Notre Dame Journal of Formal Logic
Volume 49, Number 3, 2008

The Logic of Conditional Negation

John Cantwell

Abstract It is argued that the "inner" negation $\sim$ familiar from 3-valued logic can be interpreted as a form of "conditional" negation: $\sim A$ is read 'A is false *if it has a truth value*'. It is argued that this reading squares well with a particular 3-valued interpretation of a conditional that in the literature has been seen as a serious candidate for capturing the truth conditions of the natural language indicative conditional (e.g.,"If Jim went to the party he had a good time"). It is shown that the logic induced by the semantics shares many familiar properties with classical negation, but is orthogonal to both intuitionistic and classical negation: it differs from both in validating the inference from $A \to \sim B$ to $\sim(A \to B)$.

1 Introduction

Conditional negation, sometimes referred to as "inner negation," is a form of negation that arises in the context of a semantics that allows for truth value gaps. Its truth conditions are given by

A	$\sim A$
T	F
F	T
–	–

That is, $\sim A$ is true if and only if A is false, and false if and only if A is true, lacking truth value when A lacks truth value.

There are two related but distinct reasons for calling it "conditional" negation, both involving a particular gappy set of truth conditions for the indicative conditional:

Received May 4, 2007; accepted January 24, 2008; printed June 6, 2008
2000 Mathematics Subject Classification: Primary, 03B50
Keywords: three-valued logic, inner negation, outer negation, conditionals
 10.1215/00294527-2008-010

$A \to B$	T	F	–
T	T	F	–
F	–	–	–
–	T	F	–

An important feature of these truth conditions for the indicative conditional is that when a conditional has a false antecedent it lacks truth value. This is not a new idea (it seems first to have been proposed by Quine [14] and the above truth conditions are precisely those of Belnap [2]), but it has only gradually been recognized as a serious contender for the semantics that best captures the natural language indicative conditional 'If A, B' (e.g., [2], [4], [5], [7], [8], [12], [13]).

Combining the truth tables we find that $A \to \sim B$ becomes semantically equivalent to $\sim(A \to B)$ (they have the same truth value in every assignment). This seems to correspond well with natural language where the denial of a conditional $A \to B$ does not have the strong logical implications that we find in classical logic (where $\sim(A \to B)$ entails $A \wedge \sim B$) or intuitionistic logic (where $\sim(A \to B)$ entails $\sim\sim A \wedge \sim B$). Consider, for instance, the following exchange:

Anne: If Oswald didn't kill Kennedy, Jack Ruby did.
Bill: No! You're wrong.

When Bill denies the conditional asserted by Anne, he neither asserts nor denies that Oswald did the killing (he can continue, "If Oswald didn't kill Kennedy, Castro did"); his denial seemingly amounts to no more than the assertion that if Oswald didn't shoot Kennedy then neither did Jack Ruby. This kind of "conditional *denial*" seems to be a basic move in the language game; conditional *negation* is the sentential operator that corresponds to this form of conditional denial: "*It is not the case* that if Oswald didn't shoot Kennedy, Jack Ruby did." So one reason for calling $\sim$ "conditional" negation is that it seems to capture a particular way of negating conditionals.

In a gappy setting we no longer have the equivalence:

(E) A if and only if 'A' is true.

Instead ascriptions of truth become *conditional* ascriptions of truth (the "Non-Bivalent"-Equivalence schema):

(NBE) A if and only if 'A' is true *if it has a truth value*.

To illustrate what is going on, one can introduce the operators $T(A)$ ('it is true that A'), $F(A)$ ('it is false that A'), and $TV(A)$ ('it is true or false that A') with the truth conditions:

A	$T(A)$	$F(A)$	$TV(A)$
T	T	F	T
F	F	T	T
–	F	F	F

In a bivalent setting we have the semantical equivalence (corresponding to the E-schema):

$$A \equiv T(A).$$

Here $A \equiv B$ is defined: $A \equiv B$ if and only if A and B have the same truth value in every assignment of truth values (as we will be dealing with different classes of assignments, below $\equiv$ will be indexed with respect to that class).

