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QUASINORMALITY AND FUGLEDE-PUTNAM THEOREM FOR CLASS $A(s,t)$ OPERATORS

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ABSTRACT. We investigate several properties of Aluthge transform $T(s,t) = |T|^s U|T|^t$ of an operator $T = U|T|$. We prove (1) if T is a class $A(s,t)$ operator and $T(s,t)$ is quasi-normal (resp., normal), then T is quasi-normal (resp., normal), (2) if T is a contraction with $\ker T = \ker T^2$ and $T(s,t)$ is a partial isometry, then T is a quasinormal partial isometry, (3) if T is paranormal and $T(s,t)$ is a partial isometry, then T is a quasinormal partial isometry, and (4) Fuglede-Putnam type theorem holds for a class $A(s,t)$ operator T with $s+t \leq 1$ if T satisfies a kernel condition $\ker T \subset \ker T^*$.

1. INTRODUCTION

Let $\mathcal{H}$ be a complex Hilbert space and $T = U|T|$ be the polar decomposition of a bounded linear operator $T \in B(\mathcal{H})$. An operator T is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$, where $p > 0$. In paticular, 1-hyponormal operators and 1/2-hyponormal operators are hyponormal and semi-hyponormal operators. It is known that hyponormal operators and semi-hyponormal operators enjoy some nice properties. In [1], Aluthge extended the class of hyponormal operators by introducing p-hyponormal operators and obtained some properties with the help of the transform $T(1/2,1/2) = |T|^{1/2}U|T|^{1/2}$, which now known as the Aluthge transform. The introduction of these operators by Aluthge has inspired many researchers not only to expose some important properties of p-hyponormal operators but also to introduce the number of its extensions ([2, 7, 10, 17, 23]). In this endeavor, the Aluthge transform and more generally, the generalized Aluthge transform defined as $T(s,t) = |T|^s U|T|^t$ with $s,t > 0$, have been proved to be important tools. In the present article, we investigate class $A(s,t)$ operators with the help of the generalized Aluthge transform. According to [7, 10, 11], an operator T is defined to be a class $A(s,t)$ operator if

$$|T(s,t)|^{\frac{2t}{s+t}} \geq |T|^{2t} \text{ or } \left(|T^*|^t|T|^{2s}|T^*|^t\right)^{\frac{t}{s+t}} \geq |T^*|^{2t},$$

where $s,t > 0$.

If T is p-hyponormal and $0 < q < p$, then T is q-hyponormal by Löwner-Heinz's inequality [9, 13]. If T is invertible and $\log(T^*T) \geq \log(TT^*)$, then T is said to be log-hyponormal. Invertible p-hyponormal operators are log-hyponormal, and p-hyponormal or log-hyponormal operators are class $A(s,t)$ operators for all $0 < s,t$.

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If T is a class $A(s,t)$ operator and $s \leq s', t \leq t'$, then T is a class $A(s',t')$ operator. T is called a class A operator if

$$|T^2| \geq |T|^2,$$

which means T is a class $A(1,1)$ operator. These classes are expanding for p, s, t and several authors investigated properties of these classes (see [10,11,15,17,18,23]).

We show in Section 2 that if T is a class $A(s,t)$ operator and its Aluthge transform $T(s,t)$ is quasinormal (resp. normal), then T is also quasinormal (resp. normal).

In section 3, we consider a partial isometry. Let $T = U|T|$ be a quasinormal partial isometry. Then $T(s,t) = U$, and hence $T(s,t)$ is a partial isomery. The converse does not hold in general. However we show that (1) if T is a contraction with $\ker T = \ker T^2$ and $T(s,t)$ is a partial isometry, then $T = T(s,t) = U$ and T is a quasinormal partial isometry, and (2) if T is paranormal and $T(s,t)$ is a partial isometry, then $T = T(s,t) = U$ and T is a quasinormal partial isometry.

Section 4 is devoted mainly to show that Fuglede-Putnam theorem holds for a class $A(s,t)$ operator T with $s+t=1$ if T satisfies a kernel condition $\ker T \subset \ker T^*$.

2. Quasinormality

Let $T = U|T|$ be the polar decomposition of $T \in B(\mathcal{H})$. T is said to be quasinormal if $|T|U = U|T|$, or equivalently, $TT^*T = T^*TT$. Patel [14] proved that if T is p-hyponormal and its Aluthge transform $T(1/2,1/2)$ is normal, then T is normal and $T = T(1/2,1/2)$. Aluthge and Wang [2] proved that if T is class $A(1/2,1/2)$, $\ker T \subset \ker T^*$ and its Aluthge transform $T(1/2,1/2)$ is normal, then T is normal and $T = T(1/2,1/2)$. The following is a generalization of these results.

Theorem 2.1. *Let T be a class $A(s,t)$ operator with the polar decomposition $T = U|T|$. If $T(s,t) = |T|^s U |T|^t$ is quasinormal, then T is also quasinormal. Hence T coinsides with its Aluthge transform $T(1/2,1/2) = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$.*

Proof. Since T is a class $A(s,t)$ operator,

$$|T(s,t)|^{\frac{2r}{s+t}} \geq |T|^{2r} \geq |T(s,t)^*|^{\frac{2r}{s+t}} \tag{2.1}$$

for all $r \in (0, \min\{s,t\}]$ by [11, Theorem 3] and Löwner-Heinz's inequality [9, 13]. Then Douglas's theorem [3] implies

$$[\text{ran } |T(s,t)|] = [\text{ran } |T|] \supset [\text{ran } |T(s,t)^*|] = [\text{ran } T(s,t)]$$

where $[\mathcal{M}]$ denotes the norm closure of $\mathcal{M}$. Let $T(s,t) = W|T(s,t)|$ be the polar decomposition of $T(s,t)$. Then

$$E := W^*W = U^*U \geq WW^* =: F.$$

Put

$$|T(s,t)^*|^{\frac{1}{s+t}} = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}, W = \begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}$$

on

$$\mathcal{H} = [\text{ran } T(s,t)] \oplus \ker T(s,t)^*.$$

Then X is injective and has a dense range. Since $T(s,t)$ is quasinormal, W commutes with $|T(s,t)|$ and

$$\begin{aligned}|T(s,t)|^{\frac{2r}{s+t}} &= W^*W|T(s,t)|^{\frac{2r}{s+t}} = W^*|T(s,t)|^{\frac{2r}{s+t}}W \\ &\geq W^*|T|^{2r}W \geq W^*|T(s,t)^*|^{\frac{2r}{s+t}}W = |T(s,t)|^{\frac{2r}{s+t}}.\end{aligned}$$

Hence

$$|T(s,t)|^{\frac{2r}{s+t}} = W^*|T(s,t)|^{\frac{2r}{s+t}}W = W^*|T|^{2r}W,$$

and

$$|T(s,t)^*|^{\frac{2r}{s+t}} = W|T(s,t)|^{\frac{2r}{s+t}}W^* = WW^*|T(s,t)|^{\frac{2r}{s+t}}WW^* \tag{2.2}$$

$$= WW^*|T|^{2r}WW^* = \begin{pmatrix} X^{2r} & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.3}$$

