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NON-GALOIS TRIPLE COVERING OF $\mathbb{P}^2$ BRANCHED ALONG QUINTIC CURVES AND THEIR CUBIC EQUATIONS

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ABSTRACT. Let $\varpi : S \to \mathbb{P}^2$ be a non-Galois triple covering given by the cubic equation $\zeta^3 + 3u\zeta + 2v = 0$, where u and v denote inhomogeneous coordinates of $\mathbb{P}^2$. Let $\widehat{\pi} : \widehat{X} \to \mathbb{P}^2$ be a D_6-covering of $\mathbb{P}^2$ branched along a quintic. There are two possibilities for the ramification types of $\widehat{\pi}$. One is that $\widehat{\pi}$ has the ramification index 2 (resp. 3) along a conic (resp. a cubic), and the other is that $\widehat{\pi}$ has the ramification index 2 (resp. 3) along a quartic (resp. a line). There exist 18 types in the latter case ([8]). For each $\widehat{\pi}$ of the 18 types, there exists a non-Galois triple covering $\pi : X \to \mathbb{P}^2$ with the same branch locus as $\widehat{\pi}$. In this article, we study rational maps $\Phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that the pull-backs of ϖ by Φ give rise to $\pi : X \to \mathbb{P}^2$.

1. Introduction

In this article, all varieties are defined over $\mathbb{C}$, the field of complex numbers. Let X and Y be normal projective varieties. We call X a finite covering of Y if there exists a finite surjective morphism $\pi : X \to Y$. Let $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ denote the rational function fields of X and Y, respectively. It is known that $\mathbb{C}(X)$ is a finite field extension of $\mathbb{C}(Y)$ with $[\mathbb{C}(X) : \mathbb{C}(Y)] = \deg \pi$. We say that a finite covering X is Galois if $\mathbb{C}(X)/\mathbb{C}(Y)$ is Galois. For a Galois covering whose Galois group is isomorphic to a finite group G, we call it a G-covering for simplicity. Note that, for a G-covering X, G acts on X faithfully in such a way that $Y = X/G$.

A subset of Y consisting of points $y \in Y$ such that π is not locally isomorphic over y is called the branch locus of π and we denote it by Δ_π or $\Delta(X/Y)$. By the purity of the branch locus ([10]), Δ_π is an algebraic subset of codimension 1 if Y is smooth.

We call $\pi : X \to Y$ a non-Galois triple covering if $\mathbb{C}(X)/\mathbb{C}(Y)$ is a non-Galois cubic extension. For a non-Galois triple covering $\pi : X \to Y$, $\mathbb{C}(X) = \mathbb{C}(Y)(\theta)$ where θ is a solution of a certain cubic equation $\zeta^3 + 3a\zeta + 2b = 0$, $a, b \in \mathbb{C}(Y)$. Geometrically one can regard this in the following way:

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Let $\varpi : S \to \mathbb{P}^2$ be a non-Galois triple covering given by the cubic equation

$$\zeta^3 + 3u\zeta + 2v = 0,$$

where we denote the inhomogeneous coordinates of $\mathbb{P}^2$ by (u, v). Let $\Phi_\pi : Y \dashrightarrow \mathbb{P}^2$ be a rational map given by

$$\Phi_\pi : p \mapsto (u, v) = (a(p), b(p)).$$

Then we obtain a commutative diagram as follows:

$$\begin{tikzcd}
 & X_\pi \arrow[dl] \arrow[dr] \arrow[dd] & \\
X \arrow[rr, dotted] \arrow[dd, "\pi"'] & & S \arrow[dd, "\varpi"] \\
 & Y_\pi \arrow[dl] \arrow[dr, "\nu_\pi"] & \\
Y \arrow[rr, dotted, "\Phi_\pi"'] & & \mathbb{P}^2.
\end{tikzcd}$$

Here, $\nu_\pi : Y_\pi \to \mathbb{P}^2$ is the resolution of indeterminacy of Φ_π and X_π is the normalization of the fiber product $Y_\pi \times_{\nu_\pi} S$. Note that X_π is birationally equivalent to X. In other words, X is obtained as a "rational" pull-back of $\varpi : S \to \mathbb{P}^2$. Note that Φ_π is not necessary dominant. In fact, there exist cases that Φ_π is a non-dominant rational map (see Section 6).

In this article, we are interested in the "pull-back" construction of a non-Galois triple covering as above, that is, to describe a cubic equation geometrically corresponding to a given non-Galois triple covering. This is a new approach in the study of non-Galois triple covering, which is different from that in previous papers [6] and [7].

As it is shown in [7], the study of non-Galois triple coverings is closely related to that of D_6-coverings, D_6 being the dihedral group of order 6. In fact, for a smooth projective variety Y, there exists a D_6-covering of Y along B if and only if there exists a non-Galois triple covering branched along B. In [8], such coverings (more generally D_{2p}-coverings) of $\mathbb{P}^2$ whose branch loci are quintic curves are studied. More precisely, it is as follows:

Let $\widehat{\pi} : \widehat{X} \to \mathbb{P}^2$ be a D_6-covering with $\deg \Delta_\pi = 5$. We first note that there are two possibilities with respect to the ramification indexes as follows:

Type I: The branch curve with ramification index 2 is a conic, while that with index 3 is a cubic.

Type II: The branch curve with ramification index 2 is a quartic, while that with index 3 is a line.

Here, a curve with ramification index n means that the ramification index along the smooth part of the curve is n.

In [8], D_6-coverings $\widehat{\pi} : \widehat{X} \to \mathbb{P}^2$ of type II are studied and it is given that a list of possible branch loci in terms of the configuration of singular points of the quintic and the relative position between the quintic and the line. Put $\Delta_\pi = Q + L$, where Q and L are a quartic and a line as above, respectively. The possible list of Q and a configuration of $Q + L$ is as Table 1. The second and fifth columns refer to the type

Δ_π	Q	$Q \cap L$	Δ_π	Q	$Q \cap L$
Δ_1	Q_1	(i)	Δ_{10}	Q_5	(ii)
Δ_2	Q_2		Δ_{11}	Q_6	(iii), a_3
Δ_3	Q_3		Δ_{12}	Q_{12}	
Δ_4	Q_4		Δ_{13}	Q_7	(iii), a_6
Δ_5	Q_5		Δ_{14}	Q_8	(v), a_4
Δ_6	Q_9		Δ_{15}	Q_{10}	(iv), $2a_3$
Δ_7	Q_1	(ii)	Δ_{16}	Q_{13}	
Δ_8	Q_2		Δ_{17}	Q_{11}	(v), a_7
Δ_9	Q_4		Δ_{18}	Q_{14}	(v), ordinary 4-ple point

Table 1: Possible $Q + L$

of Q (see the Table 2), the third and sixth refer to singular points of Q contained in $Q \cap L$ and the relative position between Q and L, the number being one as follows:

(i) L is a bitangent line of Q at two smooth points.

(ii) L is a tangent line of Q at a smooth point with multiplicity 4.

(iii) L is tangent to Q at one smooth point and passes through one singular point of Q.

(iv) L passes through two distinct singular points of Q.

(v) L meets Q at just one singular point.

For the types of singular points of curves, we use those in [1]. Note that we use small letters.

