3. DIRECT PRODUCTS WITH FINITE FACTORS

This section contains the fundamental results of the present work. They concern primarily direct decompositions of finite algebras; as we shall see, however, most of them also apply to direct decompositions of arbitrary algebras -- under the assumption that some of the factors involved are finite.

The notion of an indecomposable algebra will play an important part in our discussion. We define:

Definition 3.1. An algebra

$$\underline{A} = \langle A, +, O_0, O_1, \dots, O_\xi, \dots \rangle$$

is said to be indecomposable if $A \neq \{0\}$ and for any subalgebras B and C of A, $A = B \times C$ implies that $B = \{0\}$ or $C = \{0\}$.

Corollary 3.2. For every finite algebra

$$\underline{A} = \langle A, +, O_0, O_1, \dots, O_\xi, \dots \rangle$$

there exist indecomposable subalgebras $A_0, A_1, \dots, A_\kappa, \dots$ with $\kappa < \nu < \omega$ such that

$$A = \prod_{\kappa < \nu} A_\kappa .$$

Proof: obvious (by induction or by contradiction).

We now give two auxiliary theorems which concern homomorphisms of finite central subalgebras of arbitrary algebras.[13] In formulating and proving these theorems we shall use the familiar notions of the κ-th iteration f^κ of a function f. This notion is understood to be defined recursively in terms of that of the composition fg of two functions f and g; by f^0 we understand the identity function (possibly with the domain restricted to that of f), and we put

$$f^{\kappa+1} = f^\kappa f \text{ for every } \kappa < \omega.$$

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13. These theorems have been established for groups by H. Fitting; cf. Fitting [1], pp. 19 f.

Theorem 3.3. *Let* B *be a finite central subalgebra of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle.$$

If f *is a* B, B-*homomorphism, then there is a subalgebra* C *of* B *such that for some* κ *with* $0 < \kappa < \omega$ *we have*

$$B = (f^\kappa)^*(B) \times C.$$

Proof: The function f maps B upon a subset of B; consequently, for every κ,

$$(f^{\kappa+1})^*(B)) \subseteq (f^\kappa)^*(B).$$

Hence, since B is finite, we have for some κ, with $0 < \kappa < \omega$,

(1) $$(f^{\kappa+1})^*(B) = (f^\kappa)^*(B).$$

Let

(2) $$D = (f^\kappa)^*(B);$$

and let C be the set of all elements $c \varepsilon B$ such that $f^\kappa(c) = 0$. It is easily seen from 2.1 that C and D are central subalgebras of B. From (1) and (2) it follows that

(3) $$(f^\kappa)^*(D) = D.$$

Since D is finite we conclude that f^κ is a D, D-isomorphism; hence for every $d \varepsilon D$

$$f^\kappa(d) = 0 \text{ implies } d = 0.$$

In view of the definition of C this gives

$$D \cap C = \{0\}.$$

Consequently, by 1.7 (i) and 2.4 (ii), D $\times$ C exists and

(4) $$D \times C \subseteq B.$$

If b is any element in B, then, by (2), $f^\kappa(b)$ is in D, and therefore, by (3),

(5) $$f^\kappa(b) = f^\kappa(d) \text{ for some } d \varepsilon D.$$

Hence, by 2.1 (i),

$$(6) \qquad d + b' = 0 \text{ where } b' \varepsilon B.$$

By the definition of C the latter formula gives

$$(7) \qquad b + b' \varepsilon C.$$

By (5), (6), and 2.2 (i), (ii),

$$b = d + (b + b').$$

Therefore, by (5) and (7),

$$b \varepsilon D \times C.$$

This being true for every element $b \varepsilon B$, the inclusion symbol in (4) can be replaced by the equality symbol, and the proof is complete.

Theorem 3.4. *Let* B *be a finite indecomposable central subalgebra of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle;$$

let $f_0, f_1, \ldots, f_\kappa, \ldots$ *with* $\kappa < \nu < \omega$ *be* B, B-*homomorphisms; and let*

$$f(b) = \sum_{\kappa<\nu} f_\kappa(b) \text{ for } b \varepsilon B.$$

If f *is a* B, B-*isomorphism, then at least one of the functions* f_κ *is a* B, B-*isomorphism.*

Proof: We start with the following

Lemma. *If* B *is a central subalgebra of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle,$$

and if g *and* h *are* B, B-*homomorphisms such that*

$$b = g(b) + h(b) \text{ for every } b \varepsilon B,$$

then

$$gh(b) = hg(b) \text{ for every } b \varepsilon B.$$

In fact, if $b \varepsilon B$, then $h(b) \varepsilon B$; hence

$$h(b) = gh(b) + h^2(b). \tag{1}$$

But we also have

$$h(b) = h[g(b) + h(b)] = hg(b) + h^2(b). \tag{2}$$

Since all the elements involved in (1) and (2) are in B,' the conclusion follows by 2.2 (iii).

