In the case that M = A the following statements are true:

1) Ass(A) = the prime ideals (isolated and imbedded) corresponding to (0).

2) $\bigcup_{\mathfrak{p} \in \mathrm{Ass}(A)} \mathfrak{p}$ = the set of zero divisors.

3) {the minimal primes of Supp(A)} = {the isolated primes of (0)} = {the minimal primes of A}.

4) $\mathfrak{n}(A) = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(A)} \mathfrak{p}$.

We give some examples of the above notions, again without any attempts at proofs.

The <u>local rings</u> most commonly met in Algebraic Geometry are of the form $A_{\mathfrak{p}}$ where $\mathfrak{p}$ is a prime ideal of A. It is immediate to check that the complement of $\mathfrak{p}A_{\mathfrak{p}}$ in $A_{\mathfrak{p}}$ consists of units, whence $\mathfrak{p}A_{\mathfrak{p}}$ is the unique maximal ideal of $A_{\mathfrak{p}}$.

An example of a <u>Jacobson ring</u> is given by $A/\mathfrak{p}$, where A is a finitely generated algebra over an algebraically closed field k, and $\mathfrak{p}$ is a prime ideal of A.

Finally, we leave as an exercise to the reader to prove that, if M is a finitely generated A-module with annihilator $\mathfrak{a}$, then Ass(M) = {the prime ideals corresponding to an irredundant primary decomposition of $\mathfrak{a}$}.

GEOMETRIC NOTIONS

Let A be a ring. We recall that <u>Spec(A)</u> is defined, as a set, to consist of all the prime ideals $\mathfrak{p}$ of A. Such set is made into a topological space by defining a subbasis of open sets (which actually turns out to be a basis) as follows:

we define, for $t \in A$, $D(t) = \{ \mathfrak{p} \in \mathrm{Spec}(A) \mid t \notin \mathfrak{p} \}$,

and consider the collection $(D(t))_{t \in A}$ as the subbasis in question. (That it is a basis is easily seen from $D(st) = D(s) \cap D(t)$, $s, t \in A$.) The resulting topology on Spec(A) is usually called the Zariski topology.

Equivalently, we can define the Zariski topology by determining what the closed subsets are. Here we take any ideal $\mathfrak{a} \subset A$ and define

$$V(\mathfrak{a}) = \{ \mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{a} \subset \mathfrak{p} \}.$$

The collection of sets $(V(\mathfrak{a}))$ is easily seen to satisfy the axioms of closed sets in a topology, and one then shows that

$$\mathrm{Spec}(A) - D(t) = V(tA)$$
$$\mathrm{Spec}(A) - V(\mathfrak{a}) = \bigcup_{t \notin \mathfrak{a}} D(t)$$

whence the two topologies are actually the same.

Since A is noetherian, Spec(A) is a <u>noetherian topological space</u>, i.e. the open subsets of Spec(A) satisfy the maximal condition, or, equivalently, the closed subsets of Spec(A) satisfy the minimal condition. Hence Spec(A) is the finite union of its irreducible components.

We caution that Spec(A) is however highly non-Hausdorff. In fact one easily sees that $\mathfrak{a} \subset \mathfrak{b} \Rightarrow V(\mathfrak{a}) \supset V(\mathfrak{b})$, hence a point $\mathfrak{p} \in \mathrm{Spec}(A)$ has in general a closure distinct from $\mathfrak{p}$, in fact equal to $V(\mathfrak{p})$. $\mathfrak{p}$ is hence a closed point if, and only if, the ideal $\mathfrak{p}$ is maximal. However, given two distinct points $\mathfrak{p}$, $\mathfrak{q}$ of Spec(A), we can find an element $t \in A$ which belongs to one but not the other of the two ideals (we can't tell which though), whence an open subset D(t) which contains one point but not the other. In other words Spec(A) is a T_0 (Kolmogoroff) topological space. We also remark that the only closed, irreducible components of Spec(A) are precisely the closures of the minimal prime ideals of A, and that the only closed irreducible subsets of Spec(A) are precisely the subsets of the form $V(\mathfrak{a})$, where $\mathfrak{a}$ is any ideal in A with a prime radical. In fact $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$, for $\mathfrak{a} \subset A$.

With every ring A we have made correspond a certain topological space, Spec(A). We ask the question: given the topological space Spec(A), can we recover A? Unfortunately not, since, e.g., all fields have homeomorphic Spectra. The notion which is missing, in order to obtain an adequate dictionary between the algebraic and the geometric languages is the notion, due to Serre, of the <u>sheaf of local rings</u> of Spec(A).

This is a sheaf $\tilde{A}$ which can be defined in one of two equivalent ways

1) As a presheaf $\tilde{A}(D(t)) = A_t \quad t \in A$
2) As an espace étalé, the stalk $\tilde{A}_{\mathfrak{p}}$ of $\tilde{A}$ over the point $\mathfrak{p} \in \mathrm{Spec}(A)$ is given by $\tilde{A}_{\mathfrak{p}} = A_{\mathfrak{p}}$.