It is known that all configurations as above occur (see [9], for example). Hence it may be natural to rise a question as follows:

Question 1.1 Let $\pi : X \to \mathbb{P}^2$ be a non-Galois triple covering corresponding to one of the D_6-coverings of type II as above 18 types. Find a rational map $\Phi_\pi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ so that X is birationally equivalent to $\mathbb{P}^2 \times_{\nu_\pi} S$. In other words, find a cubic equation over $\mathbb{C}(\mathbb{P}^2)$ which gives X.

Our main purpose of this article is to find Φ_π explicitly. In order to explain our result, we need some more settings.

Q	Irreducible components	Singular points
Q_1	irreducible	$2a_2$
Q_2	irreducible	$a_1 + 2a_2$
Q_3	irreducible	$3a_2$
Q_4	irreducible	a_5
Q_5	irreducible	e_6
Q_6	irreducible	$a_2 + a_3$
Q_7	irreducible	a_6
Q_8	irreducible	$a_2 + a_4$
Q_9	two conics	$a_1 + a_5$
Q_{10}	two conics	$2a_3$
Q_{11}	two conics	a_7
Q_{12}	a cuspital cubic and a line	$a_1 + a_2 + a_3$
Q_{13}	a conics and two lines	$2a_3 + a_1$
Q_{14}	four lines	ordinary 4-ple point

Table 2: The list of Q

Let $\Phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a rational map. We denote the resolution of indeterminacy of Φ as follows:

$$\begin{tikzcd} & \widehat{\mathbb{P}^2} \arrow[dl, "q"'] \arrow[dr, "\nu_\Phi"] & \\ \mathbb{P}^2 \arrow[rr, dotted, "\Phi"'] & & \mathbb{P}^2, \end{tikzcd} \tag{1.1}$$

where q is a succession of blowing-ups and ν_Φ is the induced morphism.

Let $\widehat{\mathbb{P}^2} \times_{\nu_\Phi} S$ be the fiber product of $\widehat{\mathbb{P}^2}$ and S, and we denote the induced projection by $\mathrm{pr}_1 : \widehat{\mathbb{P}^2} \times_{\nu_\Phi} S \to \widehat{\mathbb{P}^2}$. If $\widehat{\mathbb{P}^2} \times_{\nu_\Phi} S$ is irreducible, then the normalization $(\widehat{\mathbb{P}^2} \times_{\nu_\Phi} S)^n$ of $\widehat{\mathbb{P}^2} \times_{\nu_\Phi} S$ is a non-Galois triple covering of $\widehat{\mathbb{P}^2}$. Hence the Stein factorization X_Φ of $(\widehat{\mathbb{P}^2} \times_{\nu_\Phi} S)^n \to \mathbb{P}^2$ is a non-Galois triple covering of $\mathbb{P}^2$. We denote its covering morphism by $\pi_\Phi : X_\Phi \to \mathbb{P}^2$.

Let $f_\Phi : Z \to \mathbb{P}^2$ be the stein factorization of $\nu_\Phi : \widehat{\mathbb{P}^2} \to \mathbb{P}^2$. Then we have

$$\begin{tikzcd} & \widehat{\mathbb{P}^2} \arrow[ddl, "q"'] \arrow[dr, "\mu"] & \\ & & Z \arrow[d, "f_\Phi"] \\ \mathbb{P}^2 \arrow[rr, dotted, "\Phi"] & & \mathbb{P}^2, \end{tikzcd}$$

where μ is a morphism with connected fibers and $\nu_\Phi = f_\Phi \circ \mu$.

For our question, it turns out to be enough to consider the following four cases:

A. The degree of f_Φ is 2 and Δ_{f_Φ} is a smooth conic.

B. The degree of f_Φ is 2 and Δ_{f_Φ} is two distinct lines.

C. The morphism f_Φ is an isomorphism, i.e., ν_Φ is birational.

D. The image of f_Φ is a curve.

In this article, for a non-Galois triple covering $\pi : X \to Y$, π is called totally branched (resp. simply branched) at $y \in \Delta_\pi$ if $\sharp\pi^{-1}(y) = 1$ (resp. $\sharp\pi^{-1}(y) = 2$).

We also note that, for the non-Galois triple covering $\varpi : S \to \mathbb{P}^2$ as before, one can easily see that

- $\Delta_\varpi = C(\varpi) + L_\infty$, where

$$\begin{array}{rcl} C(\varpi): & U^3 + V^2W & = 0 \\ L_\infty: & W & = 0, \end{array}$$

  and

- $[0 : 1 : 0]$ and $[0 : 0 : 1]$ are the only total branched points,

where we choose a homogeneous coordinate $[U : V : W]$ of $\mathbb{P}^2$ in such a way that $u = U/W, v = V/W$.

We are now in a position to state our result.

Theorem 1.1 *For each Δ_i in Table 1, the rational maps described below give rise to a non-Galois triple covering corresponding to a D_6-covering of type II with branch locus of type Δ_i.*

Δ_π	*Type of* Φ	*Relative position between* Δ_{f_Φ} *and* Δ_ϖ
Δ_1		(L1) *and* (C1)
Δ_2		(L1) *and* (C2)
Δ_3	A	(L1) *and* (C3)
Δ_4		(L1) *and* (C4)
Δ_5		(L1) *and* (C5)
Δ_6		(L1) *and* (C6)
Δ_7		(L2) *and* (C7)
Δ_8	B	(L2) *and* (C8)
Δ_9		(L2) *and* (C9)
Δ_{10}		(L2) *and* (C10)

(L1) Φ *is of type A.* Δ_{f_Φ} *is tangent to* L_∞ *at* $[0 : 1 : 0]$.

(L2) *Φ is of type B. L_∞ is an irreducible component of Δ_{f_Φ}. We write $\Delta_{f_\Phi} = L_\infty + L_o$.*

In the following, we use the following notation: For reduced curves D_1 and D_2 on $\mathbb{P}^2$, $D_1 \cdot D_2 = \sum_{i=1}^{s} m_i p_i$ means that $D_1 \cap D_2 = \{p_1,\ p_2,\ \cdots,\ p_s\}$ and the intersection multiplicity at p_i is m_i.

(C1) *Φ is of type A. $\Delta_{f_\Phi} \cdot C(\varpi) = 2[0:1:0] + p_{11} + p_{12} + p_{13} + p_{14}$.*

(C2) *Φ is of type A. $\Delta_{f_\Phi} \cdot C(\varpi) = 2[0:1:0] + 2p_{21} + p_{22} + p_{23}$.*

(C3) *Φ is of type A. $\Delta_{f_\Phi} \cdot C(\varpi) = 2[0:1:0] + 3p_{31} + p_{32}$.*

(C4) *Φ is of type A. $\Delta_{f_\Phi} \cdot C(\varpi) = 2[0:1:0] + 2[0:0:1] + p_{41} + p_{42}$.*

(C5) *Φ is of type A. $\Delta_{f_\Phi} \cdot C(\varpi) = 2[0:1:0] + 3[0:0:1] + p_{51}$.*

(C6) *Φ is of type A. $\Delta_{f_\Phi} \cdot C(\varpi) = 2[0:1:0] + 2[0:0:1] + 2p_{61}$.*

(C7) *Φ is of type B. $L_o \cdot C(\varpi) = p_{71} + p_{72} + p_{73}$.*

(C8) *Φ is of type B. $L_o \cdot C(\varpi) = 2p_{81} + p_{82}$.*

(C9) *Φ is of type B. $L_o \cdot C(\varpi) = 2[0:0:1] + p_{91}$.*

(C10) *Φ is of type B. $L_o \cdot C(\varpi) = 3[0:0:1]$.*

Here, p_{ij} $(i = 1, \dots, 9,\ j = 1, \dots, 4)$ are distinct smooth points of $C(\varpi)$.