We now turn to the main theorem. We shall prove it for $\nu = 2$; the proof can easily be completed by induction.

We put

$$g_\kappa = f^{-1} f_\kappa \text{ for } \kappa = 0, 1 \tag{1}$$

where f^{-1} is the inverse of the function f. Clearly g_0 and g_1 are ε
B, B-homomorphisms and

$$b = g_0(b) + g_1(b) \text{ for every } b \varepsilon B. \tag{2}$$

Hence, by 3.3, there exist central subalgebras C_0 and C_1 of B such that for some κ_0 and κ_1 with $0 < \kappa_0 < \omega$ and $0 < \kappa_1 < \omega$ we ε
have

$$B = (g_0^{\kappa_0})^*(B) \times C_0 = (g_1^{\kappa_1})^*(B) \times C_1. \tag{3}$$

Now assume that neither g_0 nor g_1 is a B, B-isomorphism. Then $g_0^*(B)$ and $g_1^*(B)$, and hence also $(g_0^{\kappa_0})^*(B)$ and $(g_1^{\kappa_1})^*(B)$, are proper subsets of B. By 3.1 and (3) this gives

$$(g_0^{\kappa_0})^*(B) = (g_1^{\kappa_1})^*(B) = \{0\}. \tag{4}$$

We can assume that $\kappa_1 \leq \kappa_0$. Then (4) implies

$$g_0^{\kappa}(b) = g_1^{\kappa}(b) = 0 \text{ for } b \varepsilon B \text{ and } \kappa_0 \leq \kappa < \omega. \tag{5}$$

From (2) we obtain by induction, using 2.2 (i), (ii) and the lemma previously established,

$$\sum_{\kappa<\lambda+1} g_0^{\kappa} g_1^{\lambda-\kappa}(b) = b \text{ for } b \varepsilon B. \text{ and } \lambda < \omega. \tag{6}$$

But, by (5), we have

$$\sum_{\kappa<\lambda+1} g_0^{\kappa} g_1^{\lambda-\kappa}(b) = 0 \text{ for } b \varepsilon B \text{ and } 2\kappa_0 < \lambda < \omega.$$

Therefore, by (6),

$$\underline{B} = \{0\}.$$

This, however, is impossible in view of 3.1. Hence our assumption regarding g_0 and g_1 is wrong, and either g_0 or g_1 is a B, B-isomorphism. But, as is easily seen from (1), this conclusion implies that either f_0 or f_1 is a B, B-isomorphism. The proof is thus complete.

Our first fundamental result can be referred to as the double exchange theorem. It applies to direct products which contain a finite indecomposable factor. The result will be established in 3.5 under the assumption that the factor involved is a central subalgebra; this restriction, however, will be removed in 3.7. All the remaining fundamental results of this work will be derived in a rather simple way from 3.5 by means of certain theorems stated in Section 2. Thus the proof of 3.5 constitutes a central part of our discussion.

Theorem 3.5. Let B be a finite indecomposable central subalgebra of an algebra

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle,$$

and let C as well as $D_0, D_1, \ldots, D_\kappa, \ldots$ with $\kappa < \nu < \omega$ be arbitrary subalgebras of $\underline{A}$ such that $B \times C$ exists and

$$B \times C = \prod_{\kappa<\nu} D_\kappa.$$

Then for some $\lambda < \nu$ there exist subalgebras X and Y of D_λ such that

$$D_\lambda = X \times Y \text{ and } B \times C = X \times C = B \times Y \times \prod_{\kappa<\lambda} D_\kappa \times \prod_{\kappa<\nu-\lambda-1} D_{\kappa+\lambda+1}.$$

Proof: Let

(1) $$A' = B \times C = \prod_{\kappa<\nu} D_\kappa.$$

By 1.20, there exist A', D_κ-homomorphisms f_κ with $\kappa < \nu$ such that

(2) $$a = \sum_{\kappa<\nu} f_\kappa(a) \text{ for } a \in A', \text{ and } f_\kappa^*(A') = D_\kappa \text{ for } \kappa < \nu.$$

By 1.20 and (1), there exists an A', B-homomorphism g such that for every element $a \in A'$ we have

(3) $$a = g(a) + c \text{ where } c \in C.$$

By 1.4 (ii),

(4) $$g(b) = b \text{ and } g(c) = 0 \text{ for } b \in B \text{ and } c \in C.$$

Hence, by (2),

$$b = \sum_{\kappa < \nu} gf_\kappa(b) \text{ for every } b \in B.$$

Therefore, by 3.4, at least one of the functions gf_κ, say gf_λ, is a B, B-isomorphism.

