],\ [A_0, \alpha_t(R_0)] - [\alpha_t(A_0), R_0]\,] = 0$. By the conditions ii) and iv), $[A_0 + \sqrt{-1}R_0, \alpha_t(A_0) + \sqrt{-1}\alpha_t(R_0)] = [A_0, \alpha_t(A_0)] + \sqrt{-1}\{[A_0, \alpha_t(R_0)] - [\alpha_t(A_0), R_0]\}$ is an element of $\mathfrak{h}$. We set $P = [A_0, \alpha_t(A_0)]$, $Q = -[A_0, \alpha_t(R_0)] + [\alpha_t(A_0), R_0]$. Since $\mathfrak{h}$ is a positive algebraic polarization at ω, the equation $\omega([P - \sqrt{-1}Q, P + \sqrt{-1}Q]) = 2\sqrt{-1}\omega([P, Q]) = 0$ implies $P - \sqrt{-1}Q\epsilon(\mathfrak{g}_\omega)^{\mathbf{C}}$ and hence $Q\epsilon\mathfrak{g}_\omega \cap [\mathfrak{g}, \mathfrak{g}] = \{0\}$. Since R_0 and $\alpha_t(R_0)$ are linearly independent, this implies

(2.9) $$[\alpha_t(A_0), R_0] = 0 \quad \text{for some} 0 < t < t_1 ,$$

which contradicts (2.8). Thus we proved the equation $[A, R_0] = 0$.
Second we prove the equation $[A, A_0] = 0$ for $A\epsilon\mathfrak{a}$. By the conditions vii),viii) and the result above, $[A + \sqrt{-1}R, A_0 + \sqrt{-1}R_0] = [A, A_0]$ is an element of $\mathfrak{h} \cap \mathfrak{g} \cap [\mathfrak{g}, \mathfrak{g}] = \{0\}$, which proved the proposition.

Applying recursively Propositions 1 and 2 to a solvable j-algebra $\{\mathfrak{g}, \mathfrak{h}, \omega\}$ satisfying (1.2), we obtain a direct sum decomposition of $\mathfrak{g}$ analogous to that in [11]p.55:

(2.10) $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2 + \cdots + \mathfrak{g}_p + \mathfrak{g}_\omega$,

where (2.11) $\mathfrak{g}_k = \mathbf{R}R_k + \mathbf{R}A_k + \mathfrak{z}_k$ is an elementary exponential j-algebra and $[A_k, R_k] = R_k, \omega(R_k) > 0$, $\omega(\mathfrak{z}_k) = \{0\}(\ 1 \le k \le p)$, (2.12) $[R_k, \mathfrak{g}_s] = \{0\}, [A_k, R_s] = 0$ for $1 \le k < s \le p$, (2.13) $[\mathfrak{g}_s, \mathfrak{z}_k] \subseteq \mathfrak{z}_k$ for $1 \le k < s \le p$,(2.14) $[\mathfrak{g}_\omega, \mathbf{R}R_1 + \cdots + \mathbf{R}R_p + \mathbf{R}A_1 + \cdots + \mathbf{R}A_p] = \{0\}$, and (2.15) $[\mathfrak{g}_\omega, \mathfrak{z}_k] \subseteq \mathfrak{z}_k$ $(\ 1 \le k \le p)$. We set
(2.16) $\mathcal{F} = \{\phi\epsilon\hat{\mathfrak{g}} : \phi(\mathfrak{z}_k) = \{0\}(1 \le k \le p), \phi(R_1) \neq 0, \phi(R_2) \neq 0, \ldots, \phi(R_p) \neq 0\}$,
$\mathcal{F}_+ = \{\phi\epsilon\mathcal{F} : \phi(R_1) > 0, \phi(R_2) > 0, \ldots, \phi(R_p) > 0\}$. Obviously the functional ω is an element of $\mathcal{F}_+$. Theorem 1 is an immediate consequence of the following proposition.

Proposition 3. Suppose that $\{\mathfrak{g}, \mathfrak{h}, \omega\}$ is a system satisfying the condition (1.2) and ϕ is an element of $\mathcal{F}$. Denote by G the connected, simply connected solvable Lie group

with Lie algebra $\mathfrak{g}$ and by N the analytic subgroup of G correspondind to $[\mathfrak{g},\mathfrak{g}]$. Then the following stabilizers $G_{\phi_0} = \{g \epsilon G : \phi(Ad(g)(X)) = \phi(X)\ for\ X \epsilon [\mathfrak{g},\mathfrak{g}]\}$ and $N_{\phi_0} = \{g \epsilon N : \phi(Ad(g)(X)) = \phi(X)\ for\ X \epsilon [\mathfrak{g},\mathfrak{g}]\}$
satisfy the relation

$$G_{\phi_0} = \exp(\mathfrak{g}_\omega)\mathrm{N}_{\phi_0} \tag{2.17}$$

and hence *the stabiliser* G_ϕ *is connected* for every $\phi \epsilon \mathcal{F}$.