Δ_π	*Type of* Φ	Δ_π	*Type of* Φ
Δ_{11}	C	Δ_{15}	D
Δ_{12}		Δ_{16}	
Δ_{13}		Δ_{17}	
Δ_{14}		Δ_{18}	

For Δ_i $(11 \leq i \leq 18)$, the detailed description is given in Section 5 and Section 6

In Section 2, we prepare some result used in the proof of the main result and notations used in this paper. In Section 3, 4, 5 and 6, we prove Theorem 1.1 for the case A, B, C and D, respectively.

2. Preliminaries

Let $\varpi : S \to \mathbb{P}^2$ be the non-Galois triple covering as in Introduction. Let $\widehat{\mathbb{C}(S)}$ be the Galois closure of $\mathbb{C}(S)$ over $\mathbb{C}(\mathbb{P}^2)$. Let $\widehat{S}$ be the $\widehat{\mathbb{C}(S)}$-normalization of $\mathbb{P}^2$. Since $\widehat{\mathbb{C}(S)}$ is a D_6-extension of $\mathbb{C}(\mathbb{P}^2)$, $\widehat{S}$ is a D_6-covering of $\mathbb{P}^2$. Also, $\widehat{S}$ is a double covering of S. We denote the induced covering morphisms by $\widehat{\varpi} : \widehat{S} \to \mathbb{P}^2$ and $\alpha : \widehat{S} \to S$, respectively.

Let us start with the following lemma:

Lemma 2.1 *Let Y be a normal projective variety and let $f : Y \to \mathbb{P}^2$ be a morphism. If f is either an isomorphism or a p-fold covering ($p := 2$ or odd) with $\Delta_f \neq \Delta_\varpi$, then $Y \times_f S$ is irreducible if and only if $Y \times_f \widehat{S}$ is irreducible.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
Y \times_f \widehat{S} \arrow[r] \arrow[d] & Y \times_f S \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & Y \arrow[d, "f"] \\
\widehat{S} \arrow[r, "\alpha"] & S \arrow[r, "\varpi"] & \mathbb{P}^2.
\end{tikzcd}$$

If $Y \times_f \widehat{S}$ is irreducible, $Y \times_f S$ is irreducible as $Y \times_f \widehat{S} \to Y \times_f S$ is dominant.

Conversely suppose that $Y \times_f S$ is irreducible. We assume that f is either an isomorphism or a p-fold covering ($p = 2$ or odd) with $\Delta_f \neq \Delta_\varpi$. Put $\mathrm{Fix}(D_6) := \bigcup_{\sigma \in D_6 \setminus \{\mathrm{id}\}} \{\widehat{s} \in \widehat{S} \mid \sigma(\widehat{s}) = \widehat{s}\}$. $\mathrm{pr}_2(Y \times_f S) = S \not\subset \alpha(\mathrm{Fix}(D_6))$. Since α is a double covering, $(Y \times_f S) \times_{\mathrm{pr}_2} \widehat{S}$ is irreducible ([5, Proposition 2.4]).

For all elements $((y, s), \widehat{s})$ in $(Y \times_f S) \times_{\mathrm{pr}_2} \widehat{S}$, $\alpha(\widehat{s}) = \mathrm{pr}_2(y, s) = s$. Consider the following projection:

$$\begin{array}{rcl} (Y \times_f S) \times_{\mathrm{pr}_2} \widehat{S} & \to & Y \times_f \widehat{S} \\ ((y, \alpha(\widehat{s})), \widehat{s}) & \mapsto & (y, \widehat{s}). \end{array}$$

Since this projection is surjective, $Y \times_f \widehat{S}$ is irreducible. $\square$

We also use the fact below, which can be checked easily:

Fact 2.1 *Let M be a smooth surface. Let B and C be a reduced curve on M. Assume that $B \cap C \neq \phi$ and that there exists a double covering $g : X \to M$ over M with $\Delta_g = B$. Let p be a point in $B \cap C$.*

(*i*) *Assume that both B and C are smooth at p.*

(*i*-1) *If C is tangent to B at p with multiplicity 2, then the pull-back g^*C has an a_1 singular point q with $g(q) = p$.*

(*i*-2) *If C is tangent to B at p with multiplicity 3, then the pull-back g^*C has an a_2 singular point q with $g(q) = p$.*

(*ii*) *Assume that B is smooth at p and that C has an a_2 singular point at p.*

(*ii*-1) *If B and C do not have the same tangent line at p, then the pull-back g^*C has an a_5 singular point q with $g(q) = p$.*

(*ii*-2) *If B and C have the same tangent line at p, then the pull-back g^*C has an e_6 singular point q with $g(q) = p$.*

3. The cases when the rational maps Φ are of type A

Let $f : Z \to \mathbb{P}^2$ be the double covering whose branch locus is an irreducible conic. Note that $Z \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let us start with the following lemma:

Lemma 3.1 *Let C be the branch locus of f. Assume that C is tangent to L_∞ at $[0 : 1 : 0]$. Then we have the following:*

- *The pull-back f^*L_∞ is of the form $L^+ + L^-$, and $L^\pm$ define two rulings of $Z \cong \mathbb{P}^1 \times \mathbb{P}^1$.*
- *Put $\tilde{p} = f^{-1}([0 : 1 : 0])$. The pull-back $f^*C(\varpi)$ has two local analytic branches at $\tilde{p}$. If we denote them by $C^+_{\tilde{p}}$ and $C^-_{\tilde{p}}$,suitably. Then we may assume that the local intersection numbers at $\tilde{p}$ satisfy*

$$(C^+_{\tilde{p}} \cdot L^+)_{\tilde{p}} = (C^-_{\tilde{p}} \cdot L^-)_{\tilde{p}} = 2, \quad (C^+_{\tilde{p}} \cdot L^-)_{\tilde{p}} = (C^-_{\tilde{p}} \cdot L^+)_{\tilde{p}} = 1.$$

Proof. Since both $C(\varpi)$ and L_∞ meet C at $[0 : 1 : 0]$ with multiplicity 2, one can easily see that both $f^*C(\varpi)$ and f^*L_∞ have two local analytic branches at $\tilde{p}$. Since $\sharp(L_\infty \cap C) = 1$, f^*L_∞ has two irreducible components L^+ and L^-. As $2 = (f^*L)^2 = (L^+)^2 + 2L^+ \cdot L^- + (L^-)^2 = 2(L^+)^2 + 2L^+ \cdot L^-$, $(L^\pm)^2 = 0$, and $L^\pm$ define two ruling of Z. For the local intersection number at $\tilde{p}$, it follows from equality

$$6 = (f^*C(\varpi) \cdot f^*L_\infty)_{\tilde{p}}, \quad (C^+_{\tilde{p}} \cdot L^+)_{\tilde{p}} = (C^-_{\tilde{p}} \cdot L^-)_{\tilde{p}}, \quad (C^+_{\tilde{p}} \cdot L^-)_{\tilde{p}} = (C^-_{\tilde{p}} \cdot L^+)_{\tilde{p}}.$$