Since $WW^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, (2.1), (2.2) and (2.3) imply that $|T(s,t)|^{\frac{2r}{s+t}}$ and $|T|^{2r}$ are of the forms

$$|T(s,t)|^{\frac{2r}{s+t}} = \begin{pmatrix} X^{2r} & 0 \\ 0 & Y^{2r} \end{pmatrix} \geq |T|^{2r} = \begin{pmatrix} X^{2r} & 0 \\ 0 & Z^{2r} \end{pmatrix} \tag{2.4}$$

where

$$[\text{ran } Y] = [\text{ran } Z] = [\text{ran } |T|] \ominus [\text{ran } T(s,t)] = \ker T(s,t)^* \ominus \ker T.$$

Since W commutes with $|T(s,t)|$,

$$\begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}.$$

So $W_1X = XW_1$ and $W_2Y = XW_2$, and hence $[\text{ran } W_1]$ and $[\text{ran } W_2]$ are reducing subspaces of X. Since $W^*W|T(s,t)| = |T(s,t)|$, we have $W_1^*W_1 = 1$ and

$$\begin{aligned}X^k &= W_1^*W_1X^k = W_1{}^*X^kW_1, \\ Y^k &= W_2^*W_2Y^k = W_2{}^*X^kW_2.\end{aligned}$$

Put $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$. Then $T(s,t) = |T|^sU|T|^t = W|T(s,t)|$ implies

$$\begin{pmatrix} X^s & 0 \\ 0 & Z^s \end{pmatrix}\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} X^t & 0 \\ 0 & Z^t \end{pmatrix} = \begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X^{s+t} & 0 \\ 0 & Y^{s+t} \end{pmatrix}.$$

Hence

$$\begin{aligned}X^sU_{11}X^t &= W_1X^{s+t} = X^sW_1X^t, \\ X^sU_{12}Z^t &= W_2Y^{s+t} = X^{s+t}W_2\end{aligned}$$

and

$$\begin{aligned}X^s(U_{11} - W_1)X^t &= 0, \\ X^s(U_{12}Z^t - X^tW_2) &= 0.\end{aligned}$$

Since X is injective and has a dense range, $U_{11} = W_1$ is isometry and $U_{12}Z^t = X^t W_2$. Then

$$U^*U = \begin{pmatrix} U_{11}^*U_{11} + U_{21}^*U_{21} & U_{11}^*U_{12} + U_{21}^*U_{22} \\ U_{12}^*U_{11} + U_{22}^*U_{21} & U_{12}^*U_{12} + U_{22}^*U_{22} \end{pmatrix}$$

on $\mathcal{H} = [\text{ran } T(s,t)] \oplus \ker T(s,t)^*$ is the orthogonal projection onto $[\text{ran}|T|] \supset [\text{ran } T(s,t)]$, we have $U_{21} = 0$ and

$$U^*U = \begin{pmatrix} 1 & 0 \\ 0 & U_{12}^*U_{12} + U_{22}^*U_{22} \end{pmatrix}.$$

Since $U_{12}Z^t = X^t W_2$, we have

$$Z^{2t} \geq Z^t U_{12}{}^* U_{12} Z^t = W_2{}^* X^{2t} W_2 = Y^{2t},$$

and

$$Z^{2r} \geq (Z^t U_{12}{}^* U_{12} Z^t)^{\frac{r}{t}} = W_2{}^* X^{2r} W_2 = Y^{2r} \geq Z^{2r}$$

by Löwner-Heinz inequality and (2.4). Hence

$$(Z^t U_{12}{}^* U_{12} Z^t)^{\frac{r}{t}} = Z^{2r} = Y^{2r},$$

so $Z = Y$ and $|T(s,t)| = |T|^{s+t}$. Since

$$\begin{aligned} Z^{2t} &= Z^t U_{12}{}^* U_{12} Z^t \\ &\leq Z^t U_{12}{}^* U_{12} Z^t + Z^t U_{22}{}^* U_{22} Z^t \leq Z^{2t}, \end{aligned}$$

$Z^t U_{22}{}^* U_{22} Z^t = 0$ and $U_{22}Z^t = 0$. This implies ran $U_{22}^* \subset \ker Z$. Since ran $(U_{12}^*U_{12} + U_{22}^*U_{22}) \subset [\text{ran } Z]$ and $U_{22}^*U_{22} \leq U_{12}^*U_{12} + U_{22}^*U_{22}$, we have ran $U_{22}{}^* \subset [\text{ran } Z]$. Hence $U_{22} = 0, U = \begin{pmatrix} W_1 & U_{12} \\ 0 & 0 \end{pmatrix}$ and

$$\text{ran } U \subset [\text{ran } T(s,t)] \subset [\text{ran } |T|] = \text{ran } E.$$

Since W commutes with $|T(s,t)| = |T|^{s+t}$, W commutes with $|T|$ and

$$\begin{aligned} |T|^s(W - U)|T|^t &= W|T|^s|T|^t - |T|^s U |T|^t \\ &= W|T(s,t)| - T(s,t) = 0. \end{aligned}$$

Hence $E(W - U)E = 0$ and

$$U = UE = EUE = EWE = WE = W.$$

Thus $U = W$ commutes with $|T|$ and T is quasinormal. □

Corollary 2.2. *Let $T = U|T|$ be a class $A(s,t)$ operator T. If $T(s,t) = |T|^s U |T|^t$ is normal, then T is also normal.*

Proof. Since $T(s,t)$ is normal, T is quasinormal by Theorem 2.1. Hence $T(s,t) = |T|^s U |T|^t = U|T|^{s+t}$ and $T(s,t)^* = |T|^{s+t}U^*$. hence

$$|T|^{2(s+t)} = |T(s,t)|^2 = |T(s,t)^*|^2 = |T^*|^{2(s+t)}.$$

This implies $|T| = |T^*|$ and T is normal. □

3. Partial isometry

In this section, we deals with a partial isometry, i.e., $VV^*V = V$. Let V be a quasinormal partial isometry. Then VV^* is the orthogonal projection onto $V\mathcal{H}$ and V^*V is the orthogonal projection onto $V^*\mathcal{H}$. Let $V = U|V|$ be the polar decomposition of V. Since $V = U$ and $|V| = V * V$, we have

$$V(s,t) = |V|^s U|V|^t = V^*VVV^*V = V.$$

Hence the Aluthge transform $V(s,t)$ of V is a partial isometry and coincides with V. In this section, we deal with converse situation in which either $T(s,t)$ is a partial isometry or $T(s,t) = T$. First we consider the situation in which $T(s,t)$ is a partial isometry. We start with the following lemma, which is well known.

Lemma 3.1. *If $0 \leq A \leq 1$, and $\|Ax\| = \|x\|$. Then $Ax = x$.*

Lemma 3.2. *Let $T = U|T|$ be a contraction and $T(s,t) = |T|^s U|T|^t$ a partial isometry for some $s,t > 0$. Then $T(s,t) = T(s',t')$ for all $s',t' > 0$. In particular,* $\ker T(s,t) = \ker T(1,1) = \ker T^2$.

Proof. Since $T(s,t)$ is an isometry on ran $T(s,t)^*$, $\||T|^s U|T|^t x\| = \|x\|$ for all $x \in$ ran $T(s,t)^*$. Since T is a contraction, $|T|^s$ and $|T|^t$ are also contractions, hence we have

$$|T|^t x = x, \ \ |T|^s U|T|^t x = |T|^s Ux = Ux$$

by Lemma 3.1. Hence $|T|^{t'}x = x$, $|T|^{s'}Ux = Ux$ and $|T|^{s'}U|T|^{t'}x = |T|^{s'}Ux = Ux$ for all $s',t' > 0$. Hence we have $T(s,t) = T(s',t') = U$ on ran $T(s,t)^*$. To prove the rest, it suffices to show that $\ker T(s,t) = \ker T(s',t')$ because $\mathcal{H} =$ ran $T(s,t)^* \oplus \ker T(s,t)$.