In our nonbivalent setting we instead have the semantical equivalence (corresponding to the NBE-schema):

$$A \equiv TV(A) \rightarrow T(A).$$

So a belief or an assumption that A is not to be equated with the belief or assumption that A is true, but instead with the belief or assumption that A is true *if it has a truth value*.

Classical negation is often associated with the following equivalence schema (I will throughout use $\neg$ to denote "classical" negation):

(E$\neg$) $\neg A$ if and only if 'A' is false.

Conditional negation does justice to its name by instead validating the weakened form:

(NBE$\sim$) $\sim A$ if and only if 'A' is false *if it has a truth value*.

That is, if we give classical negation the following truth conditions,

A	$\neg A$
T	F
F	T
–	F

we find that 'It is false that A' becomes semantically equivalent to $\neg A$:

$$\neg A \equiv F(A).$$

While, for conditional negation, we find that $\sim A$ amounts to a *conditional* ascription of falsity to A (conditional, that is, upon A having a truth value):

$$\sim A \equiv TV(A) \rightarrow F(A).$$

In this paper I will investigate the basic logical properties of conditional negation with a special emphasis on how it interacts with the semantically gappy conditional.

2 The Semantic Basis for Logic

A standard way of justifying classical logic is by holding (i) that every claim is either true or false (the principle of bivalence), and (ii) an inference is valid if and only if it preserves truth. Together these two principles yield the distinctive properties of classical logic; for instance, they validate the inference from $\neg\neg A$ to A and make $A \vee \neg A$ a logical truth.

In opposition to this view a wide variety of linguistic phenomena have been thought to give rise to truth value gaps and so to invalidate the principle of bivalence. Claims about the future ('There will be a sea battle tomorrow'), vagueness ('That is a heap'), nonreferring singular terms ('The present king of France'), and presupposition failure ('The accused has stopped beating his wife') are typical examples found in the literature.

A common response from the classicist camp has been that the price for these semantic analyses is too high. By allowing truth value gaps we are forced to abandon classical logic and the strictures of classical logic have an intuitive and theoretical appeal that, it is held, overrides any semantical intuitions about truth value gaps. Such a response, of course, makes sense only if one does not take classical logic to be motivated by its semantic properties alone. For if the very fact that we lose classical logic is to be taken as an argument against abandoning bivalence, then the appeal of classical logic cannot exclusively be derived from the appeal of bivalence.

I agree with those who hold that classical logic has an appeal of its own: it embodies norms of reasoning that we actually endorse and use. But by itself this does not provide an argument that every claim is either true or false. I do not think that one can coherently deny an instance of $A \vee \neg A$, and I think that the inference from $\neg\neg A$ to A is valid, but this is an argument for bivalence only if one accepts the principle that an inference is valid if and only if it preserves truth. It is possible to hold that the classically valid inferences are valid and at the same time deny that every claim is either true or false, if one questions the principle that an inference is valid if and only if it preserves truth.

For instance, say (as is suggested above) that the conditional 'If Jim took the exam he passed' is true only if Jim took the exam and passed, and lacks truth value if he didn't take the exam. In this case the inference from 'If Jim took the exam he passed' to 'Jim took the exam and passed' *preserves truth*: if the premise is true then the conclusion is true. But, of course, the inference is not *valid*—it is not *correct*—one cannot properly infer that Jim took the exam from the premise that *if* he did, he passed.

If we have good grounds for denying that every proposition is either true or false, and if we have good grounds for holding that classical logic is valid, then we have grounds for denying that valid inferences preserve truth. Instead, I suggest, we should hold that valid inferences *do not introduce falsity*: the conclusion of a valid inference is false only if one of the premises is false. Of course, in a bivalent setting the principle that valid inferences do not introduce falsity is equivalent to the principle that a valid inference preserves truth, but in a nonbivalent setting these principles do not coincide.