□

We only prove Theorem 1.1 for the case of Δ_1, as the remaining cases of type A can be proved similarly. Let $f : Z \to \mathbb{P}^2$ be a double covering whose branch locus is a conic of type (C1) in Theorem 1.1 (see Figure 1). Let $\mu : \widehat{Z} \to Z$ be the blowing-up at $L^+ \cap L^-$ (Figure 1). Let $\overline{L}^+$ and $\overline{L}^-$ be the strict transforms of

L^+ and L^-, respectively, and let E be the exceptional curve of μ. Since both $\overline{L}^+$ and $\overline{L}^-$ are the exceptional curves of the first kind, we can blow them down and the resulting surface is $\mathbb{P}^2$. We denote this construction by $q_1 : \widehat{Z} \to \mathbb{P}^2$ (Figure 1). Put $\Phi := f \circ \mu \circ q_1^{-1}$. By Lemma 2.1, $Z \times_f S$ is irreducible. Following the notation in Introduction, we have $\widehat{\mathbb{P}^2} = \widehat{Z}$, $q = q_1$, $\mu = \mu$ and $f_\Phi = f$. Hence we have the induced non-Galois triple covering $\pi_\Phi : X_\Phi \to \mathbb{P}^2$. By its construction, $\Phi^*\Delta_\varpi$ consists of a quartic Q of type Q_1 and a bitangent line $q(E)$. Since Q come from $C(\varpi)$ and $q(E)$ is mapped to $[0 : 1 : 0]$ by Φ, the branch locus of the induced non-Galois triple covering $\pi_\Phi : X_\Phi \to \mathbb{P}^2$ by Φ is a quintic of type Δ_1 such that π_Φ is simply branched along Q, while it is totally branched along $q(E)$.

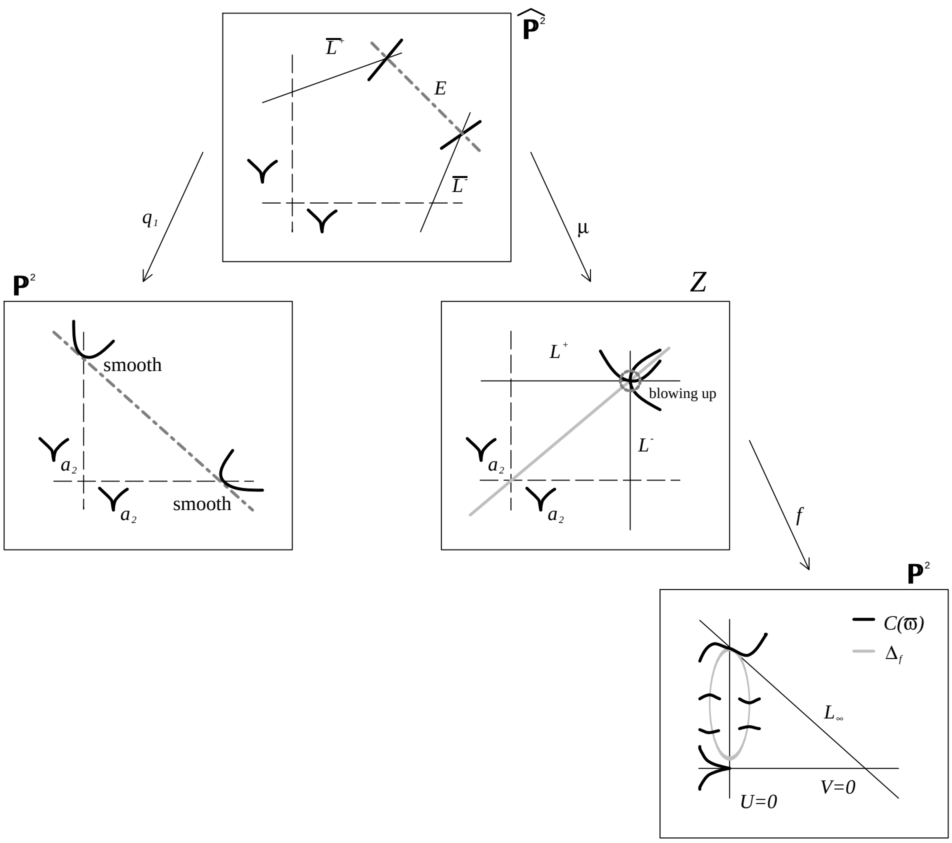

Figure 1: The case of C1

We end this section by giving explicit examples of Φ for each case.

Example 3.1 For each case, we have examples as in Table 3. In Table 3, we denote inhomogeneous coordinates of the domain $\mathbb{P}^2$ of Φ_π by (x, y).

Δ_{π_Φ}	$\Phi_\pi^* u$	$\Phi_\pi^* v$
Δ_1	x	$(y-1)(y-x)$
Δ_2	$4x-4$	$4x^2+28-2y^2$
Δ_3	$4x+36$	$x^2-108+3y^2$
Δ_4	x	$y(y-x)+2x$
Δ_5	x	$y(y-x)$
Δ_6	x	$y(y-x)+x$

Table 3: Examples for type A

4. The cases when the rational maps Φ are of type B

We only prove Theorem 1.1 for the case of Δ_7, as the remaining cases of type B can be proved similarly. Note that L_o meets L_∞ at just one point. Let $\nu : (\mathbb{P}^2)_1 \to \mathbb{P}^2$ be the blowing-up at $L_o \cap L_\infty$. We denote the exceptional curve by E_1 and the strict transforms of $C(\varpi)$, L_o and L_∞ by $\overline{C(\varpi)}$, $\overline{L_o}$ and $\overline{L_\infty}$, respectively (see Figure 2). Let $f : Z \to \mathbb{P}^2$ (resp. $g : \Sigma_2 \to (\mathbb{P}^2)_1$) be a double covering branched along $L_o + L_\infty$ (resp. $\overline{L_o} + \overline{L_\infty}$) (Figure 2). Then there exists a morphism $\nu' : \Sigma_2 \to Z$ (see [2]). Since $\overline{C(\varpi)}$ is tangent to $\overline{L_\infty}$ with multiplicity 3, by Fact 2.1, $g^*\overline{C(\varpi)}$ has an a_2 singular point p. Let $\mu : (\Sigma_2)_2 \to \Sigma_2$ be the blowing-up at p. We denote the exceptional curve by E_2 and strict transforms of $g^*\overline{L_o}$, $g^*\overline{L_\infty}$ and g^*E_1, by $\overline{L_{o2}}$, $\overline{L_{\infty 2}}$ and $\overline{E_1}$, respectively (Figure 2). Let $q_7 : (\Sigma_2)_2 \to \Sigma$ be the blowing-down the curves $\overline{L_{\infty 2}}$ and $\overline{E_{12}}$ in this order (Figure 2). Then $\Sigma = \mathbb{P}^2$. Put $\Phi := f \circ \nu' \circ \mu \circ {q_7}^{-1}$. Following the notation in Introduction, we have $\widehat{\mathbb{P}^2} = (\Sigma_2)_2$, $q = q_7$, $\nu_\Phi = \mu \circ \nu' \circ f$ and $f_\Phi = f$. By Lemma 2.1, $Z \times_f S$ is irreducible. Hence we have the induced non-Galois triple covering $\pi_\Phi : X_\Phi \to \mathbb{P}^2$. By its construction, $\Phi^*\Delta_\varpi$ consists of a quartic Q of type Q_1 and a line $q(E_2)$. Moreover $q(E_2)$ is tangent to Q with multiplicity 4. Since Q come from $C(\varpi)$ and $q(E_2)$ is mapped to $[0:1:0]$ by Φ, the branch locus of the induced non-Galois triple covering $\pi_\Phi : X_\Phi \to \mathbb{P}^2$ by Φ is a quintic of type Δ_7 such that π_Φ is simply branched along Q, while it is totally branched along $q(E_2)$.