Since

$$\begin{aligned} |T|^s U|T|^t x = 0 &\Longleftrightarrow U|T|^t x \in \ker T = \ker |T| \\ &\Longleftrightarrow |T|^{s'} U|T|^t x = 0, \end{aligned}$$

we have $T(s,t) = T(s',t)$. By using the same argument as above, we have $T(s,t)^* = T(s,t')^*$ for all $t' > 0$. Hence

$$\begin{aligned} \ker T(s,t) &= (\text{ran } T(s,t)^*)^\perp = (\text{ran } T(s,t')^*)^\perp \\ &= \ker T(s,t') = \ker T(s',t'). \end{aligned}$$

Thus $T(s,t) = T(s',t')$. It is clear that $\ker T(1,1) = \ker T^2$. □

Theorem 3.3. *Let $T = U|T|$ be a contraction such that* $\ker T = \ker T^2$. *If $T(s,t) = |T|^s U|T|^t$ is a partial isometry, then $T = T(s,t) = U$ and T is a quasinormal partial isometry.*

Proof. By Lemma 3.2,

$$\ker T(s,t) = \ker T^2 = \ker T = \ker U,$$

so ran $T(s,t)^* = [\text{ran } T^*] = [\text{ran } |T|]$. Since $T(s,t) = U$ on ran $T(s,t)^* = [\text{ran } |T|]$ and $\ker T(s,t) = \ker U = \ker T$, $T(s,t) = U$ because $\mathcal{H} = [\text{ran } |T|] \oplus \ker T$. This shows

$$\operatorname{ran} U = \operatorname{ran} T(s,t) \subset [\operatorname{ran} |T|] = \operatorname{ran} U^*U.$$

Thus $U = UU^*U = U^*UU$. Let

$$|T|^{2t} = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}, U^*U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ on } H = [\operatorname{ran} |T|] \oplus \ker |T|.$$

Since T is a contraction, we have $U^*|T|^{2s}U \leq 1$ and $0 \leq X \leq 1$. Then

$$U^*U = T(s,t)^*T(s,t) = |T|^t U^*|T|^{2s}U|T|^t \leq |T|^{2t} \leq U^*U.$$

Hence $|T| = U^*U$ and $T = U|T| = UU^*U = U = T(s,t)$. Thus T is a quasinormal partial isometry. □

Remark 3.4. *Theorem 3.3 is invalid if any one of conditions* $\ker T = \ker T^2$ *and* $||T|| \leq 1$ *is dropped.*

(Example 1)

Let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. *Then* $||T|| = 1$, $T(s,t) = 0$, $\ker T \neq \ker T^2$ *and* T *is not quasinormal.*

(Example 2)

Let $T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. *Then* $||T|| = \sqrt{2}$, $T(1/2,1/2)$ *is a projection,* $\ker T = \ker T^2$ *and* T *is not quasinormal.*

Corollary 3.5. *Let* $T = U|T| \in B(\mathcal{H})$ *be a paranormal operator, i.e.,* $||Tx||^2 \leq ||T^2x|| ||x||$ *for all* $x \in \mathcal{H}$. *If* $T(s,t) = |T|^sU|T|^t$ *is a partial isometry, then* $T = T(s,t) = U$ *and* T *is a quasinormal partial isometry.*

Proof. Since T is paranormal, $\ker T = \ker T^2$. Hence it suffices to show that T is a contraction by Theorem 3.3. Let $T \neq 0$. Then $||T|| = |\lambda|$ for some $0 \neq \lambda = |\lambda|e^{i\theta} \in \sigma(T)$. Then there exist unit vectors x_n such that

$$(T - \lambda)x_n \to 0, (T - \lambda)^*x_n \to 0.$$

Then

$$(|T| - |\lambda|)x_n \to 0, (U - e^{i\theta})x_n \to 0.$$

Hence

$$(T(s,t) - |\lambda|^{s+t}e^{i\theta})x_n \to 0$$

and $|\lambda|^{s+t}e^{i\theta} \in \sigma(T(s,t))$. Since $T(s,t)$ is a partial isometry, we have $|\lambda|^{s+t} \leq ||T(s,t)|| \leq 1$. Hence $||T|| = |\lambda| \leq 1$.

□

Corollary 3.6. *Let* $T = U|T|$ *be a class* $A(s,t)$ *operator. If* $T(s,t) = |T|^sU|T|^t$ *is a partial isometry, then* $T(s,t) = T$ *and* T *is a quasinormal partial isometry.*

Proof. Since $|T(s,t)|$ is a contraction and $|T(s,t)|^{\frac{2s}{s+t}} \geq |T|^{2s}$, it follows that T is a contraction and $\ker T = \ker T(s,t) = \ker T^2$ by Lemma 3.2. Now the result follows from Theorem 3.3. □

Now we study the situation in which $T(s,t) = T$. In case $s+t=1$, a simple argument shows that T is quasinormal. In what follows, we study cases in which $t > s+1, t = s+1$, and $t < s+1$. We begin with the following lemma.

Lemma 3.7. *Let $T = U|T|$ and $T = T(s,t) = |T|^s U|T|^t$. Then the following assertions hold.*

(i) $(T^*T)^s(TT^*)^t = TT^*$, *hence T^*T commutes with TT^*.*

(ii) $\ker T \subset \ker T^*$.

(iii) $\lambda \in \sigma(T^*T)$ *implies* $\lambda^{f(n)} \in \sigma(T^*T)$ *for each positive integer n where $f(n) = ((1-t)/s)^n$.*

Proof.

(i) Since $T = T(s,t)$,

$$U|T|U^* = |T|^s U|T|^t U^* = U|T|^t U^* |T|^s.$$

Hence $|T|$ commutes with $|T^*| = U|T|U^*$ and

$$\begin{aligned} TT^* &= U|T|U^*U|T|U^* \\ &= |T|^s|T^*|^t|T|^s|T^*|^t = (T^*T)^s(TT^*)^t. \end{aligned}$$

(ii) (i) implies $(TT^*)^t(T^*T)^s = TT^*$ and so (ii) is immediate.

(iii) Assume $0 \neq \lambda \in \sigma(T^*T)$. Then $\lambda \in \sigma(TT^*)$. Then there exist unit vectors x_n such that $(TT^* - \lambda)x_n \to 0$. Then $((TT^*)^t - \lambda^t)x_n \to 0$ and therefore $((T^*T)^s(TT^*)^t - \lambda^t(T^*T)^s)x_n \to 0$. Then $(TT^* - \lambda^t(T^*T)^s)x_n \to 0$ by (i). Since $(TT^* - \lambda)x_n \to 0$, we obtain $(\lambda^t(TT^*)^s - \lambda)x_n \to 0$. Hence, as λ is different from 0, we arrive at $\lambda^{f(1)} \in \sigma(TT^*)$ and therefore $\lambda^{f(1)} \in \sigma(T^*T)$. Now applying the same argument to $\lambda^{f(1)}$, we get $\lambda^{f(2)} \in \sigma(T^*T)$. Continuing in the same fashion, we obtain $\lambda^{f(n)} \in \sigma(T^*T)$ for each n. □

