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Decomposable Ultrafilters and Possible Cofinalities

Paolo Lipparini

Abstract We use Shelah's theory of possible cofinalities in order to solve some problems about ultrafilters. Theorem: Suppose that λ is a singular cardinal, $\lambda' < \lambda$, and the ultrafilter D is κ-decomposable for all regular cardinals κ with $\lambda' < \kappa < \lambda$. Then D is either λ-decomposable or λ^+-decomposable. Corollary: If λ is a singular cardinal, then an ultrafilter is (λ, λ)-regular if and only if it is either $\mathrm{cf}\,\lambda$-decomposable or λ^+-decomposable. We also give applications to topological spaces and to abstract logics.

Theorem 1 *Suppose that λ is a singular cardinal, $\lambda' < \lambda$, and the ultrafilter D is κ-decomposable for all regular cardinals κ with $\lambda' < \kappa < \lambda$. Then D is either λ-decomposable or λ^+-decomposable.*

Corollary 2 *If λ is a singular cardinal, then an ultrafilter is (λ, λ)-regular if and only if it is either* $\mathrm{cf}\,\lambda$*-decomposable or λ^+-decomposable.*

If F is a family of subsets of some set I and λ is an infinite cardinal, a λ-*decomposition* for F is a function $f : I \to \lambda$ such that whenever $X \subseteq \lambda$ and $|X| < \lambda$ then $\{i \in I \mid f(i) \in X\} \notin F$. The family F is λ-*decomposable* if and only if there is a λ-decomposition for F. If D is an ultrafilter (that is, a maximal proper filter) let us define the *decomposability spectrum* K_D of D by $K_D = \{\lambda \geq \omega \mid D \text{ is } \lambda\text{-decomposable}\}$.

The question of the possible values the spectrum K_D may take is particularly intriguing. Even the old problem from Prikry [14] and Silver [17] of characterizing those cardinals μ for which there is an ultrafilter D such that $K_D = \{\omega, \mu\}$ is not yet completely solved (Sheard [15], p. 1007).

The case when K_D is infinite is even more involved. Prikry studied the situation in which λ is limit and $K_D \cap \lambda$ is unbounded in λ; he found some assumptions

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which imply that $\lambda \in K_D$. This is not always the case; if μ is strongly compact and $\mathrm{cf}\,\lambda < \mu < \lambda$, then there is an ultrafilter D such that $K_D \cap \lambda$ is unbounded in λ, and D is not λ-decomposable. If we are in the above situation, D is necessarily λ^+-decomposable (by Solovay [18], Lemma 3, and the proof of [14], Proposition 2).

The above examples suggest the problem (implicit in [14]) whether $K_D \cap \lambda$ unbounded in λ implies that either $\lambda \in K_D$ or $\lambda^+ \in K_D$. In general, the problem is still open; here we solve it affirmatively in the particular case when there is $\lambda' < \lambda$ such that K_D contains all regular cardinals in the interval $[\lambda', \lambda)$. This is sufficient for all applications we know of; see Corollaries 2, 7, 8, and 9 and Theorem 10.

We briefly review some known results on K_D. If κ is regular and $\kappa^+ \in K_D$, then $\kappa \in K_D$. If $\kappa \in K_D$ is singular, then $\mathrm{cf}\,\kappa \in K_D$. Results from Donder [4] imply that if there is no inner model with a measurable cardinal then K_D is always an interval with minimum ω. On the other hand, it is trivial that $K_D = \{\mu\}$ if and only if μ is either ω or a measurable cardinal. If a measurable cardinal μ is made singular by Prikry forcing, then in the resulting model we have an ultrafilter D such that $K_D = \{\omega, \mu\}$. Further comments and constraints on K_D are given in Lipparini [12] and [9]. Apparently the problem of determining which sets of cardinals can be represented as $K_F = \{\lambda \geq \omega | F \text{ is } \lambda\text{-decomposable}\}$ for a filter F has not been studied.

If $(\lambda_j)_{j\in J}$ are regular cardinals, the *cofinality* $\mathrm{cf} \prod_{j\in J} \lambda_j$ of the product $\prod_{j\in J} \lambda_j$ is the smallest cardinality of a set $G \subseteq \prod_{j\in J} \lambda_j$ having the property that for every $f \in \prod_{j\in J} \lambda_j$ there is $g \in G$ such that $f(j) \leq g(j)$ for all $j \in J$. We shall state our results in a quite general form, involving arbitrary filters rather than ultrafilters. In what follows, the reader interested in ultrafilters only can always assume that F is an ultrafilter.