It will be shown below that if we interpret negation as *outer* negation or *classical* negation, that is, if we interpret '$\neg A$' as 'A is false', the resulting logic induced by the principle that valid inferences do not introduce falsity is *classical* logic, even though the conditional $\rightarrow$ introduces truth value gaps. It is classical logic, not in the sense that it is based on a bivalent semantics or in the sense that it preserves truth, but in the sense that all and only classically valid inferences turn out to be valid. Of course, if we think of "classical" logic as a package, representing not only a particular collection of valid inferences or valid inference rules, but also as a particular way of justifying these inferences and inference rules (every claim is true or false, valid inferences preserve truth), then the result is not classical logic. But if we keep apart the extension of the classical inference relation from the orthodox way of justifying how this inference relation characterizes the valid inferences, there should be no grounds for confusion.

Conditional negation, the topic of this paper, is not a "classical" negation. For instance, it validates the inference from $A \rightarrow \sim B$ to $\sim(A \rightarrow B)$. What makes it interesting is that there seems to be a form of negation in natural language that works in this way. By this I do not mean to deny that there is a form of negation in natural language that works as "classical" negation (which validates the inference from $\neg(A \rightarrow B)$ to $A \wedge \neg B$), that is, a form of negation which amounts to the claim that the negated sentence is false. The two can happily coexist.

3 Motivating Nonbivalent Conditionals

Conditional negation differs semantically from classical negation only when the negated sentence lacks truth value; thus some motivation should be given for the claim that propositions can lack truth value to begin with. In the present setting it is the conditional that introduces truth value gaps so this needs some motivation.

The primary motivation, as I see it, is that there is just no evidence that we as speakers take a conditional to be true (or, for that matter, false) on the grounds that its antecedent is false. The very fact that one believes that Jim didn't take the exam does not by itself provide grounds for holding that *if* he took the exam, he passed. Indeed, if I am reasonably confident that Jim didn't take the exam, but completely certain that Jim was not smart enough to pass the exam, I can even *deny* that he passed the exam *if* he took it. If the standard material analysis of the conditional is correct, this is puzzling indeed.

Even if the material analysis is incorrect, it doesn't immediately follow that a conditional like 'If Jim took the exam he passed' lacks truth value if Jim didn't take the exam. There are several other analyses around that do not introduce truth value gaps. For instance, one popular strategy is to hold that 'If Jim took the exam he passed is true' if Jim passed in every possible world maximally similar to the present world where Jim took the exam. Typically, the "similarity" involved in this analysis is "epistemic" similarity (which sets the indicative conditional apart from the *counterfactual* conditional 'If Jim had taken the exam he would have passed' where the similarities involved are "objective").

There are numerous problems involved with these alternate analyses and I will not rehearse them all here (see [8]). Instead I will focus on one problem: a number of classically valid inferences turn out to be invalid on these analyses. For instance, I think that the classically valid inference from 'Either gardener or the butler did it' to 'If the gardener didn't do it, the butler did' is indeed valid in the following sense: upon making the *supposition* for the sake of the argument (upon *assuming*) that either the gardener or the butler did it, one is committed to accepting that if the gardener didn't do it, the butler did. It is a valid inference, indeed a *classically* valid inference, but it is not validated by accounts that analyze the conditional in terms of "epistemic" similarities between possible worlds.

Of course, one can deny that the inference from 'Either gardener or the butler did it' to 'If the gardener didn't do it, the butler did' is valid. In particular, say that I believe that the gardener did it and that the butler is innocent. From 'The gardener did it' I can validly infer 'Either gardener or the butler did it' but we do not want to say that I am thereby committed to believing that if the gardener didn't do it, the butler did. Indeed, the inference in question is commonly invoked as an argument against the material analysis of the indicative conditional.