Proof: First we show that the equation (2.17) implies the connectedness of the stabiliser G_ϕ. It is obvious that $\exp(\mathfrak{g}_\omega)$ is a subgroup of G_ϕ and G_ϕ is a subgroup of G_{ϕ_0} . By (2.17) every element g of G_ϕ is written as $g = g_1 \cdot h$ where $g_1 \epsilon \exp(\mathfrak{g}_\omega) \subseteq G_\phi, h \epsilon N_{\phi_0} \cap G_\phi$. Since h is an element of the nilpotent Lie group N, the condition $h \epsilon G_\phi$ implies $h = \exp(X)$ for some $X \epsilon [\mathfrak{g},\mathfrak{g}]$ satisfying $\phi([X,Y]) = 0$ for every $Y \epsilon \mathfrak{g}$. Therefore G_ϕ is connected.
Second we prove the relation (2.17). It is obvious that the inclusion $G_{\phi_0} \supseteq \exp(\mathfrak{g}_\omega)N_{\phi_0}$ holds. We prove the inverse inclusion. We define a linear functional R_k^* on $[\mathfrak{g},\mathfrak{g}]$ by the relations $R_k^*(\mathfrak{z}_s) = \{0\}$ $(1 \le k, s \le p), R_k^*(R_k) = 1, R_k^*(R_s) = 0$ $(1 \le k \ne s \le p)$. For every $g \epsilon G$, we define a transformation $Ad(g)^*$ on $\widehat{[\mathfrak{g},\mathfrak{g}]}$ by the equation $Ad(g)^*\ (\psi)(X) = \psi(Ad_{[\mathfrak{g},\mathfrak{g}]}(g)(X))$ for $X \epsilon [\mathfrak{g},\mathfrak{g}]$. We suppose that g is an element of G_{ϕ_0} for $\phi = a_1R_1^* + a_2R_2^* + \cdots + a_pR_p^*$ where $a_k's$ are real numbers with $a_1a_2\cdots a_p \ne 0$. Then the element g is written as $g = g_1\ \exp(t_1A_1 + +t_2A_2 + \cdots + t_pA_p)\, g_2$ where $g_1 \epsilon N$, $g_2 \epsilon \exp(\mathfrak{g}_\omega)$. Hence we have the equation (2.18) $Ad(g_1)^*(\phi) = Ad(g_2^{-1})^*\ Ad(exp\{-(t_1A_1 + t_2A_2 + \cdots + t_pA_p)\})^*(a_1R_1^* + a_2R_2^* + \cdots + a_pR_p^*) = a_1e^{-t_1}R_1^* + a_2e^{-t_2}R_2^* + \cdots + a_pe^{-t_p}R_p^*$. Next we prove that if $n \epsilon N$ satisfies the equation (2.19) $Ad(n)^*(\phi) = a_1c_1R_1^* + a_2c_2R_2^* + \cdots + a_pc_pR_p^*$ for some real numbers c_k's, then all c_k's are equal to 1. We decompose n as $n = \exp(\alpha R_1)exp(Z_1)n_2$ where $Z_1 \epsilon \mathfrak{z}_1 \subseteq \mathfrak{g}_1$ and $n_2 \epsilon \exp([\mathfrak{g}_2 + \mathfrak{g}_3 + \cdots + \mathfrak{g}_p] \cap [\mathfrak{g},\mathfrak{g}])$. Since R_1 is an central element of $[\mathfrak{g},\mathfrak{g}]$, $Ad(exp(Z_1)n_2)^*(\phi) = a_1c_1R_1^* + a_2c_2R_2^* + \cdots + a_pc_pR_p^*$. By (2.11),(2.12) and (2.13), $Ad(\exp\ Z_1)^*\ (\phi)$ is written as $a_1R_1^* + \eta$ where $\eta(R_1) = 0$ and $Ad(n_2^{-1})^*\ (a_1c_1R_1^* + a_2c_2R_2^* + \cdots + a_pc_pR_p^*)$ is written as $a_1c_1R_1^* + a_2c_2R_2^* + \xi$ where $\xi(R_1) = \xi(R_2) = 0$, $\xi(\mathfrak{z}_1) = \{0\}$, and $Ad(\exp\ Z_1)^*(a_2R_2^* + \cdots + a_pR_p^*) = a_2R_2^* + \cdots + a_pR_p^*$. Therefore we have the equations $c_1 = 1$, and $R_1^*(Ad(\exp\ Z_1)(Y_1))$

$=R_1^*(Y_1 + [Z_1,Y_1] + \frac{1}{2}[Z_1,[Z_1,Y_1]] + \cdots) = R_1^*(Y_1) + R_1^*([Z_1,Y_1]) = R_1^*(Y_1)$ for every $Y_1 \in \mathfrak{z}_1$, which implies $\omega([Z_1,\mathfrak{z}_1]) = \{0\}$ and hence $Z_1 = 0$. So the equation (2.19) implies $Ad(n_2)^*(a_2R_2^* + \cdots + a_pR_p^*) = a_2c_2R_2^* + \cdots + a_pc_pR_p^*$. By inductive arguments we obtain the conclusion $c_1 = c_2 = \cdots = c_p = 1$. Taking account of (2.18) we find that $G_{\phi_0} \subseteq N_{\phi_0}\exp(\mathfrak{g}_\omega) = \exp(\mathfrak{g}_\omega)N_{\phi_0}$.