We end this section by giving explicit examples of Φ for each case.

Example 4.1 . For each case, we have example as Table 4. In Table 4, we use the same notation as Example 3.1.

5. The cases when the rational maps Φ are of type C

We first introduce some notation. Let $\mu_1 : (\mathbb{P}^2)_1 \to \mathbb{P}^2$ be the blowing-up at $[0:1:0]$. We denote the strict transform of $C(\varpi)$ and L_∞ by $\overline{C(\varpi)}$ and $\overline{L_\infty}$, respectively. $\overline{C(\varpi)}$ is tangent to $\overline{L_\infty}$ at a point p with $\mu_1(p) = [0:1:0]$. The exceptional curve E_1 of μ_1 meets $\overline{C(\varpi)}$ at p. Let $\mu_2 : (\mathbb{P}^2)_2 \to (\mathbb{P}^2)_1$ be the blowing-up at p. We denote the exceptional curve of μ_2 by E_2 and the strict transform of E_1, $\overline{C(\varpi)}$ and $\overline{L_\infty}$ by $\overline{E_{12}}$, $\overline{C(\varpi)_2}$ and $\overline{L_{\infty 2}}$, respectively.

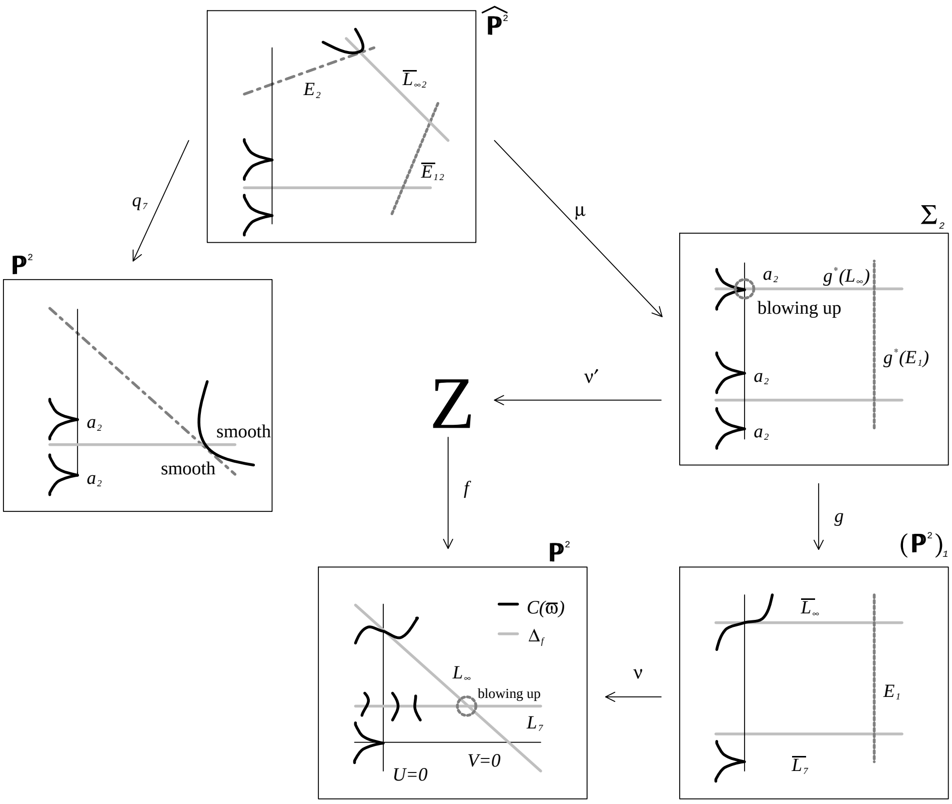

Figure 2: The case of C7

Δ_{π_Φ}	$\Phi_\pi^* u$	$\Phi_\pi^* v$
Δ_7	x	$y^2 - y$
Δ_8	x	$y^2 - 3x - 4$
Δ_9	x	$y^2 - x$
Δ_{10}	x	y^2

Table 4: Examples for type B

We prove Theorem 1.1 for Δ_i $(11 \leq i \leq 14)$.

The case of Δ_{11}

Choose a general line L_1 on $\mathbb{P}^2$ passing through $[0:1:0]$ (see Figure 3). We denote the strict transform of L_1 by $\overline{L_{12}}$ with respect to $\mu_1 \circ \mu_2$. Choose a general point $p_{\Delta_{11}}$ in $\overline{L_{12}} \setminus \overline{C(\varpi)}_2$. Let $\mu_3 : (\mathbb{P}^2)_3 \to (\mathbb{P}^2)_2$ be the blowing-up at $p_{\Delta_{11}}$. We denote the exceptional curve by E_3, and the strict transform of $\overline{L_{12}}$ by $\overline{L_{13}}$. For the strict transforms of $\overline{C(\varpi)}_2$, $\overline{L_{\infty 2}}$, $\overline{E_{12}}$ and E_2, we use the same notations as μ_3 has nothing to do with these curves. Let $q_{11} : (\mathbb{P}^2)_3 \to \Sigma$ be the blowing-down the curves $\overline{L_{\infty 2}}$, $\overline{L_{13}}$ and $\overline{E_{12}}$ in this order (Figure 3). Since Σ is a rational surface with Picard number one, $\Sigma \simeq \mathbb{P}^2$. Define a birational map $\Phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ by $\mu_1 \circ \mu_2 \circ \mu_3 \circ {q_{11}}^{-1}$. Following the notation in Introduction, we have $\widehat{\mathbb{P}^2} = (\mathbb{P}^2)_3$, $Z = \mathbb{P}^2$, $q = q_{11}$ and $\mu = \mu_1 \circ \mu_2 \circ \mu_3$. By Lemma 2.1 and [5, Proposition 2.4], $Z \times_f S$ is irreducible. Hence we have the induced non-Galois triple covering $\pi_\Phi : X_\Phi \to \mathbb{P}^2$. By its construction, $\Phi^*\Delta_\varpi$ consists of an irreducible quartic Q of type Q_6 and a line $q(E_2)$. Moreover $q(E_2)$ is tangent to Q at a smooth point of Q and pass through an a_3 singular point of Q. Since Q comes from $C(\varpi)$ and $q(E_2)$ is mapped to $[0:1:0]$ by Φ, the branch locus of the induced non-Galois triple covering $\pi_\Phi : X_\Phi \to \mathbb{P}^2$ is a quintic of type Δ_{11} such that π_Φ is simply branched along Q, while it is totally branched along $q(E_2)$.