I think the problem here is that people tend not to recognize the importance of the difference between *assuming* or *supposing* for the sake of the argument that something is the case, and *believing* that something is the case (although see Levi [10]). I believe that the gardener did it, and so I believe that either the gardener or the butler did it. But upon *supposing* for the sake of the argument that the gardener didn't do it, I no longer accept that either the gardener or the butler did it as I no longer accept that the gardener did it. The weatherman tells me that it will snow tomorrow, and I believe him. But on the supposition that the weatherman is lying, I no longer accept that it will snow tomorrow. Suppositions can *undermine* beliefs, upon making a supposition for the sake of the argument one will reject (hypothetically reject) things that one otherwise believes, and accept (hypothetically accept) things that one does not otherwise believe. By contrast, assumptions do not "undermine" other assumptions. If I first *assume* that the butler did it, and then *assume* that the butler didn't do it, I am committed to accepting a contradiction. Assumptions remain in force until they are explicitly dropped, not by adding further assumptions that make earlier assumptions implausible.

This distinction between *supposing* and *believing* that something is the case becomes particularly important in the context of conditionals. For, paradigmatically, one accepts 'If A, B' if and only if one accepts B on the *supposition* that A. So upon *assuming* both that the butler did it and that the butler didn't do it I am committed to accepting a contradiction and hence (via ex falso quodlibet) am committed to accepting that the gardener did it. So, upon *assuming* that the gardener did it, I am committed to accepting that *if* the gardener didn't do it, the butler did. This does not entail that when I *believe* that the gardener did it, I am committed to accepting that if the gardener didn't do it, the butler did. If one keeps in mind the distinction between *supposing* or *assuming* that something is the case and *believing* that something is the case, the "paradoxes" of material implication cease to be paradoxical.

In holding that the "paradoxes" of material implication are not paradoxical in the context of suppositional reasoning, one does not reinstate the *semantical* thesis according to which a conditional is true if the antecedent is true. I began this section by noting that a major problem with the material analysis is that people do not in general take the falsity of the antecedent of a conditional as grounds for accepting the conditional; indeed, there are countless examples where it seems perfectly coherent to reject both a conditional and its antecedent. The point being that ultimately ascriptions of truth conditions must answer to usage; if people who seemingly know all the relevant facts, who are rational, and who are not linguistically confused are willing to reject a conditional, what grounds do we have for holding that the conditional is true?

Indeed the added structure we get from allowing conditionals to lack truth value turns out to be suitable for characterizing the *acceptability* conditions for the indicative conditional. One key to a full understanding of indicative conditionals is to understand the conditions under which one accepts or rejects a conditional that may lack truth value. For instance, why do I *accept* 'If Oswald didn't kill Kennedy someone else did' but *reject* 'If Oswald didn't kill Kennedy, my grandmother did' even though I believe that the antecedents of both these conditionals are false?

The acceptance conditions for the indicative conditional, I have argued elsewhere [8] (and others before me, e.g., [1], [9]), cannot exclusively be given by appealing to the semantic value of the indicative conditional. The fact that conditionals can lack truth value opens an explanatory gap between the semantics of the conditional and its assertibility or acceptability conditions. This gap cannot be filled by semantic means but must instead invoke epistemic considerations. One of the strong points of the present gappy truth conditions for indicative conditionals is that, when combined with an interpretation of probability that is suitable for gappy propositions ([3], [6], [5]), we get the much discussed identity (sometimes called Adams Thesis after Adams [1]):

$$\Pr(A \to B) = \Pr(B|A),$$

while avoiding the "triviality" results that behest standard bivalent accounts (e.g., Lewis [11]).

4 A Semantic Characterization

It is time to become more specific about the precise structure of the object language. Let $\mathcal{L}_\sim$ be a language containing a countable set of propositional atoms $p, q, r, \ldots,$ closed under the connectives $\sim$, $\wedge$, $\vee$, and $\rightarrow$, together with the contradictory sentence $\bot$. The language $\mathcal{L}_\neg$ is like $\mathcal{L}_\sim$ except that we replace $\sim$ by $\neg$.