Theorem 3.8. *Let $T = U|T|$ and $T = T(s,t) = |T|^sU|T|^t$ for some $0 < s,t$ with $t > s+1$. Then T is a quasinormal partial isometry.*

Proof. By Lemma 3.7 and our assumption on t,

$$TT^*((TT^*)^{t-1}(T^*T)^s - 1) = 0$$

or equivalently,

$$T^*((TT^*)^{t-1}(T^*T)^s - 1) = 0.$$

This implies

$$|T|U^*((TT^*)^{t-1}(T^*T)^s - 1) = 0$$

and hence

$$|T|U^*(U|T|^{2t-2}U^*|T|^{2s} - 1) = 0.$$

Then $|T|^{2t-1}U^*|T|^{2s} = |T|U^*$. Since $t > 1$, we have $U|T|^{2t-2}U^*|T|^{2s} = UU^*$. In consequence of this, we find $|T|^{2t-2}U^*|T|^{2s} = U^*$. This shows that the generalized Aluthge Transform $T(2s, 2t-2) = |T|^{2s}U|T|^{2t-2}$ is a partial isometry and $\ker T = \ker U = \ker T(2s, 2t-2) = \ker T^2$ by Lemma 3.2. In view of Theorem 3.3, the proof is over once we establish the inequality $\|T\| \leq 1$. Choose a non-negative real number $\lambda \in \sigma(|T|)$ such that $\lambda = \||T|\| = \|T\|$. Then there exist unit vectors x_n

such that $(|T| - \lambda)x_n \to 0$. By Lemma 3.7 (iii), we have $\lambda^{f(n)} \in \sigma(T^*T)$ for each positive integer n. In particular $\lambda^{f(2n)} \in \sigma(T^*T)$. If $\lambda > 1$, then the assumption that $t > s+1$ will show that $f(2n) \to \infty$ and so $\lambda^{f(2n)} \to \infty$ as $n \to \infty$. This is clearly impossible. Therefore $\|T\| = \lambda \leq 1$. □

Remark 3.9. *If $t < s+1$, then Theorem 3.8 does not hold.*

(i. **In case of** $1 < t < s+1$)

Let $p = (t-1)/s \in (0,1)$. Let $\{e_n\}_{n=1,2,\cdots}$ be an orthonormal base of $\mathcal{H}$ and $0 < a$. Define a weighted shift T by

$$Te_n = a^{(-p)^{n-1}} e_{n+1}.$$

Since $a^{(-p)^{n-1}} \to a^0 = 1$, T is bounded. Let $T = U|T|$ be the polar decomposition of T. Then U is a unilateral shift (i.e., $Ue_n = e_{n+1}$) and $|T| = \sum a^{(-p)^{n-1}} P_n$ where P_n is the orthogonal projection onto $\mathbb{C}e_n$. Then

$$\begin{aligned} T(s,t)e_n &= |T|^s U |T|^t e_n = |T|^s U a^{t(-p)^{n-1}} e_n \\ &= a^{t(-p)^{n-1} + s(-p)^n} e_{n+1} = a^{(-p)^{n-1}} e_{n+1} = Te_n. \end{aligned}$$

Hence $T(s,t) = T$. Since U does not commute with $|T|$, T is not quasinormal. Since a or a^{-p} is larger than 1, $\|T\| > 1$, so T is not a partial isometry.

(ii. **In case of** $0 < s, t = 1$)

Let $0 < a \neq 1$. Define a weighted shift T by

$$Te_n = \begin{cases} ae_2 & \textit{if } n = 1 \\ e_{n+1} & \textit{if } n > 1. \end{cases}$$

Then $T(s,t) = |T|^s U |T| = T$, but T is neither quasinormal nor a partial isometry.

(iii. **In case of** $0 < t < 1, 1 < s+t$)

Let $p = (1-t)/s \in (0,1)$ and $0 < a \neq 1$. Define a weighted shift T by

$$Te_n = a^{p^n} e_{n+1}.$$

Then $T(s,t) = |T|^s U |T|^t = T$, but T is neither quasinormal nor a partial isometry.

(iv. **In case of** $0 < s, t, s+t = 1$)

Let $0 < a \neq 1$. Define a weighted shift T by

$$Te_n = ae_{n+1}.$$

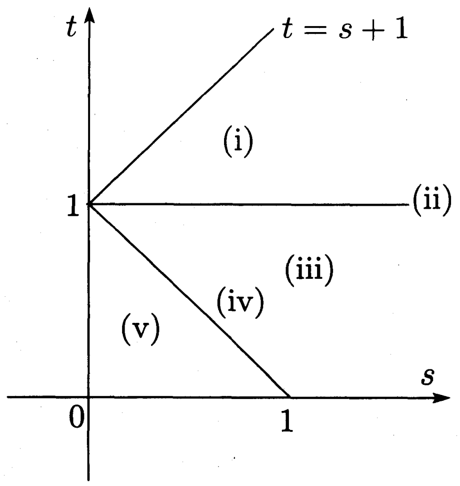
Then $|T| = a$ and $T(s,t) = a^s U a^t = T$, but T is not a partial isometry.

(v. **In case of** $0 < s+t < 1$)

Let $p = (1-t)/s > 1$ and $0 < a < 1$. Define a weighted shift T by

$$Te_n = a^{p^n} e_{n+1}.$$

Then $T(s,t) = |T|^s U |T|^t = T$ and T is quasinilpotent because $a^{p^n} \to 0$. Since U does not commute with $|T|$, T is not quasinormal. Since $\|Te_1\| = \|a^p e_2\| = a^p \neq \|e_1\|$, T is not a partial isometry.



The preceding remarks suggest that additional restrictions on T are required to insure the validity of Theorem 3.8 in case $t \leq s+1$.

Theorem 3.10. *Let T be a contraction with $T = T(s,t)$, where $t = s+1$. Then T is a quasinormal partial isometry.*

Proof. Since $U|T| = |T|^s U|T|^{s+1}$, we have $U = |T|^s U|T|^s$ as $\ker |T| = \ker U$. Then

$$UU^* = U|T|^s U^*|T|^s = |T^*|^s|T|^s = |T|^s|T^*|^s.$$

Hence $|T|$ commutes with $|T^*|$. Since UU^* is the orthogonal projection, $(UU^*)^{1/s} = UU^* = |T||T^*| = |T^*||T|$. Then $U = UU^*U = |T||T^*|U = |T|U|T|U^*U = |T|U|T| = T(1,1)$. Hence $T(1,1)$ is a partial isometry and $\ker T^2 = \ker T(1,1) = \ker U = \ker T$. Thus T is a quasinormal partial isometry by Theorem 3.3. $\square$

Remark 3.11. *Theorem 3.10 does not hold if $s = 0$. In this case $T(0,1) = T$ for any invertible operator T. Also the condition that $\|T\| \leq 1$ cannot be removed. For if $T = \begin{pmatrix} 0 & 2 \\ 1/2 & 0 \end{pmatrix}$ on $\mathcal{H} = \mathbb{C}^2$, then it has the polar decomposition $T = U|T|$ with $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $|T| = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$. Also $T(1,2) = T$, $\|T\| > 1$ and T is neither a partial isometry nor quasinormal.*

Theorem 3.12. *Let T be a contraction with $T = T(s,t)$, where $t < s+1$.*

(i) *If $t > 1$, then T is a quasinormal partial isometry.*

(ii) *If $s + t < 1$ and 0 is not a limit point of $\sigma(T^*T)$, then T is a quasinormal partial isometry.*

(iii) *If $1 - s < t < 1$ and 1 is not a limit point of $\sigma(T^*T)$, then T is a quasinormal partial isometry.*

Proof. (i) Since $U|T| = |T|^s U|T|^t$,

$$U = |T|^s U|T|^{t-1} = T(s, t-1).$$

Hence $T(s,t-1)$ is a partial isometry and $UU^* = U|T|^{t-1}U^*|T|^s$. Then $\ker U = \ker T = \ker|T| \subset \ker U^* = \ker T^*$ and $\ker T = \ker T^2$. Thus T is a quasinormal partial isometry by Theorem 3.3.