Let

(5) $$X = f_\lambda{}^*(B).$$

Then

(6) $$g^*(X) = B.$$

Furthermore, g is an X, B-isomorphism, and consequently, by (4),

(7) $$X \cap C = \{0\}.$$

It is easily seen that X is a central subalgebra of A', while, by (1) and 1.16, C is a subtractive subalgebra of A'. Therefore, by 2.5 (ii) and (7), X × C exists and, by (1), (2), and (5),

(8) $$X \times C \subseteq B \times C.$$

Let

(9) $$D'_\kappa = f_\kappa{}^*(B) \text{ for } \kappa < \nu, \text{ and } C' = C \cap \prod_{\kappa<\nu} D'_\kappa.$$

We then have, by (1) and (2),

(10) $$B \subseteq \prod_{\kappa<\nu} D'_\kappa \subseteq B \times C.$$

From (1), (2), and (9) it follows by 2.6 (ii) and 2.7 that D'_κ is a central subalgebra of A' for $\kappa < \nu$. Hence, by 2.4 (iv), $\prod_{\kappa<\nu} D'_\kappa$ is a central subalgebra of A', and consequently, by 2.3 (i), a subtractive subalgebra of A'. Therefore, by (9), (10), and 1.18,

$$(11) \qquad B \times C' = \prod_{\kappa<\nu} D'_{\kappa}.$$

(5) and (9) imply that $X = D'_{\lambda}$. We conclude hence by (11) that

$$X \subseteq B \times C'.$$

Consequently,

$$(12) \qquad X \times C' \subseteq B \times C'$$

By (5) and (6), X and B clearly have the same number of elements, and from (9) we see that C' is finite. Hence $X \times C'$ and $B \times C'$ have the same number of elements, and inclusion (12) can be replaced by the corresponding equation. Thus, by (9),

$$B \subseteq X \times C' \subseteq X \times C.$$

Therefore

$$B \times C \subseteq X \times C.$$

Together with (8) this gives

$$(13) \qquad B \times C = X \times C.$$

From (1) it follows by 1.16 that D_{λ} is a subtractive subalgebra of A'. By (1), (2), and (5), X is a subalgebra of D_{λ}. Hence, by (1), (13), and 1.18,

$$(14) \qquad D_{\lambda} = X \times Y \text{ where } Y = C \cap D_{\lambda}.$$

Let

$$(15) \quad D = \prod_{\kappa<\lambda} D_{\kappa} \times \prod_{\kappa<\nu-\lambda-1} D_{\kappa+\lambda+1} \text{ and } D' = \prod_{\kappa<\lambda} D'_{\kappa} \times \prod_{\kappa<\nu-\lambda-1} D'_{\kappa+\lambda+1}.$$

Then, by (1) and (14),

$$(16) \qquad X \times D \subseteq D_{\lambda} \times D = A'.$$

Furthermore, by (2), (14), and (15), we have for every element a in A':

$$(17) \qquad a \in Y \times D \text{ if, and only if, } f_{\lambda}(a) \in Y;$$

and from (14) we obtain by 1.8 (i)

(18) $$X \cap Y = \{0\}.$$

For any given element b in $\underset{\cdot}{B}$, $f_\lambda(b)$ is in X by (5); therefore, by (17) and (18),

$$b \in Y \times D \text{ if, and only if, } f_\lambda(b) = 0.$$

Since f_λ is a B, X-isomorphism, this implies that

(19) $$B \cap (Y \times D) = \{0\}.$$

C and D_λ being subtractive subalgebras of A', we easily see from (14) and 1.15 that Y is a subtractive subalgebra of A'. Consequently, by (17) and 1.15, Y × D is a subtractive subalgebra of A'. Therefore, by (19) and 2.5 (ii), B × (Y × D) exists and

(20) $$B \times (Y \times D) \subseteq A'.$$

By (1), (2), (9), and (15), D' is a subalgebra of D. Hence, by (11), (15), and (20), B × D' exists and

$$B \times D' \subseteq \prod_{\kappa<\nu} D'_\kappa.$$

Therefore, by (5), (9), and (15),

$$B \times D' \subseteq X \times D'.$$

Since D' is clearly finite while B and X have the same number of elements, this implies that

$$B \times D' = X \times D'.$$

Consequently,

(21) $$X \subseteq B \times D' \subseteq B \times (Y \times D).$$

By (14) and (16),

$$A' = X \times (Y \times D).$$

Hence, by (20) and (21),

(22) $$A' = B \times (Y \times D).$$

The conclusion now follows from (1), (13)-(15), and (22).

Another proof of Theorem 3.5, which has a somewhat more elementary character, is also available; it does not, in particular, involve the notion of homomorphism (and hence is independent of Theorem 3.4), and the properties of isomorphism which are used in it are all of an obvious nature. This proof, however, is longer and more involved in details than the original one, and it will not be given here.