Definition 4.1 A *classical valuation* is an assignment I of the truth values {T, F} to the sentences of $\mathcal{L}$ satisfying

A	$\neg A$
T	F
F	T

$A \rightarrow B$	T	F
T	T	F
F	T	T

$A \wedge B$	T	F
T	T	F
F	F	F

$A \vee B$	T	F
T	T	T
F	T	F

Definition 4.2 The classical consequence relation $\models_C$ is a relation from sets of sentences to single sentences satisfying

$\Gamma \models_C A$ if and only if for every classical valuation I, if $I(B) = \mathrm{T}$ for all $B \in \Gamma$, then $I(A) = \mathrm{T}$.

Definition 4.3 An NBC-*valuation* (a Non-Bivalent Classical valuation) is an assignment I of the truth values {T, F, –} to the sentences of $\mathcal{L}$ satisfying

A	$\neg A$
T	F
F	T
–	F

$A \rightarrow B$	T	F	–
T	T	F	–
F	–	–	–
–	T	F	–

$A \wedge B$	T	F	–
T	T	F	–
F	F	F	F
–	–	F	–

$A \vee B$	T	F	–
T	T	T	T
F	T	F	–
–	T	–	–

Theorem 4.4 *$\Gamma \models_C A$ if and only if for every* NBC-*valuation I, if $I(B) \neq$* F *for all $B \in \Gamma$, then $I(A) \neq$* F.

Proof The proof is quite trivial, just think of – as representing 'true' and we have the standard truth conditions for the connectives. □

That is, a nonbivalent valuation characterizes the classical consequence relation if one takes the defining feature of a classical inference to be that it does not introduce falsity: in a valid inference the conclusion is false only if one of the premises is false. Indeed, in the bivalent case the property of not introducing falsity is coextensional with the property of preserving truth:

$\Gamma \models_C A$ if and only if for every classical valuation I, if $I(B) \neq \mathrm{F}$ for all $B \in \Gamma$, then $I(A) \neq \mathrm{F}$.

Definition 4.5 A CN-*valuation* is an assignment I of the truth values {T, F, –} to the sentences of $\mathcal{L}_\sim$ satisfying

A	$\sim A$
T	F
F	T
–	–

$A \rightarrow B$	T	F	–
T	T	F	–
F	–	–	–
–	T	F	–

$A \wedge B$	T	F	–
T	T	F	–
F	F	F	F
–	–	F	–

$A \vee B$	T	F	–
T	T	T	T
F	T	F	–
–	T	–	–

Definition 4.6 The consequence relation $\models_{CN}$ of conditional negation is a relation from sets of sentences to single sentences satisfying

$\Gamma \models_{\mathrm{CN}} A$ if and only if for every CN-valuation I, if $I(B) \neq \mathrm{F}$ for all $B \in \Gamma$, then $I(A) \neq \mathrm{F}$.

Again the defining feature of the consequence relation is taken to be that it does not introduce falsity.

4.1 Discussion Define the following notion of *logical* equivalence:

$$A \Leftrightarrow_{\mathrm{CN}} B \text{ if and only if } A \models_{\mathrm{CN}} B \text{ and } B \models_{\mathrm{CN}} A.$$

Note that (this is trivial) semantic equivalence entails logical equivalence: if $A \equiv_{\mathrm{CN}} B$, then $A \Leftrightarrow_{\mathrm{CN}} B$. The converse, however, need not hold. For instance, we have $\sim A \vee B \Leftrightarrow_{\mathrm{CN}} A \rightarrow B$ but it is not, in general, the case that $\sim A \vee B \equiv_{\mathrm{CN}} A \rightarrow B$ (take the case when A is false). At the level of acceptance and rejection $\sim A \vee B$ is quite distinct from $A \rightarrow B$ which explains why they are not semantically equivalent, but at the level of suppositional reasoning which seeks only to avoid introducing falsity, these differences are washed out.