(ii) Since $|T|^sU|T|^t = U|T|$, we have $|T|^sU = U|T|^{1-t}$. Then $|T^*|^{1-t} = U|T|^{1-t}U^* = |T|^sUU^* = UU^*|T|^s$. Hence

$$|T|^s \geq UU^*|T|^s = |T^*|^{1-t}.$$

Let $\lambda \in \sigma(T^*T)$. Since T is a contraction, $0 \leq \lambda \leq 1$. Then $\lambda^{f(2n)} \in \sigma(T^*T)$ for each positive integer n, where $f(2n) = \left(\frac{1-t}{s}\right)^{2n}$ by Lemma 3.7. Assume $0 < \lambda < 1$. Then $\sigma(T^*T) \ni \lambda^{f(2n)} \to 0$ as $1 < \left(\frac{1-t}{s}\right)^2$. This is a contradiction. Hence $\sigma(T^*T) \subset \{0,1\}$ and T^*T is the orthogonal projection. Thus T is a partial isometry and $|T|U = U|T|$.

(iii) The proof is similar to (ii).

□

Remark 3.13. *Theorem 3.12* (i) *is not true in case* $t = 1$. *For the counter example, refer to Remark 3.9* (ii). *Theorem 3.12* (ii) *is not valid if 0 is not a limit point of* $\sigma(T^*T)$ *as can be seen in Remark 3.9* (V). *Also Theorem 3.12* (iii) *is not valid if 1 is not a limit point of* $\sigma(T^*T)$ *as can be seen in Remark 3.9* (iii).

4. Fuglede-Putnam type Theorem

Our basic aim in this section is to extend the Fuglede-Putnam Theorem [6, 16], one of the celebrated theorems in the subject of operator theory. We would like to state the theorem.

Proposition 4.1 (Fuglede-Putnam). *Let $S \in B(\mathcal{H})$ and $T^* \in B(\mathcal{K})$ be normal operators and $SX = XT$ for some operator $X \in B(\mathcal{H},\mathcal{K})$. Then $S^*X = XT^*$, $[\operatorname{ran} X]$ reduces S, $(\ker X)^{\perp}$ reduces T, and $S|_{[\operatorname{ran} X]}$, $T|_{(\ker X)^{\perp}}$ are unitarily equivalent normal operators.*

Various extensions of the Fuglede-Putnam Theorem can be found in the literature. (See [5], [12], [15]). Recently Uchiyama and Tanahashi [20] generalized the theorem for p-hyponormal operators and log-hyponormal operators, a subclass of $A(s,t)$ operators with $s = t = 1/2$. In the present section, we extend the above theorem for class $A(s,t)$ operators with $s+t=1$ with reducing kernels. Further extensions for class A operators and more generally for class $A(s,t)$ operators remain as an open problem. Here we wish to give two alternate proofs.

1. First Proof.

First we start with establishing several lemmas.

Lemma 4.2. ([22]) *Let A, B and C be positive operators, $0 < p$ and $0 < r \le 1$. If $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \ge B^r$ and $B \ge C$, then $(C^{\frac{r}{2}}A^pC^{\frac{r}{2}})^{\frac{r}{p+r}} \ge C^r$.*

Lemma 4.3. *Let T be a class $A(s,t)$ operator for some $s,t \in (0,1]$ and $\mathcal{M}$ an invariant subspace of T. Then the restriction $T|_{\mathcal{M}}$ is also a class $A(s,t)$ operator.*

Proof. Let $T = \begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix}$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$ and P the orthogonal projection onto $\mathcal{M}$. Let $T_0 = TP = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$|T_0|^{2s} = (P|T|^2P)^s \ge P|T|^{2s}P$$

by Hansen's inequality, and

$$|T^*|^2 = TT^* \ge TPT^* = |T_0^*|^2.$$

Hence,

$$\begin{aligned}
&T \text{ is a class } A(s,t) \text{ operator}\\
&\iff (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \ge |T^*|^{2t}\\
&\Longrightarrow (|T_0^*|^t|T|^{2s}|T_0^*|^t)^{\frac{t}{s+t}} \ge |T_0^*|^{2t} \quad \text{(by Lemma 4.2)}\\
&\Longrightarrow (|T_0^*|^t|T_0|^{2s}|T_0^*|^t)^{\frac{t}{s+t}} \ge |T_0^*|^{2t} \quad (\text{since } |T_0^*|^t = |T_0^*|^tP = P|T_0^*|^t)\\
&\iff T_{\mathcal{M}} \text{ is a class } A(s,t) \text{ operator .}
\end{aligned}$$

□

Lemma 4.4. *Let $T \in L(\mathcal{H})$ be a class A operator. Let $\mathcal{M}$ be an invariant subspace of T and $T = \begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix}$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. If $T_1 = T|_{\mathcal{M}}$ is quasinormal, then* ran $S \subset \ker T_1^*$. *Moreover, if $\ker T \subset \ker T^*$ and $T_1 = T|_{\mathcal{M}}$ is normal, then $\mathcal{M}$ reduces T.*

Proof. Let P be the orthogonal projection onto $\mathcal{M}$. Then we have,

$$\begin{aligned}\begin{pmatrix} {T_1}^*T_1 & 0 \\ 0 & 0 \end{pmatrix} &= PT^*TP \leq P|T^2|P \quad \text{(since } T \text{ is class } A)\\ &\leq \begin{pmatrix} ({T_1}^{*2}{T_1}^2)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{(by Hansen's inequality [8])}\\ &= \begin{pmatrix} {T_1}^*T_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{(since } T_1 \text{ is quasinormal).}\end{aligned}$$

Let $|T^2| = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$. Then $X = {T_1}^*T_1$ by the above inequality. Since $|T^2|^2 = T^{*2}T^2$, we have

$$\begin{aligned}&\begin{pmatrix} X^2 + YY^* & XY + YZ \\ ZY^* + Y^*X & Y^*Y + Z^2 \end{pmatrix}\\ &= \begin{pmatrix} {T_1}^{*2}{T_1}^2 & {T_1}^{*2}(T_1S + ST_2) \\ (S^*{T_1}^* + T_2^*S^*)T_1^2 & (S^*{T_1}^* + T_2^*S^*)(T_1S + ST_2) + {T_2}^{*2}{T_2}^2 \end{pmatrix}\end{aligned}$$

and hence

$$X^2 + YY^* = {T_1}^{*2}{T_1}^2 = (T_1^*T_1)^2 = X^2.$$

This implies that $Y = 0$. Then

$$|T^2| = \begin{pmatrix} {T_1}^*T_1 & 0 \\ 0 & Z \end{pmatrix} \geq T^*T = \begin{pmatrix} {T_1}^*T_1 & {T_1}^*S \\ S^*T_1 & S^*S + {T_2}^*T_2 \end{pmatrix}$$

and ${T_1}^*S = 0$. This implies

$$\text{ran } S \subset \ker T_1^*.$$

Moreover, assume T_1 is normal. Then

$$S(\mathcal{M}^\perp) \subset \ker T_1^* = \ker T_1 \subset \ker T \subset \ker T^*.$$