Theorem 3.6 (First exchange theorem). Let B be a finite subalgebra of an algebra

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle,$$

and let C as well as $D_0, D_1, \ldots, D_\kappa, \ldots$ with $\kappa < \nu < \omega$ be arbitrary subalgebras of $\underline{A}$ such that $B \times C$ exists and

$$B \times C = \prod_{\kappa<\nu} D_\kappa.$$

Then there exist subalgebras $D'_0, D'_1, \ldots, D'_\kappa, \ldots$ such that

$$D'_\kappa \subseteq D_\kappa \text{ for every } \kappa < \nu, \text{ and } B \times C = B \times \prod_{\kappa<\nu} D'_\kappa.$$

Proof: We first assume B to be a central subalgebra of $B \times C$. By 3.2 there exist indecomposable subalgebras $B_0, B_1, \ldots, B_\kappa, \ldots$ with $\kappa < \pi < \omega$ such that

(1) $$B = \prod_{\kappa<\pi} B_\kappa.$$

We shall prove the theorem by induction with respect to π. If $\pi = 0$, then, by (1) and 1.10, $B = \{0\}$. We then put $D'_\kappa = D_\kappa$ for $\kappa < \nu$, and the conclusion follows by 1.8 (ii). Suppose that $\mu < \omega$, and assume the theorem to hold for $\pi < \mu$. If $\pi = \mu + 1$, then, by (1),

(2) $$B \times C = \prod_{\kappa<\mu} B_\kappa \times (B_\mu \times C).$$

Hence, by the inductive premise,

(3) $$B \times C = \prod_{\kappa<\mu} B_\kappa \times \prod_{\kappa<\nu} D''_\kappa \text{ where } D''_\kappa \subseteq D_\kappa \text{ for } \kappa < \nu.$$

Let

(4) $$D''_\nu = \prod_{\kappa<\mu} B_\kappa.$$

Then, by (2) and (3),

$$(5) \qquad B \times C = B_\mu \times D''_\nu \times C = \prod_{\kappa<\nu+1} D''_\kappa.$$

By 3.5 and (5), and in view of the fact that B_μ is indecomposable, there exist for some $\lambda < \nu + 1$ two subalgebras X and Y of D''_λ such that

$$(6) \quad D''_\lambda = X \times Y \text{ and } B \times C = X \times D''_\nu \times C = B_\mu \times Y \times \prod_{\kappa<\lambda} D''_\kappa \times \prod_{\kappa<\nu-\lambda} D''_{\kappa+\lambda+1}.$$

(5) and (6) imply by 2.18 that $B_\mu \cong X$; hence, by 3.1, $X \neq \{0\}$. From (6) we further conclude that

$$X \cap D''_\lambda = X \text{ and } X \cap D''_\nu = \{0\}.$$

Therefore $\lambda \neq \nu$. Let

$$D'_\lambda = Y \text{ and } D'_\kappa = D''_\kappa \text{ for } \kappa < \nu \text{ and } \kappa \neq \lambda.$$

Then, by (1), (4), and (6),

$$B \times C = B \times \prod_{\kappa<\nu} D'_\kappa.$$

Thus the theorem holds for $\pi = \mu + 1$; and therefore it holds for every π in (1)--under the assumption that B is a central subalgebra of $B \times C$.

Turning now to the general case, we notice that, by 2.11 (i), 2.6 (ii), and the hypothesis, B^c is a finite central subalgebra of $B \times C$. Hence, by 2.13,

$$B^c \times C = \prod_{\kappa<\nu} [(B^c \times C) \cap D_\kappa].$$

Consequently, by the first part of the proof, we have

$$B^c \times C = B^c \times \prod_{\kappa<\nu} D'_\kappa \text{ where } D'_\kappa \subseteq (B^c \times C) \cap D_\kappa, \text{ for } \kappa < \nu.$$

Therefore, by 2.14,

$$B \times C = B \times \prod_{\kappa<\nu} D'_\kappa.$$

This completes the proof.

Theorem 3.7 (Double exchange theorem). The conclusion of Theorem 3.5 holds also in case B is an arbitrary finite indecomposable subalgebra of A (and not necessarily a central subalgebra).

Proof: We can assume without loss of generality that

$$(1) \qquad A = B \times C = \prod_{\kappa<\nu} D_\kappa .$$

If B is a central subalgebra of $B \times C$, then the conclusion holds by 3.5. Assume that B is not a central subalgebra of $B \times C$. By 3.6 and (1),

$$(2) \qquad A = B \times \prod_{\kappa<\nu}{}^{\cdot} D'_\kappa \text{ where } D'_\kappa \subseteq D_\kappa \text{ for } \kappa < \nu.$$

Hence, by (1), 1.16, and 1.18, there are subalgebras D''_κ such that

$$(3) \qquad D_\kappa = D'_\kappa \times D''_\kappa \text{ for } \kappa < \nu.$$

Consequently, by (1),

$$A = \prod_{\kappa<\nu} D''_\kappa \times \prod_{\kappa<\nu} D'_\kappa .$$

This implies, by (2) and 2.17,

$$(4) \qquad B \cong \prod_{\kappa<\nu} D''_\kappa .$$

Since B is indecomposable, we conclude that all but one of the algebras D''_κ, say, all except D''_λ, must be equal to $\{0\}$, and therefore $D'_\kappa = D_\kappa$ for $\kappa < \nu$ and $\kappa \neq \lambda$. Hence, if we put $X = D''_\lambda$ and $Y = D'_\lambda$, we have by (2) and (3),

$$(5) \qquad A = B \times Y \times \prod_{\kappa<\lambda} D_\kappa \times \prod_{\kappa<\nu-\lambda-1} D_{\kappa+\lambda+1} \text{ and } D_\lambda = X \times Y.$$