3.Some Examples

In this section we give some examples to understand the significance of Question B) and the difference between exponential and non-exponential solvable j-algebras.

Example 1(communicated by H.Fujiwara[5]) An example of positive algebraic polarizations which are not globally invariant under the stabiliser. Suppose that $\mathfrak{g}$ is a 6-dimensional nilpotent Lie algebra with a basis $\{T, X_1, X_2, Z, Y_1, Y_2\}$ satisfying $[T,X_1] = X_2, [T,X_2] = -X_1, [T,Y_1] = 4Y_2, [T,Y_2] = -4Y_1, [X_1,X_2] = Z, [T,Z] = [Y_1,Y_2] = 0, [X_j,Y_k] = [X_j,Z] = [Y_k,Z] = 0 (1 \leq j,k \leq 2)$. Denote by $\{T^*, X_1^*, \ldots\}$ the dual basis of $\{T, X_1, \ldots\}$ .Set $\omega = Z^* + Y_1^*$, $\mathfrak{h} = \mathbf{C}Z + \mathbf{C}Y_1 + \mathbf{C}Y_2 + \mathbf{C}X_2$. Then $\mathfrak{g}_\omega = \mathbf{R}Z + \mathbf{R}Y_1$ and the abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{\mathbf{C}}$ is a positive algebraic polarization at ω. But $Ad(exp[\frac{\pi}{2}T])(\mathfrak{h}) \neq \mathfrak{h}$ for the element $exp[\frac{\pi}{2}T]$ of G_ω

Example 2.An example of solvable j-algebras of rank 2 which have no non- trivial Kähler ideal. Suppose that $\mathfrak{g}$ is a 10-dimensional solvable Lie algebra with a basis $\{A, B, X, Y, Z, E_1, E_2, E_3, E_4, E_5\}$ satisfying the following relations : $[A, E_1] = \frac{1}{2}E_1, [A, E_3] = \frac{1}{2}E_3$, $[A, E_2] = \frac{1}{2}E_2 + E_4, [A, E_4] = -E_2 + \frac{1}{2}E_4, [A, E_5] = E_5$, $[A, X] = Y, [A, Y] = -X, [A, Z] = 0$, $[B, E_1] = E_1, [B, E_3] = -E_3, [B, E_2] = [B, E_4] = [B, E_5] = 0, [B, X] = X, [B, Y] = Y$, $[B, Z] = 2Z, [A, B] = 0, [X, E_2] = E_1, [X, E_3] = -E_4, [X, E_1] = [X, E_4] = [X, E_5] = 0$, $[Y, E_3] = E_2, [Y, E_4] = E_1, [Y, E_1] = [Y, E_2] = 0$, $[Y, E_5] = 0, [Z, E_3] = 2E_1, [Z, E_1] = [Z, E_2] = [Z, E_4] = [Z, E_5] = 0, [X, Y] = Z$, $[E_1, E_3] = [E_2, E_4] = E_5$, $[X, Z] = [Y, Z] = [E_1, E_2] = [E_1, E_4] = [E_2, E_3] = [E_3, E_4] = 0$, $[E_1, E_5] = [E_2, E_5] = [E_3, E_5] = [E_4, E_5] = 0$.

Define $\omega \in \hat{\mathfrak{g}}$ by the relations $\omega(\mathbf{R}E_1 + \mathbf{R}E_2 + \mathbf{R}E_3 + \mathbf{R}E_4 + \mathbf{R}X + \mathbf{R}Y) = 0$, $\omega(E_5) > 0, \omega(Z) > 0$. ($\omega(A)$ and $\omega(B)$ are arbitrary real numbers.)Define $j : \mathfrak{g} \to \mathfrak{g}$ by the relations $j(A) = -E_5, j(E_5) = A, j(B) = -Z, j(Z) = B, j(E_1) = -E_3, j(E_3) = E_1$

$j(E_2) = -E_4, j(E_4) = E_2, j(X) = -Y, j(Y) = X$.

Set $\mathfrak{h}=\{X - \sqrt{-1}j(X) : X \in \mathfrak{g}\}$. Then $\mathfrak{h}$ is a totally complex positive polarization at ω and $\mathfrak{h} \cap \bar{\mathfrak{h}} = \{0\}$. The space $\mathbf{R}A + \mathbf{R}B$ is orthogonal to $[\mathfrak{g}, \mathfrak{g}]$ with respect to the inner product $<,>:< V, W >= \omega([jV, W])$. The algebra $\mathfrak{g}$ has no non-trivial j-invariant ideal.

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Received September 13, 1993, Revised February 3, 1994