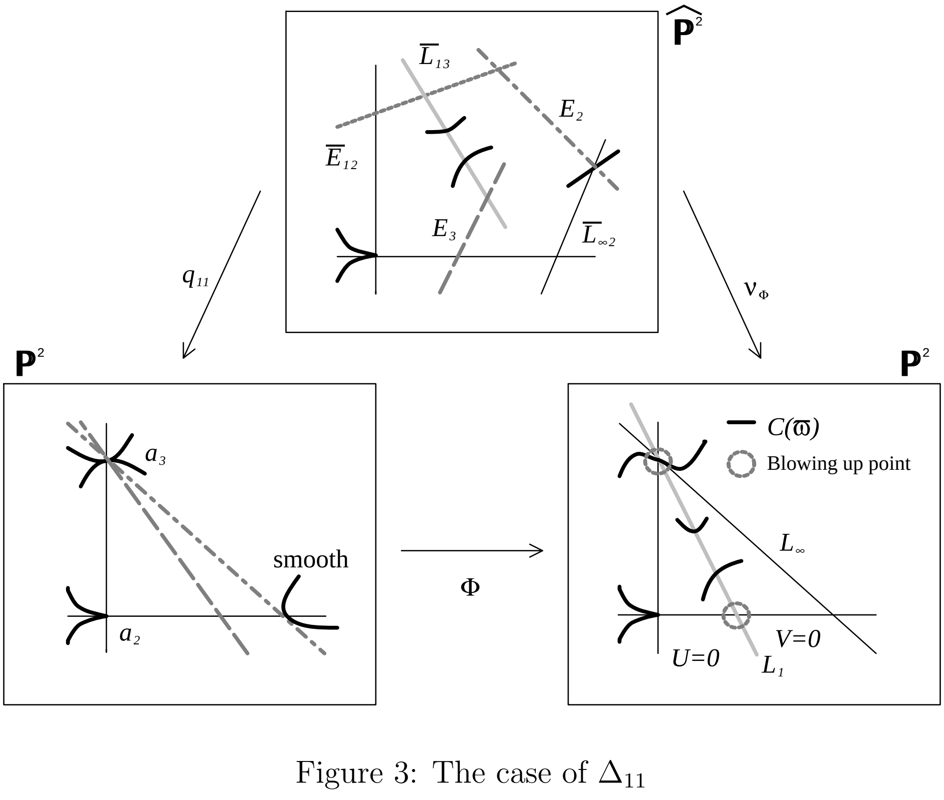

Figure 3: The case of Δ_{11}

The case of Δ_{12}

Choose a general line L_2 on $\mathbb{P}^2$ through $[0:1:0]$ (Figure 4). We denote the strict transform of L_2 by $\overline{L_{22}}$ with respect to $\mu_1 \circ \mu_2$. Choose a point $p_{\Delta_{12}}$ in $\overline{L_{22}} \cap \overline{C(\varpi)}_2$. Let $\mu_4 : (\mathbb{P}^2)_4 \to (\mathbb{P}^2)_2$ be the blowing-up at $p_{\Delta_{12}}$. We denote the exceptional curve by E_4 and the strict transforms $\overline{L_{22}}$ and $\overline{C(\varpi)}_2$ by $\overline{L_{24}}$ and $\overline{C(\varpi)}_4$, respectively. For the strict transforms of $\overline{L_{\infty 2}}$, $\overline{E_{12}}$ and E_2, we use the same notations as μ_4 has nothing to do with these curves. Let $q_{12} : (\mathbb{P}^2)_4 \to \Sigma$ be the blowing-down the curves $\overline{L_{\infty 2}}$, $\overline{L_{24}}$ and $\overline{E_{12}}$ in this order (Figure 4). Likewise the previous case, since Σ is a rational surface with Picard number one, $\Sigma \simeq \mathbb{P}^2$. Define a birational map $\Phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ by $\mu_1 \circ \mu_2 \circ \mu_4 \circ {q_{12}}^{-1}$. We have the induced non-Galois triple covering $\pi_\Phi : X_\Phi \to \mathbb{P}^2$. Following the notation in Introduction, we have $\widehat{\mathbb{P}^2} = (\mathbb{P}^2)_4$, $Z = \mathbb{P}^2$, $q = q_{12}$ and $\mu = \mu_1 \circ \mu_2 \circ \mu_4$. Then $\Phi^*\Delta_\varpi$ consists of a cuspital cubic C_{12}, two distinct line $q(E_2)$ and $q(E_4)$. $q(E_4)$ is tangent to C_{12} at a smooth point p. $q(E_2)$ is tangent to C_{12} at a smooth point and pass through p. Since $C_{12} \cup q(E_4)$ come from $C(\varpi)$ and $q(E_2)$ is mapped to $[0:1:0]$ by Φ. Hence the branch locus of the induced non-Galois triple covering $\pi_\Phi : X_\Phi \to \mathbb{P}^2$ is a quintic of type Δ_{12} such that π_Φ is simply branched along $C_{12} \cup q(E_4)$, while it is totally branched along $q(E_2)$.