Hence, we have

$$0 = T^*Sx = \begin{pmatrix} T_1^* & 0 \\ S^* & T_2^* \end{pmatrix} \begin{pmatrix} Sx \\ 0 \end{pmatrix} = \begin{pmatrix} T_1^*Sx \\ S^*Sx \end{pmatrix}$$

for $x \in \mathcal{M}^\perp$. This implies $S^*S = 0$ and $S = 0$. Thus $\mathcal{M}$ reduces T. □

Remark 4.5. *The following example shows that there exists a class A operator T such that $T|_{\mathcal{M}}$ is quasinormal but $\mathcal{M}$ does not reduce T.*

Let T be a bilateral shift on $\ell^2(\mathbb{Z})$ definded by $Te_n = e_{n+1}$ and $\mathcal{M} = \vee_{0 \leq n}\mathbb{C}e_n$. Then T is unitary and $T|_{\mathcal{M}}$ is isometry. However $\mathcal{M}$ does not reduce T.

The next lemma is a simple consequence of the preceding one.

Lemma 4.6. *Let $T \in L(\mathcal{H})$ be a class A operator with* $\ker T \subset \ker T^*$. *Then $T = T_1 \oplus T_2$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where T_1 is normal,* $\ker T_2 = \{0\}$ *and T_2 is pure class A, i.e., T_2 has no non-zero invariant subspace $\mathcal{M}$ such that $T_2|_{\mathcal{M}}$ is normal.*

Lemma 4.7. *Let $T = U|T| \in B(\mathcal{H})$ be a class $A(s,t)$ operator with $s+t=1$ and* $\ker T \subset \ker T^*$. *Let $T(s,t) = |T|^s U |T|^t$. Suppose $T(s,t)$ be of the form $N \oplus T'$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, where N is a normal operator on $\mathcal{M}$. Then $T = N \oplus T_1$ and $U = U_{11} \oplus U_{22}$, where T_1 is a class $A(s,t)$ operator with* $\ker T_1 \subset \ker T_1^*$ *and $N = U_{11}|N|$ is the polar decomposition of N.*

Proof. Since

$$|T(s,t)|^{2r} \geq |T|^{2r} \geq |T(s,t)^*|^{2r}$$

for $r \in (0, \min\{s,t\}]$, we have

$$|N|^{2r} \oplus |T'|^{2r} \geq |T|^{2r} \geq |N|^{2r} \oplus |T'^*|^{2r}$$

by assumption. This implies that $|T|$ is of the form $|N| \oplus L$ for some positive operator L. Let $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ be 2×2 matrix representation of U with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. Then the definition $T(s,t) = |T|^s U |T|^t$ means

$$\begin{pmatrix} N & 0 \\ 0 & T' \end{pmatrix} = \begin{pmatrix} |N|^s & 0 \\ 0 & L^s \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} |N|^t & 0 \\ 0 & L^t \end{pmatrix}.$$

Hence, we have

$$N = |N|^s U_{11} |N|^t, |N|^s U_{12} L^t = 0, L^s U_{21} |N|^t = 0.$$

Since $\ker T \subset \ker T^*$,

$$[\text{ran } U] = [\text{ran } T] = (\ker T^*)^\perp \subset (\ker T)^\perp = [\text{ran } |T|].$$

Let $Nx = 0$ for $x \in \mathcal{M}$. Then $x \in \ker |T| = \ker U$, and

$$Ux = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11}x \\ U_{21}x \end{pmatrix} = 0.$$

Hence

$$\ker N \subset \ker U_{11} \cap \ker U_{21}.$$

Let $x \in \mathcal{M}$. Then

$$U \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11}x \\ U_{21}x \end{pmatrix} \in [\text{ran } |T|] = [\text{ran } (|N| \oplus L)].$$

Hence

$$\text{ran } U_{11} \subset [\text{ran}|N|], \text{ ran } U_{21} \subset [\text{ran} L].$$

Similarly

$$\text{ran } U_{12} \subset [\text{ran}|N|], \text{ ran } U_{22} \subset [\text{ran} L].$$

Let $Lx = 0$ for $x \in \mathcal{M}^\perp$. Then $x \in \ker |T| = \ker U$ and

$$U \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} U_{12}x \\ U_{22}x \end{pmatrix} = 0.$$

Hence

$$\ker L \subset \ker U_{12} \cap \ker U_{22}.$$

Let $N = V|N|$ be the polar decomposition of N. Then

$$(V|N|^s - |N|^s U_{11})|N|^t = 0.$$

Hence $V|N|^s - |N|^s U_{11} = 0$ on $[\text{ran } |N|]$. Since $\ker N \subset \ker U_{11}$, this implies $0 = V|N|^s - |N|^s U_{11} = |N|^s (V - U_{11})$. Hence

$$\text{ran } (V - U_{11}) \subset \ker |N| \cap [\text{ran } |N|] = \{0\}.$$

Hence $V = U_{11}$ and $N = U_{11}|N|$ is the polar decomposition of N. Since $|N|^s U_{12} L^t = 0$,

$$\text{ran } U_{12} L^t \subset \ker |N| \cap [\text{ran } |N|] = \{0\}.$$

Hence $U_{12}L^t = 0$ and $U_{12} = 0$. Similarly we have $U_{21} = 0$ by $L^s U_{21} |N|^t = 0$. Hence $U = U_{11} \oplus U_{22}$. So we obtain

$$T = U|T| = U_{11}|N| \oplus U_{22} L = N \oplus T_1,$$

where $T_1 = U_{22}L$. □

Theorem 4.8. *Let $S \in B(\mathcal{H})$ and $T^* \in B(\mathcal{K})$ are class $A(s,t)$ operators with $s + t \leq 1$ and $\ker S \subset \ker S^*, \ker T^* \subset \ker T$. Let $SX = XT$ for some operator $X \in B(\mathcal{K}, \mathcal{H})$. Then $S^*X = XT^*, [\text{ran } X]$ reduces $S, (\ker X)^\perp$ reduces T, and $S|_{[\text{ran } X]}, T|_{(\ker X)^\perp}$ are unitarily equivalent normal operators.*

Proof. We may assume $s + t = 1$ by [11, Theorem 4]. Decompose S, T^* into normal parts and pure parts as in Lemma 4.6, i.e., $S = S_1 \oplus S_2$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $T^* = T_1^* \oplus T_2^*$ on $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ where S_1, T_1^* are normal and S_2, T_2^* are pure. Let $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$. Then $SX = XT$ implies

$$\begin{pmatrix} S_1 X_{11} & S_1 X_{12} \\ S_2 X_{21} & S_2 X_{22} \end{pmatrix} = \begin{pmatrix} X_{11} T_1 & X_{12} T_2 \\ X_{21} T_1 & X_{22} T_2 \end{pmatrix}.$$

Let $S_2 = U_2|S_2|, T_2^* = V_2^*|T_2^*|$ be the polar decompositions and

$$S_2(s,t) = |S_2|^s U_2 |S_2|^t, T_2^*(s,t) = |T_2^*|^s V_2^* |T_2^*|^t, W = |S_2|^s X_{22} |T_2^*|^s.$$