Furthermore, by (4),

$$(6) \qquad B \cong X.$$

By (1), (3), and (5) we obtain:

$$(7) \qquad A = X \times Y \times \prod_{\kappa<\lambda} D_\kappa \times \prod_{\kappa<\nu-\lambda-1} D_{\kappa+\lambda+1} .$$

(6) implies that X is finite. Consequently, by (1), (7), and 3.6,

$$(8) \qquad A = X \times B' \times C' \text{ where } B' \subseteq B \text{ and } C' \subseteq C.$$

Hence, by (1), 1.16, and 1.18, there are subalgebras B'' and C'' such that

(9) $$B = B' \times B'' \text{ and } C = C' \times C''.$$

Therefore, by (1), (6), (8), 2.16 (iii), and 2.17,

(10) $$B \cong B'' \times C''.$$

B being indecomposable, this shows that either $B'' = \{0\}$ or $C'' = \{0\}$. In the first case we have $B \cong C''$. By (1), (9), and 1.16, $B \times C''$ exists and is a subtractive subalgebra of $\underline{A}$; therefore, by 2.19, B is a central subalgebra of $B \times C''$. Consequently, by 2.6 (i), (ii), B is a central subalgebra of $B \times C$; this, however, contradicts our original assumption. Hence we must have $C'' = \{0\}$, and therefore, by (9) and (10), and in view of the finiteness of B,

$$B' = \{0\} \text{ and } C = C'.$$

Consequently, by (8),

(11) $$A = X \times C.$$

The conclusion follows from (1), (5), and (11).

<u>Theorem</u> 3.8 (<u>Second exchange theorem</u>). <u>Under the assumptions of Theorem</u> 3.6 <u>there exist subalgebras</u> D'_κ <u>for</u> $\kappa < \nu$ <u>such that</u>

$$B \times C = \prod_{\kappa<\nu} D'_\kappa \times C.$$

Proof: by induction, using 3.7.

In connection with the last few theorems it would be interesting to see whether the following <u>general double exchange theorem</u> holds:

<u>Under the assumptions of Theorem 3.6 there exist subalgebras</u> D'_κ <u>and</u> D''_κ <u>such that</u>

$$B \times C = \prod_{\kappa<\nu} D'_\kappa \times C = B \times \prod_{\kappa<\nu} D''_\kappa \text{ \underline{and} } D_\kappa = D'_\kappa \times D''_\kappa \text{ \underline{for} } \kappa < \nu.$$

If a proof of this conjecture were at hand, Theorems 3.5-3.8 could clearly be derived from it as immediate consequences.

Theorem 3.9 (*Refinement theorem*).[14] *Let* $B_0, B_1, \ldots, B_\kappa, \ldots$ *with* $\kappa < \nu < \omega$ *and* $C_0, C_1, \ldots, C_\lambda, \ldots$ *with* $\lambda < \pi < \omega$ *be subalgebras of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

such that the algebras B_κ *with* $\kappa < \nu-1$ *are finite and*

$$A = \prod_{\kappa<\nu} B_\kappa = \prod_{\lambda<\pi} C_\lambda.$$

Then there exist subalgebras $B_{\kappa\lambda}$ *and* $C_{\kappa\lambda}$ *of* $\underline{A}$ *such that*

$$B_\kappa = \prod_{\lambda<\pi} B_{\kappa\lambda},\ C_\lambda = \prod_{\kappa<\nu} C_{\kappa\lambda},\ \textit{and}\ B_{\kappa\lambda} \cong C_{\kappa\lambda} \text{ for } \kappa < \nu \text{ and } \lambda < \pi.$$

Proof: With the help of 3.8 we easily show by induction that there exist subalgebras $C_{\kappa\lambda}$ for $\kappa < \nu-1$ and $\lambda < \pi$ such that

$$(1) \qquad A = \prod_{\kappa<\iota} \prod_{\lambda<\pi} C_{\kappa\lambda} \times \prod_{\kappa<\nu-\iota} B_{\kappa+\iota} \text{ for } \iota < \nu.$$

By considering two successive values for ι, we conclude by 2.17 that

$$(2) \qquad B_\kappa \cong \prod_{\lambda<\pi} C_{\kappa\lambda} \text{ for } \kappa < \nu-1.$$