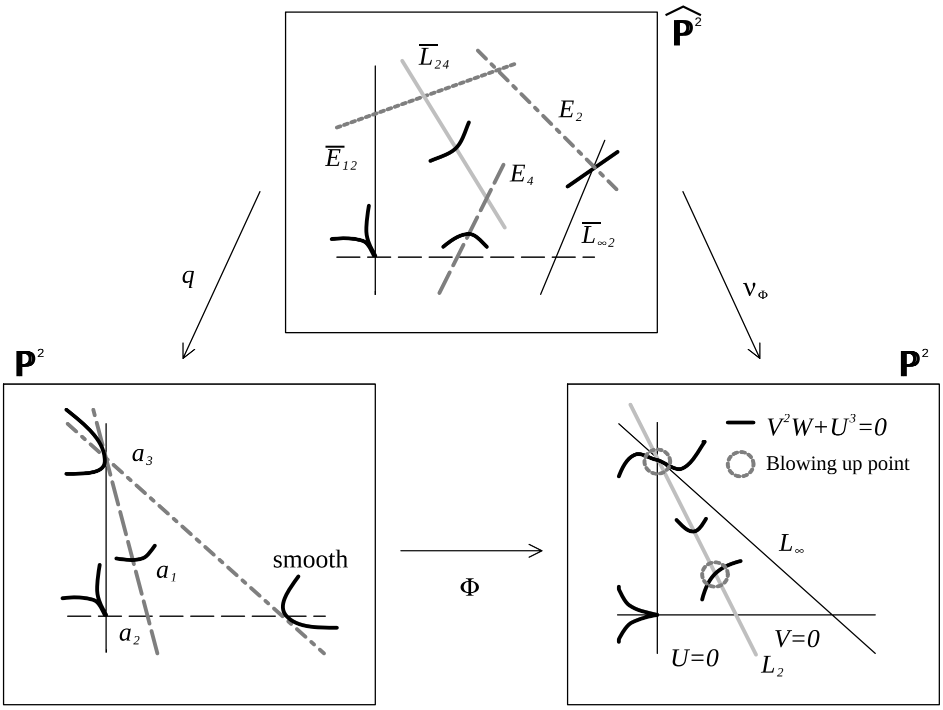

Figure 4: The case of Δ_{12}

The case of Δ_{13}

Let $\overline{L_{32}}$ be the strict transform of $U = 0$ by $\mu_1 \circ \mu_2$. Choose a general point $p_{\Delta_{13}}$ in $\overline{L_{32}}$. Let $\mu_5 : (\mathbb{P}^2)_5 \to (\mathbb{P}^2)_2$ be the blowing-up at $p_{\Delta_{13}}$. We denote the exceptional curve by E_5 and the strict transform of $\overline{L_{32}}$ by $\overline{L_{35}}$. For the strict transforms of $\overline{L_{\infty 2}}$, $\overline{C(\varpi)_2}$, $\overline{E_{12}}$ and E_2, we use the same notations as μ_5 has nothing to do with these curves. Let $q_{13} : (\mathbb{P}^2)_5 \to \Sigma$ be the blowing-down the curves $\overline{L_{35}}$, $\overline{E_{12}}$ and $\overline{L_{\infty 2}}$ in this order (Figure 5). Again, $\Sigma \simeq \mathbb{P}^2$ and put $\widehat{\mathbb{P}^2} = (\mathbb{P}^2)_5$, $Z = \mathbb{P}^2$, $q = q_{13}$, $\mu = \mu_1 \circ \mu_2 \circ \mu_5$ and $\Phi = \mu \circ q^{-1}$. We have the induced non-Galois triple covering $\pi_\Phi : X_\Phi \to \mathbb{P}^2$. Then $\Phi^*\Delta_\varpi$ consists of an irreducible quartic Q of type Q_7 and a line $q(E_2)$ which is tangent to Q at a smooth point and pass through an a_6 singular point of Q. Since Q comes from $C(\varpi)$ and $q(E_2)$ is mapped to $[0 : 1 : 0]$ by Φ, the branch locus of the induced non-Galois triple covering $\pi_\Phi : X_\Phi \to \mathbb{P}^2$ is a quintic of type Δ_{13} such that π_Φ is simply branched along Q, while totally branched along $q(E_2)$.

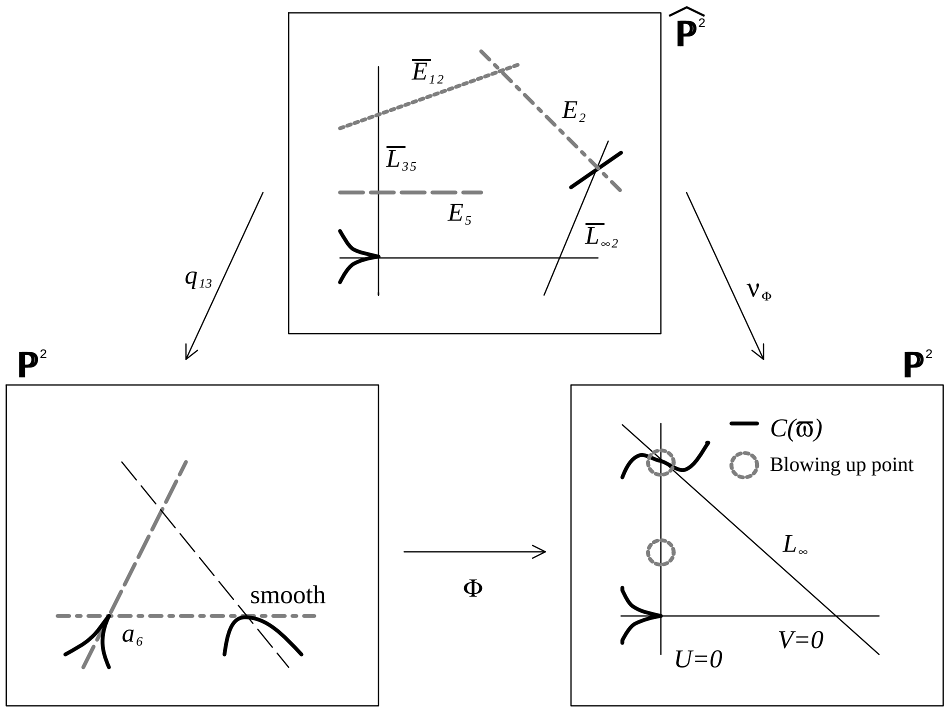

Figure 5: The case of Δ_{13}

The case of Δ_{14}

Choose a general point $p_{\Delta_{14}}$ in E_2 on $(\mathbb{P}^2)_2$. Let $\mu_6 : (\mathbb{P}^2)_6 \to (\mathbb{P}^2)_2$ be the blowing-up at $p_{\Delta_{14}}$. We denote the exceptional curve by E_6 and the strict transform of E_2 by

$\overline{E_{26}}$. For the strict transforms of $\overline{C(\varpi)}_2$, $\overline{L_{\infty 2}}$, and $\overline{E_{12}}$, we use the same notations as μ_6 has nothing to do with these curves. Let $q_{14} : (\mathbb{P}^2)_6 \to \Sigma$ be the blowing-down the curves $\overline{L_{\infty 2}}$, $\overline{E_{26}}$ and $\overline{E_{12}}$ in this order (Figure 6). Again, $\Sigma \simeq \mathbb{P}^2$ and put $\widehat{\mathbb{P}^2} := (\mathbb{P}^2)_6$, $Z := \mathbb{P}^2$, $q := q_{14}$, $\mu := \mu_1 \circ \mu_2 \circ \mu_6$ and $\Phi := \mu \circ q^{-1}$. We have the induced non-Galois triple covering $\pi_\Phi : X_\Phi \to \mathbb{P}^2$. By its construction, $\Phi^*\Delta_\varpi$ consists of an irreducible quartic Q of type Q_8 and a line $q(E_6)$ which is tangent to Q at an a_4 singular point. Since Q comes from $C(\varpi)$ and $q(E_6)$ is mapped to $[0:1:0]$ by Φ, the branch locus of the induced non-Galois triple covering $\pi_\Phi : X_\Phi \to \mathbb{P}^2$ is a quintic of type Δ_{14} such that π_Φ is simply branched along Q, while it is totally branched along $q(E_6)$.