Then

$$\begin{aligned} S_2(s,t) W &= |S_2|^s S_2 X_{22} |T_2^*|^s \\ &= |S_2|^s X_{22} T_2 |T_2^*|^s = W(T_2^*(s,t))^*. \end{aligned}$$

Since S_2, T_2^* are class $A(s,t)$ operators, $S_2(s,t), T_2^*(s,t)$ are $\min\{s,t\}$-hyponormal. Hence $[\text{ran } W]$ reduces $S_2(s,t)$, $(\ker W)^\perp$ reduces $T_2^*(s,t)$ and

$$S_2(s,t)|_{[\text{ran } W]} \simeq T_2^*(s,t)|_{(\ker W)^\perp}$$

are unitarily equivalent normal operators by [5]. Since S_2, T_2^* are pure, we have $W = 0$ by Lemma 4.7. Then $X_{22} = 0$ as S_2, T_2^* are injective by Lemma 4.6. Since

$S_2X_{21} = X_{21}T_1$ and $S_1X_{12} = X_{12}T_2$, we have $X_{21}T_1 = 0$ and $S_1X_{12} = 0$ by similar arguments. Then $SX = XT$ implies

$$\begin{pmatrix} S_1X_{11} & 0 \\ S_2X_{21} & 0 \end{pmatrix} = \begin{pmatrix} X_{11}T_1 & X_{12}T_2 \\ 0 & 0 \end{pmatrix}$$

and $X_{12} = 0, X_{21} = 0$. Hence $X = \begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix}$ and

$$\text{ran } X = \text{ran } X_{11} \oplus \{0\}, \ (\ker X)^{\perp} = (\ker X_{11})^{\perp} \oplus \{0\}.$$

Since $S_1X_{11} = X_{11}T_1$, we have $S_1^*X_{11} = X_{11}T_1^*$, [ran X_{11}] reduces S_1, $S_1|_{[\text{ran } X_{11}]}$ and $T_1|_{(\ker X_{11})^{\perp}}$ are unitarily equivalent normal operators by Proposition 4.1. Then $S|_{[\text{ran } X]} \simeq S_1|_{[\text{ran } X_{11}]}$, $T_1|_{(\ker X_{11})^{\perp}} \simeq T|_{(\ker X)^{\perp}}$ imply that $S^*X = XT^*$, [ran X] reduces S, $(\ker X)^{\perp}$ reduces T, and $S|_{[\text{ran } X]}, T|_{(\ker X)^{\perp}}$ are unitarily equivalent normal operators. □

Remark 4.9. *The authors* [19, Example 13] *made a class* $A(1/2, 1/2)$ *operator* A *such that* $\ker A$ *does not reduce* A. *Let* $S = T^* = A$ *and* $X = P$ *be the orthogonal projection onto* $\ker S$. *Then* $SX = 0 = XT$, *but* $S^*X \neq XT^*$. *Hence the kernel condition is necessary for Theorem 4.8.*

2. Second Proof.

Theorem 4.10. *Let* $T \in B(\mathcal{H})$ *be a class* $A(s,t)$ *operator with* $s + t \leq 1$ *and* $\ker T \subset \ker T^*$. *If* L *is self-adjoint and* $TL = LT^*$, *then* $T^*L = LT$.

Proof. We may assume $s + t = 1$ by [11, Theorem 4]. Since $\ker T \subset \ker T^*$ and $TL = LT^*$, $\ker T$ reduces T and L. Hence

$$T = T_1 \oplus 0, \ L = L_1 \oplus L_2 \text{ on } \mathcal{H} = [\text{ran } T^*] \oplus \ker T,$$

$T_1L_1 = L_1T_1{}^*$ and $\{0\} = \ker T_1 \subset \ker T_1^*$. Since [ran L_1] is invariant under T_1 and reduces L_1,

$$T_1 = \begin{pmatrix} T_{11} & S \\ 0 & T_{22} \end{pmatrix}, L_1 = L_{11} \oplus 0 \text{ on } [\text{ran } T^*] = [\text{ran } L_1] \oplus \ker L_1.$$

T_{11} is an injective class $A(s,t)$ operator by Lemma 4.3 and L_{11} is an injective self-adjoint operator (hence it has dense range) such that $T_{11}L_{11} = L_{11}T_{11}{}^*$. Let $T_{11} = V_{11}|T_{11}|$ be the polar decomposition of T_{11} and $T_{11}(s,t) = |T_{11}|^sV_{11}|T_{11}|^t, W = |T_{11}|^sL_{11}|T_{11}|^s$. Then

$$\begin{aligned} T_{11}(s,t)W &= |T_{11}|^sV_{11}|T_{11}|^t|T_{11}|^sL_{11}|T_{11}|^s \\ &= |T_{11}|^sT_{11}L_{11}|T_{11}|^s = |T_{11}|^sL_{11}T_{11}^*|T_{11}|^s \\ &= |T_{11}|^sL_{11}|T_{11}|^s|T_{11}|^tV_{11}^*|T_{11}|^s = WT_{11}(s,t)^*. \end{aligned}$$

Since $T_{11}(s,t)$ is $\min\{s,t\}$-hyponormal and ran W is dense (because $\ker W = \{0\}$), $T_{11}(s,t)$ is normal by [5, Theorem 7]. Hence T_{11} is normal and $T_{11} = T_{11}(s,t)$ by

Corollary 2.2. Then [ran L_1] reduces T_1 by Lemma 4.4 and ${T_{11}}^*L_{11} = L_{11}T_{11}$ by Proposition 4.1. Hence

$$T = T_{11} \oplus T_{22} \oplus 0,$$
$$L = L_{11} \oplus 0 \oplus L_2$$

and

$$T^*L = T_{11}^*L_{11} \oplus 0 \oplus 0 = L_{11}T_{11} \oplus 0 \oplus 0 = LT.$$

□

Remark 4.11. *Let $T = A$ be a class $A(1/2, 1/2)$ operator as in Remark 4.9. Let $X = P$ be the orthogonal projection onto* $\ker T$. *Then T is a class A operator and $TL = 0 = LT^*$, but $T^*L \neq LT$. Hence the kernel condition* $\ker T \subset \ker T^*$ *is necessary for Theorem 4.10.*

Corollary 4.12. *Let $T \in B(\mathcal{H})$ be a class $A(s,t)$ operator with $s + t \leq 1$ and* $\ker T \subset \ker T^*$. *If $TX = XT^*$ for some $X \in B(\mathcal{H})$, then $T^*X = XT$.*

Proof. Let $X = L + iK$ be the Cartesian decomposition of X. Then we have $TL = LT^*$ and $TJ = JT^*$ by the assumption. By Theorem 4.10, we have $T^*L = LT$ and $T^*J = JT$. This implies that $T^*X = XT$. □

Corollary 4.13. *Let $S \in B(\mathcal{K}), T^* \in B(\mathcal{H})$ be class $A(s,t)$ operators with $s+t \leq 1$ and* $\ker S \subset \ker S^*, \ker T^* \subset \ker T$. *If $SX = XT$ for some $X \in B(\mathcal{K}, \mathcal{H})$, then $S^*X = XT^*$.*

Proof. Put $A = \begin{pmatrix} T^* & 0 \\ 0 & S \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}$ on $\mathcal{H} \oplus \mathcal{K}$. Then A is a class $A(s,t)$ operator with $\ker A \subset \ker A^*$, which satisfies $AB = BA^*$. Hence we have $A^*B = BA$ by Corollary 4.12, and therefore $S^*X = XT^*$. □

As an application of Corollary 4.13, we establish below Corollary 4.14; thus completing the second proof.