From (1) (with $\iota = \nu-1$) it follows that, for every $\lambda < \pi$, $\prod_{\kappa<\nu-1} C_{\kappa\lambda}$ exists and is a subalgebra of C_λ. We see from the hypothesis and 1.16 that C_λ is a subtractive subalgebra of $\underline{A}$. Hence, by (1) and 1.18, there exists a subalgebra $C_{\nu-1\,\lambda}$ of C_λ such that

$$C_\lambda = \prod_{\kappa<\nu} C_{\kappa\lambda}.$$

Substituting this in the formula

$$A = \prod_{\kappa<\nu} B_\kappa = \prod_{\lambda<\pi} C_\lambda$$

and comparing the resulting formula with (1) (for $\iota = \nu-1$), we conclude by 2.17 that (2) holds also for $\kappa = \nu-1$. Hence, by 2.16 (iv), there exist subalgebras $B_{\kappa\lambda}$ of B_κ such that

14. Theorems 3.9 and 3.11 even when applied to groups cannot be derived from the results known in the literature; the reason is that the subgroups $B_{\nu-1}$ in 3.9 and C in 3.11 are not assumed to be finite. (Of course, if these subgroups, and thus also the whole group $\underline{A}$, were assumed to be finite, then the group-theoretical implications of 3.9 and 3.11 would present rather trivial consequences of the unique factorization theorem for finite groups.)

$$B_\kappa = \prod_{\lambda<\pi} B_{\kappa\lambda} \text{ and } B_{\kappa\lambda} \cong C_{\kappa\lambda} \text{ for } \kappa < \nu \text{ and } \lambda < \pi.$$

This completes the proof.

Theorem 3.10 (Unique factorization theorem).[15] *Every finite algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

has, up to isomorphism, just one representation as a direct product of indecomposable subalgebras:

$$A = \prod_{\kappa<\nu} A_\kappa$$

(cf. Corollary 3.2). *More specifically, if*

$$A = \prod_{\lambda<\pi} B_\lambda$$

is another representation of this kind, then $\nu = \pi$ *and there exists a permutation* φ *of the ordinals* $0, 1, \ldots, \nu-1$ *such that*

(i) $$A = \prod_{\kappa<\mu} B_{\varphi(\kappa)} \times \prod_{\kappa<\nu-\mu} A_{\kappa+\mu} \text{ for } \mu \le \nu$$

and

(ii) $$A_\kappa \cong B_{\varphi(\kappa)} \text{ for } \kappa < \nu.$$

Proof: Suppose we have defined the values $\varphi(\kappa)$ for all $\kappa < \mu$ in such a way that (i) holds. By 3.7, there exist for some $\lambda < \pi$ two subalgebras X and Y of B_λ such that

(1) $$B_\lambda = X \times Y \text{ and } A = \prod_{\kappa<\mu} B_{\varphi(\kappa)} \times X \times \prod_{\kappa<\nu-\mu-1} A_{\kappa+\mu+1}.$$

Hence, by (i) and 2.17, $X \cong A_\mu$. Since A_μ and B_λ are indecomposable, we conclude by 3.1 and (1) that $X \neq \{0\}$, and consequently $Y = \{0\}$. Therefore $B_\lambda = X$. If we put $\varphi(\mu) = \lambda$ we can thus write (1) in the form

15. Theorem 3.10 for groups follows from the main result in Schmidt [1]. (In Maclagen-Wedderburn [1] the unique factorization theorem for groups lacks the "exchange" conclusion (i), while in Remak [1] this conclusion is stated in somewhat weaker form.)

$$A = \prod_{\kappa<\mu+1} B_{\varphi(\kappa)} \times \prod_{\kappa<\nu-\mu-1} A_{\kappa+\mu+1}.$$

It is now easy to see how φ can be defined recursively so as to satisfy (i). Furthermore, φ is clearly a permutation and therefore $\nu = \pi$. Finally, by considering (i) with two successive values for μ, we conclude by 2.17 that (ii) is also satisfied, and the proof is complete.

Theorem 3.11 (*Cancellation theorem*).[14] *Let* B *be a finite subalgebra of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle,$$

and let B', C, *and* C' *be arbitrary subalgebras of* $\underline{A}$ *such that* B × C *exists and*

$$B \times C = B' \times C'.$$

Then

(i) $A = B \times C$ *and* $B \cong B'$ *imply* $C \cong C'$;

(ii) $B \simeq B'$ *implies* $C \simeq C'$.

Proof: We shall actually prove a somewhat stronger statement, namely the following

Lemma. Let B *be a finite subalgebra of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle,$$

and let B', C, C', *and* D *be arbitrary subalgebras of* $\underline{A}$ *such that* B × C × D *exists and*

$$B \times C \times D = B' \times C' \times D.$$

Then

(i) $A = B \times C \times D$ *and* $B \cong B'$ *imply* $C \cong C'$;

(ii) $B \simeq B'$ *implies* $C \simeq C'$.

In fact, suppose that

(1) $A = B \times C \times D = B' \times C' \times D$ and $B \cong B'$.