Indeed, we find that there are a number of connectives that are logically equivalent (in the sense that they are interderivable in the logic of suppositional reasoning) without being semantically equivalent. Compare, for instance, the following two connectives (which we temporarily add to $\mathcal{L}_{\neg}$):

A	$\ominus A$
T	F
F	–
–	F

$A \oplus B$	T	F	–
T	–	F	–
F	F	F	F
–	–	F	–

Note that in spite of their differences in truth tables, we have $\ominus A \Leftrightarrow_{\mathrm{NBC}} \neg A$ and $A \oplus B \Leftrightarrow_{\mathrm{NBC}} A \wedge B$.

The consequence relation seemingly treats "lacks truth value" as "true" (for instance, in a valid inference it can happen that we go from premises that lack truth value to a conclusion that lacks truth value), thus should we not view "lacks truth value" as another "kind" of truth? No. It can be reasonable to *reject* a claim that one believes lacks truth value, as in 'If Oswald didn't kill Kennedy my grandmother did', just as it can be reasonable to *accept* a claim that one believes lacks truth value, as in 'If Oswald didn't kill Kennedy someone else did'. The key here being that although I believe that 'If Oswald didn't kill Kennedy my grandmother did' lacks truth value, I also believe that *if* I am wrong and the conditional has a truth value (which it does only if Oswald didn't kill Kennedy) then the conditional is false. By comparison it is not reasonable to reject a claim that one believes is true, just as it is not reasonable to accept a claim that one believes is false.

So the connective $\oplus$ would give an incorrect rendition of the English 'and': when someone asserts 'A and B' we take that claim to be *correct* if both A and B are true, given that one accepts A and B one is committed to accepting 'A and B' and we are willing to say that the speaker (who uttered the conjunction) was *right*. This speaks against treating 'A and B' as lacking truth value (and so speaks against interpreting 'and' as $\oplus$) when both A and B are true. Similarly, if one believes that A is false, then it is not reasonable to *deny* 'It is not the case that A' and so 'It is not the case that A' is *true* when A is false, thus $\ominus$ is not a correct semantic rendition of 'It is not the case that…'.

An important point here is that truth conditions must answer not only to the logic of suppositional reasoning, but also to language use. The appeal to logical intuitions when settling semantic issues must be tempered by the realization that the logic of suppositional reasoning does not necessarily coincide with the logic of belief or acceptance (this becomes particularly important when dealing with conditionals).

Note also that while we have $\ominus A \Leftrightarrow_{\mathrm{NBC}} \neg A$ we do not have $\neg \ominus A \Leftrightarrow_{\mathrm{NBC}} \neg\neg A$; thus $\ominus A$ and $\neg A$ are not substitutable in the logic of suppositional reasoning. Indeed, we get a similar phenomenon in $\mathcal{L}_{\sim}$. We have $p \to q \Leftrightarrow_{\mathrm{CN}} {\sim}p \vee q$, but we do not have ${\sim}(p \to q) \Leftrightarrow_{\mathrm{CN}} {\sim}({\sim}p \vee q)$. For ${\sim}(p \to q)$ is semantically equivalent to $p \to {\sim}q$ whereas ${\sim}({\sim}p \vee q)$ is semantically equivalent to $p \wedge {\sim}q$.

On the other hand, we have the following theorem.

Theorem 4.7 *If $A \Leftrightarrow_{\mathrm{CN}} B$ and ${\sim}A \Leftrightarrow_{\mathrm{CN}} {\sim}B$, then $A \equiv_{\mathrm{CN}} B$.*

Proof Assume (i) $A \Leftrightarrow_{\mathrm{CN}} B$ and (ii) ${\sim}A \Leftrightarrow_{\mathrm{CN}} {\sim}B$. Assume that A is true in some CN-valuation I, then ${\sim}A$ is false in I and so, by (ii), ${\sim}B$ is false in I and B is true in I. Assume instead that A is false in I, then, by (i), B is false in I. Similarly, if B is true in I, then A is true in I and if B is false in I, then A is false in I. Thus A and B have the same truth value in every CN-valuation. □

That is, if A and B are logically equivalent and ${\sim}A$ and ${\sim}B$ are also logically equivalent, then A and B are semantically equivalent (and so fully substitutable).