Proposition 3 *If $(\lambda_j)_{j\in J}$ are infinite regular cardinals, $\mu = \mathrm{cf} \prod_{j\in J} \lambda_j$, and the filter F is λ_j-decomposable for all $j \in J$, then F is μ'-decomposable for some μ' with $\sup_{j\in J} \lambda_j \leq \mu' \leq \mu$.*

Proof Let F be over I, and let $(g_\alpha)_{\alpha\in\mu}$ witness $\mu = \mathrm{cf} \prod_{j\in J} \lambda_j$. For every $j \in J$, let $f(j, -) : I \to \lambda_j$ be a λ_j-decomposition for F. For any fixed $i \in I$, $f(-, i) \in \prod_{j\in J} \lambda_j$; thus there is $\alpha(i) \in \mu$ such that $f(j, i) \leq g_{\alpha(i)}(j)$ for all $j \in J$.

Let X be a subset of μ with minimal cardinality with respect to the property that $Y = \{i \in I | \alpha(i) \in X\} \in F$. Let $\mu' = |X|$. Thus, whenever $X' \subseteq \mu$ and $|X'| < \mu'$, we have $Y' = \{i \in I | \alpha(i) \in X'\} \notin F$. Define $h(i) = \alpha(i)$ for $i \in Y$, and $h(i) = 0$ for $i \notin Y$. Thus, $h : I \to X \cup \{0\}$.

If $|X'| < \mu'$, then $\{i \in I | h(i) \in X'\} \subseteq Y' \cup (I \setminus Y) \notin F$ (otherwise, since F is a filter, $Y' \supseteq Y \cap Y' = Y \cap (Y' \cup (I \setminus Y)) \in F$, contradiction). This shows that, modulo a bijection from $X \cup \{0\}$ onto μ', h is a μ'-decomposition for F. Trivially, $\mu' \leq \mu$.

Hence, it remains to show that $\sup_{j\in J} \lambda_j \leq \mu'$. Suppose to the contrary that $\mu' < \lambda_{\bar{j}}$ for some $\bar{j} \in J$. Then $|\{g_{\alpha(i)}(\bar{j}) | i \in Y\}| \leq |\{\alpha(i) | \alpha(i) \in X\}| \leq |X| = \mu' < \lambda_{\bar{j}}$. Since $\lambda_{\bar{j}}$ is regular, we have that $\beta = \sup_{i\in Y} g_{\alpha(i)}(\bar{j}) < \lambda_{\bar{j}}$. Hence, if $i \in Y$, then $f(\bar{j}, i) \leq g_{\alpha(i)}(\bar{j}) \leq \beta < \lambda_{\bar{j}}$. Thus, $|[0, \beta]| < \lambda_{\bar{j}}$, but $\{i \in I | f(\bar{j}, i) \in [0, \beta]\} \supseteq Y \in F$, and this contradicts the assumption that $f(\bar{j}, -)$ is a $\lambda_{\bar{j}}$ decomposition for F. $\square$

Proposition 3 has not the most general form: we have results dealing with the cofinality μ of reduced products $\mathrm{cf} \prod_E \lambda_j$, where E is a filter on J. We shall not need this more general version here.

Recall from Shelah [16] that if $\mathfrak{a}$ is a set of regular cardinals, then $\mathrm{pcf}\,\mathfrak{a}$ is the set of regular cardinals which can be obtained as $\mathrm{cf} \prod_E \mathfrak{a}$, for some ultrafilter E on $\mathfrak{a}$.

Corollary 4 *If $\mathfrak{a}$ is a set of infinite regular cardinals, $|\mathfrak{a}|^+ < \min \mathfrak{a}$, and the filter F is λ-decomposable for all $\lambda \in \mathfrak{a}$, then F is μ'-decomposable for some μ' with* $\sup \mathfrak{a} \leq \mu' \leq \max \mathrm{pcf}\,\mathfrak{a}$.

Proof By [16], II, Lemma 3.1, if $|\mathfrak{a}|^+ < \min \mathfrak{a}$, then $\max \mathrm{pcf}\,\mathfrak{a} = \mathrm{cf} \prod_{\lambda \in \mathfrak{a}} \lambda$; thus the conclusion is immediate from Proposition 3. □

Recall that an ultrafilter D is (μ, λ)-regular if and only if there is a family of λ members of D such that the intersection of any μ members of the family is empty. We list below the properties of decomposability and regularity we shall need. Much more is known; see Deiser and Donder [3], Foreman [6], and Woodin [19], pp. 427–31, for recent results. See Lipparini [11] and [9] for more references.

Properties 5

(a) Every λ-decomposable ultrafilter is $\mathrm{cf}\,\lambda$-decomposable.