By 3.2. B can be represented in the form

(2) $$B = \prod_{\kappa<\nu} B_\kappa$$

where the subalgebras B_κ with $\kappa < \nu$ are finite and indecomposable. We shall establish the first part of our lemma by induction with respect to ν. If $\nu = 0$, then the conclusion follows from 2.17. Suppose that $\pi < \omega$, and assume the lemma to hold whenever $\nu \leq \pi$. Let $\nu = \pi + 1$. By (1), (2), and 2.16 (v), B' has a representation

(3) $$B' = \prod_{\kappa<\nu} B'_\kappa \text{ with } B_\kappa \cong B'_\kappa \text{ for } \kappa < \nu.$$

The subalgebras B' are clearly indecomposable. Hence, by (1), (2), and 3.7, we either have

(4) $$A = \prod_{\kappa<\nu-1} B_\kappa \times B'_\lambda \times C \times D \text{ for some } \lambda < \nu,$$

or else there exist subalgebras X and Y of C' such that

(5) $$A = \prod_{\kappa<\nu-1} B_\kappa \times X \times C \times D = \prod_{\kappa<\nu} B'_\kappa \times B_{\nu-1} \times Y \times D \text{ and } C' = X \times Y.$$

If (4) holds, then, by (1), (2), and 2.17,

$$B_{\nu-1} \cong B'_\lambda.$$

We then conclude from (3), using various parts of 2.16, that

$$\prod_{\kappa<\nu-1} B_\kappa \cong \prod_{\kappa<\lambda} B'_\kappa \times \prod_{\kappa<\nu-\lambda-1} B'_{\kappa+\lambda+1}.$$

Consequently, by (1)-(4) and the inductive premise,

(6) $$C \cong C'.$$

If (5) holds, then, by (1), (2), (3), and the inductive premise,

(7) $$C \cong B'_{\nu-1} \times Y.$$

Furthermore, by (1), (3), (5), and 2.17,

$$B_{\nu-1} \cong X.$$

Using (3), (5), (7), and the various parts of 2.16, we see that (6) holds in this case as well. Thus, part (i) of the lemma has been established. To obtain part (ii) we proceed in an analogous way; we use, however, 2.18 instead of 2.17, and various elementary properties of isomorphism instead of 2.16. Since Theorem 3.11 is but a particular case of our lemma, the proof is complete.

To conclude this section we want to discuss briefly certain generalizations of the results obtained in Theorems 3.5-3.11. In each of these theorems some of the algebras involved have been assumed to be finite. Actually all these theorems with the exception of 3.11 (ii), remain valid if we assume only that the algebras in question satisfy the so-called double chain condition for central subalgebras--i.e., that they contain no infinite sequence of central subalgebras which is strictly increasing or strictly decreasing. We shall refer to these generalizations as Theorems 3.5', 3.6', etc.; in 3.10' we state only that the algebra $\underline{A}$ has, up to isomorphism, at most one (and not just one) representation as a product of indecomposable factors.

By analyzing the proofs of Theorems 3.5-3.11, we notice that the condition of finiteness is needed only to enable us to apply 2.11 (i) and 3.2-3.4. As is easily seen, the assumption that A^c be finite can be replaced in 2.11 (i) by the double chain condition for central subalgebras of $\underline{A}$ (in fact, this condition is needed merely for increasing sequences of central subalgebras). Similarly we notice that Theorems 3.3 and 3.4 remain valid if the assumption of the finiteness of B is replaced by the double chain condition. Having made these observations, we meet with no difficulties in verifying Theorems 3.5'-3.7'. Certain complications arise in connection with Theorems 3.8' and 3.11'(i); this is due to the fact that Corollary 3.2 can no longer be applied; these complications, however, are not of a very serious nature. Consider, for instance, Theorem 3.8'. From the restriction imposed on B we conclude that B can be represented in the form

$$B = B' \times B''$$

where B' is a central subalgebra of B, while no factor of B'', except {0}, is a central subalgebra of B. By an argument similar to that applied in the proof of Theorem 3.7 we show that there exist subalgebras D''_κ of D_κ for $\kappa < \nu$ such that

$$B \times C = B' \times \prod_{\kappa<\nu} D''_\kappa \times C.$$

The algebra B' is clearly the direct product of indecomposable subalgebras, and hence we can complete the proof by reasoning in the same way as in the proof of Theorem 3.8 (we use, of course, 3.7' instead of 3.7). The difficulties in the proof of Theorem 3.11' (i) can be overcome in a similar way. The proofs of the remaining two theorems, 3.9' and 3.10', are analogous to those of 3.9 and 3.10.[16]

On the other hand, it is readily seen that Corollary 3.2 and Theorem 3.11 (ii) cannot be generalized in the same way as the other theorems of this section. For instance, we can easily construct an algebra whose center consists of just one element and which is isomorphic with a proper factor of itself; obviously, the conclusions of 3.2 and 3.11 (ii) do not hold for such an algebra.