One can easily prove that $A_{\mathfrak{p}} = \varinjlim_{t \notin \mathfrak{p}} A_t$, where the homomorphisms $A_s \to A_{st}$ are given by $a/s^n \rightsquigarrow at^n/(st)^n$. Hence the two definitions are indeed equivalent.

We now have associated with every ring A two objects, namely the topological space Spec(A) and the sheaf of local rings $\tilde{A}$ over Spec(A). Given the pair (Spec(A), $\tilde{A}$), it is now easy to recover A, namely $A = A_1 = \tilde{A}(D(1)) = \tilde{A}(\mathrm{Spec}(A))$, which is the totality of sections of $\tilde{A}$ over Spec(A).

The pairs (Spec(A), $\tilde{A}$) are the objects in the category of affine schemes, whose morphisms we now discuss.

To describe the morphisms in the category of affine schemes, let (Spec(A), $\tilde{A}$), (Spec(B), $\tilde{B}$) be two objects in the category. Let $\varphi: A \to B$ be a ring homomorphism. Over Spec(A) we define the sheaf of rings $\varphi_*(\tilde{B})$, given by $\varphi_*(\tilde{B})(D(t)) = B_{\varphi(t)} = \tilde{B}(D(\varphi(t)))$, $t \in A$. Then the function $\varphi^a: \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ given by $\varphi^a(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ is continuous, as is seen from the formula $(\varphi^a)^{-1}(D(t)) = D(\varphi(t))$.

Furthermore define $\tilde{\varphi}: \tilde{A} \to \varphi_*(\tilde{B})$ by defining $\tilde{\varphi}(D(t)): A_t \to B_{\varphi(t)}$ as follows: $a/t^n \rightsquigarrow \varphi(a)/\varphi(t^n)$.

To the ring homomorphism φ we have associated a pair of functions $(\varphi^a, \tilde{\varphi})$. Such pairs are precisely the morphisms in the category of affine schemes.

Our dictionary is now adequate, since in fact one can prove that the category of affine schemes is the dual (in the categorical sense) of the category of rings.

APPENDIX

Let $\mathbb{C}$ be the field of complex numbers, $R = \mathbb{C}[X_1,\ldots,X_n]$, $\mathfrak{a}$ an ideal of R such that $\mathfrak{a} = \sqrt{\mathfrak{a}}$. Define $V(\mathfrak{a})$ as follows

$$V(\mathfrak{a}) = \{(x_1,\ldots,x_n) \in \mathbb{C}^n \mid f(x_1,\ldots,x_n) = 0 \text{ for all } f \in \mathfrak{a}\}.$$

This is the classical notion of an affine variety (in fact, to be strictly classical one should take $\mathfrak{a}$ to be a prime ideal), and it is well known that the points of $V(\mathfrak{a})$ are in a 1-1, onto correspondence with the maximal ideals of the ring $A = R/\mathfrak{a}$. So the classical notion of an affine variety corresponds simply to the set of closed points of Spec(A). In defining Spec(A) as we have, we have in fact added to the classical notion of point a lot of other "undrawable" points, namely the prime ideals of A which are not maximal. We can ask:

a) What are the advantages of such addition?

b) If such addition is indeed advantageous, how could classical geometers get along without it?

The answer to b) is simple: $R/\mathfrak{a}$ is a Jacobson ring, and the knowledge of its maximal ideals determine its prime ideals.

To answer a), at the moment, we make the following four observations:

1) We are not limited to rings of the form $R/\mathfrak{a}$, and, were it so, $\mathfrak{a}$ can be arbitrary, whence $R/\mathfrak{a}$ may have zero divisors (Serre's point of view) and, more strikingly, nilpotent elements.

2) Prime ideals have a "good" functorial behavior (e.g., the inverse image of a prime ideal under a ring homomorphism is again prime), while maximal ideals do not.

3) The notion of a "ringed space", i.e. a topological space X and a sheaf of rings over X, is the natural tool to give an intrinsic geometric definition of projective varieties (which are definitively not affine).

4) The possibility that $R/\mathfrak{a}$ have nilpotent elements has brought the solutions of long standing conjectures, unsolved until now.

There is one notion that seems to be lost in the transition from the classical case to Spec(A). In the classical case an element of A identifies a regular function over $V(\mathfrak{a})$, with a well defined value $f(x) \in \mathbb{C}$, at each $x \in V(\mathfrak{a})$. Can an element $f \in A$ be considered as a function over Spec(A)? Most definitely, but the value field may change with the point $\mathfrak{p} \in \text{Spec}(A)$. More precisely, $A_{\mathfrak{p}} / \mathfrak{p} A_{\mathfrak{p}}$ is a field, which we denote by $k(\mathfrak{p})$, and we define <u>the value of f at $\mathfrak{p}$</u> as the image of f/1 under the canonical morphism $A_{\mathfrak{p}} \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. It is trivial to see that, when $A = R/\mathfrak{a}$, and $\mathfrak{p}$ is a maximal ideal of A, then $k(\mathfrak{p}) = \mathbb{C}$, which throws a better light on the classical situation. We point out that, if $f \in A$ is nilpotent, we have the highly non-classical situation of getting $f(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \text{Spec}(A)$, but $f \neq 0$.