(b) Every $\mathrm{cf}\,\lambda$-decomposable ultrafilter is (λ, λ)-regular.

(c) If $\mu' \geq \mu$ and $\lambda' \leq \lambda$, then every (μ, λ)-regular ultrafilter is (μ', λ')-regular.

(d) If λ is singular, D is a λ^+-decomposable ultrafilter, and D is not $\mathrm{cf}\,\lambda$-decomposable, then D is (λ', λ^+)-regular for some $\lambda' < \lambda$. (Cudnovskii and Cudnovskii [2], Theorem 1; Kunen and Prikry [8], Theorem 2.1)

(e) If λ is singular, then every λ^+-decomposable ultrafilter is (λ, λ^+)-regular. (Kanamori [7], Corollary 2.4)

(f) If λ is singular, then every (λ, λ)-regular ultrafilter is either $\mathrm{cf}\,\lambda$-decomposable or (λ', λ)-regular for some $\lambda' < \lambda$. (Lipparini [10], Corollary 1.4)

(g) If λ is regular, then an ultrafilter is λ-decomposable if and only if it is (λ, λ)-regular.

Theorem 6 *Suppose that λ is a singular cardinal, F is a filter, and either*

(a) *there is $\lambda' < \lambda$ such that F is κ-decomposable for all regular cardinals κ with $\lambda' < \kappa < \lambda$, or*

(b) $\mathrm{cf}\,\lambda > \omega$ *and $S = \{\kappa < \lambda | F$ is κ^+-decomposable$\}$ is stationary in λ.*

Then F is either λ-decomposable or λ^+-decomposable.

If $F = D$ is an ultrafilter, then D is (λ, λ)-regular. Moreover, D is either (i) λ-decomposable or (ii) (λ', λ^+)-regular for some $\lambda' < \lambda$ or (iii) $\mathrm{cf}\,\lambda$*-decomposable and (λ, λ^+)-regular.*

Proof If $\mathrm{cf}\,\lambda = \nu > \omega$, then, by [16], II, Claim 2.1, there is a sequence $(\lambda_\alpha)_{\alpha \in \nu}$ closed and unbounded in λ and such that, letting $\mathfrak{a} = \{\lambda_\alpha^+ | \alpha \in \nu\}$, we have $\lambda^+ = \max \mathrm{pcf}\,\mathfrak{a}$. If $\mathrm{cf}\,\lambda = \omega$, then we have $\lambda^+ = \max \mathrm{pcf}\,\mathfrak{a}$ for some $\mathfrak{a}$ of order type ω unbounded in λ as a consequence of [16], II, Theorem 1.5. (Since $\mathfrak{a}$ has order type ω, any ultrafilter over $\mathfrak{a}$ is either principal or extends the dual of the ideal of bounded subsets of $\mathfrak{a}$.)

Letting $\mathfrak{b} = \mathfrak{a} \cap [\lambda', \lambda)$ in case (a) and $\mathfrak{b} = \mathfrak{a} \cap \{\kappa^+ | \kappa \in S\}$ in case (b), we still have $\max \operatorname{pcf} \mathfrak{b} = \lambda^+$, because $\mathfrak{b}$ is unbounded in λ, hence $\max \operatorname{pcf} \mathfrak{b} \geq \lambda^+$, and because $\max \operatorname{pcf} \mathfrak{b} \leq \max \operatorname{pcf} \mathfrak{a} = \lambda^+$, since $\mathfrak{b} \subseteq \mathfrak{a}$.

Assume, without loss of generality, that $\lambda' > (\operatorname{cf} \lambda)^+$ in (a) and that $\inf S > (\operatorname{cf} \lambda)^+$ in (b). Since $|\mathfrak{b}| \leq |\mathfrak{a}| = \operatorname{cf} \lambda$, then $|\mathfrak{b}|^+ < \min \mathfrak{b}$; hence Corollary 4 with $\mathfrak{b}$ in place of $\mathfrak{a}$ implies that F is either λ-decomposable or λ^+-decomposable. The last statements follow from Properties 5(a) – (e). □

Corollary 7 *If λ is a singular cardinal and the ultrafilter D is not* cf λ*-decomposable, then the following conditions are equivalent:*

(a) *There is $\lambda' < \lambda$ such that D is κ-decomposable for all regular cardinals κ with $\lambda' < \kappa < \lambda$.*

(a′) *(Only in case* $\operatorname{cf} \lambda > \omega$*) $\{\kappa < \lambda | D$ is κ^+-decomposable$\}$ is stationary in λ.*