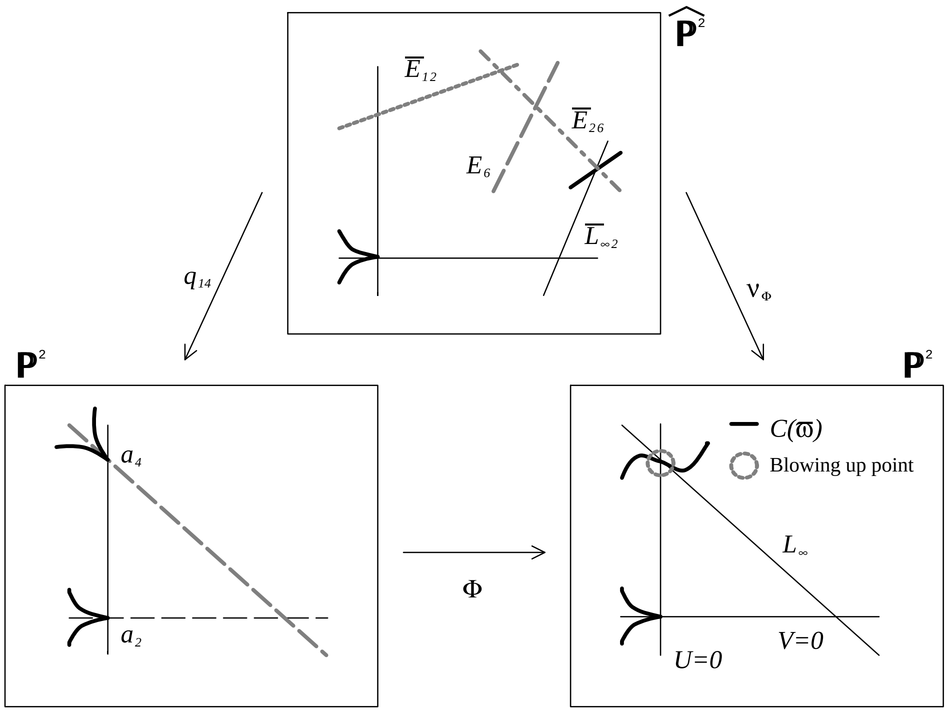

Figure 6: The case of Δ_{14}

We end this section by giving explicit examples of Φ for each case.

Example 5.1 For each case, we have examples as in Table 5. In Table 5, we use the same notation as Example 3.1.

6. The cases when the rational maps Φ are of type D

We consider four rational maps $\Phi_i : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ $(i = 1. \ldots, 4)$ as in Table 6. In Table 6, we denote homogeneous coordinates of the domain $\mathbb{P}^2$ of rational maps Φ_i by $[X : Y : Z]$.

Δ_{π_Φ}	$\Phi_\pi^* u$	$\Phi_\pi^* v$
Δ_{11}	x	$y - xy$
Δ_{12}	$x - 1$	$1 - xy$
Δ_{13}	x	$1 - xy$
Δ_{14}	x	$y - x^2$

Table 5: Examples for type C

Φ_1	$[-Z^2 : XY : Z^2]$
Φ_2	$[-Z^2 : Z^2 - XY : Z^2]$
Φ_3	$[-Z^2 : XZ - Y^2 : Z^2]$
Φ_4	$[-Z^2 : Y(Y - 3Z) : Z^2]$

Table 6: The rational maps Φ_i

For each case, $\operatorname{Im}\Phi_i = \{[U : V : W] \mid U = -W\}$ and $\Delta_\varpi \cap \operatorname{Im}\Phi_i = \{[-1 : 1 : 1], [-1 : -1 : 1], [0 : 1 : 0]\}$. Consider the diagram (1.1) in Introduction. Put $\Phi := \Phi_i$, $C_1 := {\nu_\Phi}^{-1}([-1 : 1 : 1])$, $C_2 := {\nu_\Phi}^{-1}([-1 : -1 : 1])$ and $C_\infty := {\nu_\Phi}^{-1}([0 : 1 : 0])$ (Figure 7).

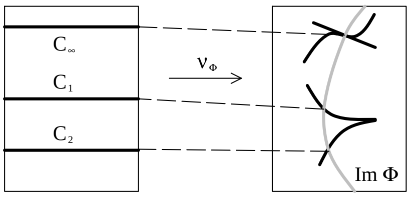

Figure 7: A map $\nu_\Phi : (\mathbb{P}^2)_* \to \mathbb{P}^2$

We only prove Theorem 1.1 for the case Δ_{15}, as the remaining cases of type D can be proved similarly. Consider the rational map Φ_1. The points of indeterminacy of Φ_1 are $[0 : 1 : 0]$ and $[1 : 0 : 0]$. q consists of four blowing-ups. We denote the exceptional curves by E_i, $(i = 1, \ldots, 4)$ (see Figure 8). In this case, C_1 is an irreducible curve, C_2 is also an irreducible curve and C_∞ consists of three irreducible curves E_1, E_3 and C_3 with $E_1 \cap E_3 = \phi$. Then $q(C_1 \cup C_2 \cup C_\infty)$ consists of a line $q(C_\infty)$ and two irreducible conics $q(C_1)$ and $q(C_2)$. $q(C_1)$ is tangent to $q(C_2)$ at two distinct points p and p'. The line $q(C_\infty)$ pass through p and p'. So, $q(C_1 \cup C_2 \cup C_\infty)$ is a quintic of type Δ_{15} (Figure 8).

Consider a morphism $\overline{\nu_\Phi} : \widehat{\mathbb{P}^2} \ni\ a \mapsto \nu_\Phi(a) \in \operatorname{Im}\Phi$. Since $\overline{\nu_\Phi}$ is dominant and the general fiber is connected, the induced field extension $\mathbb{C}(\widehat{\mathbb{P}^2} \times_{\mathbb{P}^2} S)/\mathbb{C}(\operatorname{Im}\Phi)$ by

$\overline{\nu_\Phi} \circ \mathrm{pr}_1$ is a regular extension ([3, Lemma 9.3]). As $\widehat{\mathbb{P}^2} \times_{\mathbb{P}^2} S = \widehat{\mathbb{P}^2} \times_{\mathrm{Im}\,\Phi} \varpi^{-1}(\mathrm{Im}\,\Phi)$, $\widehat{\mathbb{P}^2} \times_{\mathbb{P}^2} S$ is irreducible.

Hence we have the induced non-Galois triple covering $\pi_\Phi : X_{\Phi_1} \to \mathbb{P}^2$. By the construction, Δ_{π_Φ} is a quintic of type Δ_{15} such that π_Φ is simply branched along $q(C_1) \cup q(C_2)$, while it is totally branched along $q(C_\infty)$.

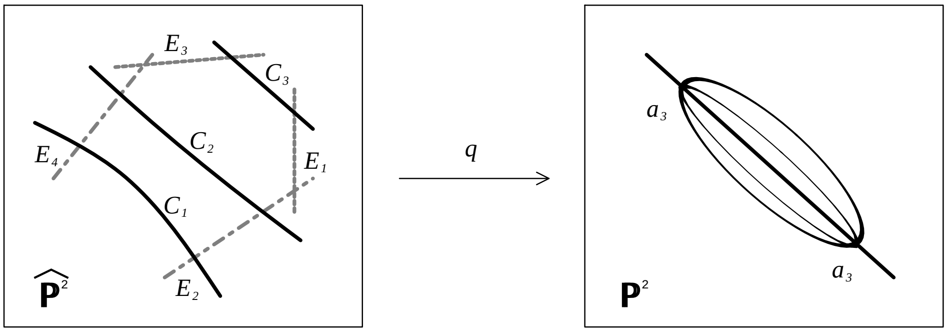

Figure 8: The case of $\Phi = \Phi_1$

The remaining cases are as in Table 7.

Φ_i	Δ_{π_Φ}
Φ_2	Δ_{16}
Φ_3	Δ_{17}
Φ_4	Δ_{18}

Table 7: The remaining cases of type D

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