Corollary 4.14. *Let $S \in B(\mathcal{H})$ and $T^* \in B(\mathcal{K})$ are class $A(s,t)$ operators with $s + t \leq 1$ and* $\ker S \subset \ker S^*, \ker T^* \subset \ker T$. *Let $SX = XT$ for some operator $X \in B(\mathcal{K}, \mathcal{H})$. Then* [ran X] *reduces S,* $(\ker X)^\perp$ *reduces T and $S|_{[\text{ran } X]}, T|_{(\ker X)^\perp}$ are unitarily equivalent normal operators.*

Proof. By Corollary 4.13, $S^*X = XT^*$. Therefore $S^*SX = XT^*T$ and so $|S|X = X|T|$. Let $S = U|S|, T = V|T|$ be polar decomposition. Then $UX|T| = U|S|X = SX = XT = XV|T|$. Let $x \in \ker|T|$. Then $Vx = 0$ and $SXx = XTx = 0$. Hence $Xx \in \ker S = \ker U$ and $UXx = 0$. Hence $UX = XV$. Since $\ker U = \ker T \subset \ker T^* = \ker U^*$, $UU^* \leq U^*U$. Hence $U^*UU = U^*UUU^*U = UU^*U = U$. This implies U and V^* are quasinormal. Hence $U^*X = XV^*$, [ran$X$] reduces $U, |S|$, $(\ker X)^\perp$ reduces $V, |T|$. We may assume $t < s$. Then S, T^* are class $A(s,s)$ operators with reducing kernels. Let $S(s,s) = |S|^sU|S|^s, T(s,s) = |T|^sV|T|^s$. Then $S(s,s), T^*(s,s) = |T^*|^sV^*|T^*|^s = VT(s,s)^*V^*$ are semi-hyponormal. Also, since

$$|T(s,s)^*| - |T(s,s)| = V^*(|T^*(s,s)| - |T^*(s,s)^*|)V \geq 0,$$

$T(s,s)^*$ is semi-hyponormal, too. Then

$$\begin{aligned} S(s,s)X &= |S|^s U|S|^s X = |S|^s UX|T|^s \\ &= |S|^s XV|T|^s = XT(s,s), \end{aligned}$$

hence $S(s,s)^*X = XT(s,s)^*$, [ran X] reduces $S(s,s)$, $(\ker X)^\perp$ reduces $T(s,s)$ and

$$S|_{[\text{ran } X]}(s,s) = S(s,s)|_{[\text{ran } X]} \simeq T(s,s)|_{(\ker X)^\perp} = T|_{(\ker X)^\perp}(s,s)$$

are unitarily equivalent normal operators. Hence $S|_{[\text{ran } X]}, T|_{(\ker X)^\perp}$ are normal by Corollary 2.2, and that they are unitarily equivalent follows from the fact that if $N = U|N|$ are $M = W|M|$ are normal operators, then for a unitary operator V, $N = V^*MV$ if and only if $U = V^*WV$ and $|N|^s = V^*|M|^s V$ for any $s > 0$. □

Theorem 4.15. *Let $T = U|T| \in B(\mathcal{H})$ be a class $A(s,t)$ operator with $s+t \leq 1$ and N a normal operator. Let $TX = XN$. Then the following assertions hold.*

(i) *If the range* ran X *is dense, then T is normal.*

(ii) *If* $\ker X^* \subset \ker T^*$, *then T is quasinormal.*

Proof. Let $Z = |T|^s X$. Then

$$\begin{aligned} T(s,t)Z &= |T|^s U|T|^t|T|^s X = |T|^s TX \\ &= |T|^s XN = ZN. \end{aligned}$$

Since $T(s,t)$ is $\min\{s,t\}$-hyponormal, we have

$$T(s,t)^*Z = ZN^*$$

by [20]. Hence

$$\begin{aligned} &(T(s,t)^*T(s,t) - T(s,t)T(s,t)^*)\,|T|^s X \\ &= T(s,t)^*T(s,t)Z - T(s,t)T(s,t)^*Z \\ &= T(s,t)^*ZN - T(s,t)ZN^* = ZN^*N - ZNN^* = 0. \end{aligned}$$

(i) If ran X is dense, then

$$(T(s,t)^*T(s,t) - T(s,t)T(s,t)^*)\,|T|^s = 0.$$

Since

$$\ker |T|^s \subset \ker T(s,t) \cap \ker T(s,t)^*,$$

this implies $T(s,t)$ is normal. Hence T is normal by Corollary 2.2.

(ii) Let $X^*|T|^s x = 0$. Then $|T|^s x \in \ker X^* \subset \ker T^* = \ker U^*$ and $T(s,t)^*x = |T|^t U^*|T|^s x = 0$. Hence $\ker(X^*|T|^s) \subset T(s,t)^*$ and [ran $T(s,t)$] $\subset$ [ran $|T|^s X$]. Hence

$$(T(s,t)^*T(s,t) - T(s,t)T(s,t)^*)\,T(s,t) = 0$$

by (i). This implies $T(s,t)$ is quasinormal, and T is quasinormal by Theorem 2.1. □

Next theorem is an extension of Theorem 3 of [20].

Theorem 4.16. *Let $S \in B(\mathcal{H})$ be dominant and $T^* \in B(\mathcal{K})$ a class $A(s,t)$ operator with $s+t \leq 1$ and $\ker T^* \subset \ker T$. Let $SX = XT$ for some operator $X \in B(\mathcal{K}, \mathcal{H})$. Then $S^*X = XT^*$, $[\operatorname{ran} X]$ reduces S, $(\ker X)^\perp$ reduces T, and $S|_{[\operatorname{ran} X]}, T|_{(\ker X)^\perp}$ are unitarily equivalent normal operators.*

Proof. Decompose S, T^* into normal parts and pure parts as in Lemma 4.6 and [4], i.e., $S = S_1 \oplus S_2$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $T^* = T_1^* \oplus T_2^*$ on $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ where S_1, T_1^* are normal and S_2, T_2^* are pure. Let $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$. Let $T_2^* = U_2^*|T_2^*|$ be the polar decomposition of T_2^* and $T_2^*(s,t) = |T_1^*|^s U_2^* |T_1^*|^t$. Let $T_2^*(s,t) = V_2^*|T_2^*(s,t)|$ be the polar decomposition of $T_2^*(s,t) = W$ and $W(s,t) = |T_2^*(s,t)|^s V_2^* |T_2^*(s,t)|^t$. Since $SX = XT$, we have

$$S_2 X_{21} = X_{21} T_1,$$
$$S_2 X_{22} |T_2^*|^s |T_2^*(s,t)|^s = X_{22} |T_2^*|^s |T_2^*(s,t)|^s W(s,t)^*$$
$$S_1 X_{12} = X_{12} T_2.$$

Then $X_{21}, X_{22}, X_{12} = 0$ by [4, Corollary 1] and Theorem 4.10. The rest of the proof is similar to the proof of Theorem 4.10. □

Remark 4.17. *Let $T^* = A$ as in Remark 4.9. Let $X = P$ be the orthogonal projection onto $\ker T^*$ and $S = 1 - P$. Then $SX = 0 = XT$, but $0 = S^*X \neq XT^*$. Hence the kernel condition is necessary for Theorem 4.16.*

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