(b) *D is λ^+-decomposable.*

(c) *There is $\lambda' < \lambda$ such that D is (λ', λ^+)-regular.*

(d) *D is (λ, λ)-regular.*

(e) *There is $\lambda' < \lambda$ such that D is (λ', λ)-regular.*

(f) *There is $\lambda' < \lambda$ such that D is (λ'', λ'')-regular for every λ'' with $\lambda' < \lambda'' < \lambda$.*

Proof (a) $\Rightarrow$ (b) and (a′) $\Rightarrow$ (b) are immediate from Theorem 6 and Property 5(a). In case $\operatorname{cf} \lambda > \omega$, (a) $\Rightarrow$ (a′) is trivial.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (a) are given, respectively, by Properties 5(d), (c), (f), (c), (g). □

Proof of Corollary 2 The proof is immediate from Corollary 7(d) $\Rightarrow$ (b) and Properties 5(b) – (d). □

A topological space is $[\mu, \lambda]$-*compact* if and only if every open cover by λ many sets has a subcover by $< \mu$ many sets. A family $\mathcal{F}$ of topological spaces is *productively* $[\mu, \lambda]$-compact if and only if every (Tychonoff) product of members of $\mathcal{F}$ (allowing repetitions) is $[\mu, \lambda]$-compact.

Corollary 8 *If λ is a singular cardinal, then a family of topological spaces is productively $[\lambda, \lambda]$-compact if and only if it is either productively* $[\operatorname{cf} \lambda, \operatorname{cf} \lambda]$*-compact or productively $[\lambda^+, \lambda^+]$-compact.*

Proof Caicedo [1], Theorem 1.7, proved that, for every infinite cardinals μ and λ, a family $\mathcal{F}$ of topological spaces is productively $[\mu, \lambda]$-compact if and only if there exists a (μ, λ)-regular ultrafilter D such that every member of $\mathcal{F}$ is D-compact (see [1] for the definition and references). The corollary is then immediate from Corollary 2, using Property 5(g). □

Henceforth, by a *logic*, we mean a *regular logic* in the sense of Ebbinghaus [5]. Typical examples of regular logics are infinitary logics, or extensions of first-order logic obtained by adding new quantifiers, for example, cardinality quantifiers asserting "there are at least ω_α xs such that ... ."

A logic L is $[\lambda, \mu]$-*compact* if and only if for every pair of sets Γ and Σ of sentences of L, if $|\Sigma| \leq \lambda$ and if $\Gamma \cup \Sigma'$ has a model for every $\Sigma' \subseteq \Sigma$ with $|\Sigma| < \mu$, then $\Gamma \cup \Sigma$ has a model (see [1] and Makowsky [13] for some history and further comments).

Corollary 9 *If λ is a singular cardinal, then a logic is $[\lambda, \lambda]$-compact if and only if it is either $[\operatorname{cf}\lambda, \operatorname{cf}\lambda]$-compact or $[\lambda^+, \lambda^+]$-compact.*

Proof Makowski and Shelah defined what it means for an ultrafilter to be *related* to a logic and showed that a logic $\mathcal{L}$ is $[\lambda, \mu]$-compact if and only if there exists some (μ, λ)-regular ultrafilter related to $\mathcal{L}$ (see [13], Theorem 1.4.4; notice that the order of the parameters is reversed in the definition of (λ, μ)-regularity as given by [13]). The corollary is then immediate from Corollary 2 and Property 5(g). □

Theorem 10 *Suppose that $(\lambda_i)_{i\in I}$ and $(\mu_j)_{j\in J}$ are sets of infinite cardinals. Then the following are equivalent:*

(i) *for every $i \in I$ there is a (λ_i, λ_i)-regular ultrafilter which for no $j \in J$ is (μ_j, μ_j)-regular;*
(ii) *there is a logic which is $[\lambda_i, \lambda_i]$-compact for every $i \in I$ and which for no $j \in J$ is $[\mu_j, \mu_j]$-compact;*
(iii) *for every $i \in I$ there is a $[\lambda_i, \lambda_i]$-compact logic which for no $j \in J$ is $[\mu_j, \mu_j]$-compact.*

The logics in (ii) and (iii) can be chosen to be generated by at most $2 \cdot |J|$ cardinality quantifiers (at most $|J|$ cardinality quantifiers if all μ_js are regular).

Proof In the case when all the μ_js are regular, the theorem is proved in [10], Theorem 4.1. The general case follows from the above particular case by applying Corollaries 2 and 9. □

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Dipartimento di Matematica
II Università di Roma (Tor Vregata)
Viale della Ricerca Scientifica
I-00133 Rome ITALY
http://www.mat.uniroma2.it/~lipparin