The generalizations of Theorems 3.5–3.11 discussed above apply in particular to those algebras which have a finite center They have various simple and interesting implications for the so-called <u>centerless algebras</u>, i.e., for those algebras in which the center consists of the zero element only. The relation of central isomorphism between subalgebras of a centerless algebra obviously reduces to logical identity. Hence it is clear that our results take on a simple form when applied to this class of algebras. However, the fundamental results for centerless algebra can be more readily derived from certain theorems of Section 2.

In fact, we obtain directly from 2.13 the following corollary:

I <u>Let</u> B <u>be a centerless subalgebra of an algebra</u>

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle.$$

16. It may be interesting to compare Theorems 3.5'–3.11' in their applicatiins to groups with related results which are explicitly or implicitly contained in group-theoretical literature, e.g., in Fitting [1], Golowin [1], Kořinek [1], Krull [1], Kurosh [1], and Schmidt [1]. Our theorems seem to give some essentially new results even when applied to groups (with or without operators). One of the reasons is that in 3.5'–3.9' and 3.11' not the whole algebras, but only certain subalgebras are assumed to satisfy the double chain condition. The main results in Fitting [1], Krull [1], and Schmidt [1] immediately follow from Theorem 3.10'. On the other hand, the results in Golowin [1], Kořinek [1], and Kurosh [1] cannot be derived from the theorems established in this work. Nevertheless these results can be extended to arbitrary algebras as well; in the case of Golowin [1] this requires an extension of the notion of a direct product to infinite systems of subalgebras. We may add that some of our results, e.g., Theorems 3.9' and 3.10', can also be extended to infinite direct products.

If $C, D_0, D_1, \ldots, D_\kappa, \ldots$ *with* $\kappa < \nu < \omega$ *are arbitrary subalgebras of* $\underline{A}$ *such that* $B \times C$ *exists and*

$$B \times C = \prod_{\kappa<\nu} D_\kappa,$$

then

$$C = \prod_{\kappa<\nu} (C \cap D_\kappa).$$

From this we get in turn the following theorems:

II (*Strict refinement theorem*). *Let*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots$$

be a centerless algebra and let B_0, $B_1, \ldots, B_\kappa, \ldots$ *with* $\kappa < \nu < \omega$ *and* C_0, $C_1, \ldots, C_\lambda, \ldots$ *with* $\lambda < \pi < \omega$ *be subalgebras of* $\underline{A}$ *such that*

$$A = \prod_{\kappa<\nu} B_\kappa = \prod_{\lambda<\pi} C_\lambda.$$

Then

$$B_\kappa = \prod_{\lambda<\pi} (B_\kappa \cap C_\lambda) \text{ for } \kappa < \nu, \text{ and } C_\lambda = \prod_{\kappa<\nu} (B_\kappa \cap C_\lambda) \text{ for } \lambda < \pi.$$

III (*Strict unicity theorem*). *If, under the assumptions of* II, *all the algebras* B_κ *and* C_λ *are indecomposable, then* $\nu = \pi$, *and there exists a permutation* φ *of the ordinals* $0, 1, \ldots, \nu - 1$ *such that*

$$B_\kappa = C_{\varphi(\kappa)} \text{ for } \kappa < \nu.$$

IV (*Strict cancellation theorem*). *Let* B *be a centerless subalgebra of an algebra*

$$\underline{A} = \langle A, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle.$$

If C *and* D *are any subalgebras of* $\underline{A}$ *such that* $B \times C$ *exists and*

$$B \times C = B \times D,$$

then

$$C = D.$$

The theorems just stated can be applied in particular to various algebras discussed in Examples I-IV of Section 2; they thus contain as special cases most of the results of a related nature which can be found in the literature and concern centerless groups, rings with unit elements, and lattices.[17]

In a certain sense Theorems I and IV are the best results obtainable. In fact, the following result can be established:

V. For every algebra

$$\underline{B} = \langle B, +, O_0, O_1, \ldots, O_\xi, \ldots \rangle$$

the following conditions are equivalent:

(i)' B is centerless;

(ii) B satisfies Theorem I for every "superalgebra" $\underline{A}$;

(iii) B satisfies Theorem IV for every "superalgebra" $\underline{A}$.

On the other hand, Theorems II and III can be extended to a wider class of algebras. They apply to every algebra $\underline{A}$ in which all factors in direct decompositions constitute a Boolean algebra under inclusion (and Theorem II expresses even a characteristic property of such algebras). This class of algebras is rather comprehensive; it contains, for instance, all centerless algebras and, more generally, every algebra which cannot be mapped homomorphically on any subalgebra of its center with at least two different elements. Thus, e.g., the strict unicity theorem for groups which coincide with their commutator groups[18] appears as a particular case of the generalized Theorem III.

17. We have in mind the results stated in Birkhoff [1], p. 23, Fitting [1] p. 29, Golowin [1], p 424, Jacobson [1], pp. 62 f. and Remak [1], p.304. Although Theorems II and III are formulated in the text only for finite direct products, they can be extended to infinite products; cf. the preceding footnote. Compare here, Jónsson-Tarski [2] and [3].

18. Cf. Speiser [1], pp. 136 f.