5 A Proof-Theoretical Characterization

Definition 5.1 An *inferential relation* $\vdash$ is a relation from finite sets of sentences to sentences satisfying the *structural* conditions,[1]

Reflexivity $A \vdash A$;
Monotonicity If $\Delta \vdash B$, then $\Delta \cup \Delta' \vdash B$;
Cut If $\Delta \vdash A$ for each $A \in \Delta'$, and $\Delta \cup \Delta' \vdash B$, then $\Delta \vdash B$.

Definition 5.2 An inferential relation $\vdash$ is standard if it satisfies

Rules for ∧ $A \wedge B \vdash A$ and $A \wedge B \vdash B$;
$A, B \vdash A \wedge B$;

Rules for ∨ $A \vdash A \vee B$ and $A \vdash B \vee A$;
If $\Gamma, A \vdash C$ and $\Gamma, B \vdash C$, then $\Gamma, A \vee B \vdash C$;

Rules for → $A, A \to B \vdash B$;
If $\Gamma, A \vdash B$, then $\Gamma \vdash A \to B$;

Rules for ⊥ $p, \neg p \vdash \bot$, for any propositional letter p;
$\bot \vdash A$, for any A;

LEM $\vdash A \vee \neg A$.

Definition 5.3 A standard inferential relation $\vdash$ embodies classical negation if it satisfies

$(\neg \to)$ $\neg(A \to B) \dashv\vdash A \wedge \neg B$;
$(\neg\vee)$ $\neg(A \vee B) \dashv\vdash \neg A \wedge \neg B$;
$(\neg\wedge)$ $\neg(A \wedge B) \dashv\vdash \neg A \vee \neg B$;
$(\neg\neg)$ $\neg\neg A \dashv\vdash A$.

Theorem 5.4 *Let $\vdash$ be the smallest standard inferential relation embodying classical negation:*

$$\Gamma \vdash A \textit{ if and only if } \Gamma \models_C A.$$

Proof Assume that $\vdash$ is the smallest standard inferential relation embodying classical negation. Clearly, all the inference rules of $\vdash$ are classically valid; thus we need to show that it captures all classically valid inference rules. Due to well-known properties of classical logic we need only show the following:

1. If $\Gamma, A \vdash \bot$, then $\Gamma \vdash \neg A$;
2. $A, \neg A \vdash \bot$.

(1) Assume that $\Gamma, A \vdash \bot$. By cut and the rule for $\bot$, $\Gamma, A \vdash \neg A$. By reflexivity and monotonicity $\Gamma, \neg A \vdash \neg A$. So $\Gamma, A \vee \neg A \vdash \neg A$. By LEM $\Gamma \vdash A \vee \neg A$. Thus by cut (and the rules for $\vee$) $\Gamma \vdash \neg A$.

(2) We derive $A, \neg A \vdash \bot$ by induction over the length of A.

$A = p$. Follows directly from the rule for $\bot$.

Assume that it has been shown that $C, \neg C \vdash \bot$ and $D, \neg D \vdash \bot$.

$A = C \vee D$. We need to show $C \vee D, \neg(C \vee D) \vdash \bot$, that is, by $\neg\vee$ and the standard rules that $C \vee D, \neg C, \neg D \vdash \bot$. We know that $C, \neg C, \neg D \vdash \bot$ and $D, \neg C, \neg D \vdash \bot$, but then $C \vee D, \neg C, \neg D \vdash \bot$.

$A = C \wedge D$. We need to show $C \wedge D, \neg(C \wedge D) \vdash \bot$, that is, by $\neg\wedge$ and the standard rules that $C \wedge D, \neg C \vee \neg D \vdash \bot$. We know that $C, D, \neg C \vdash \bot$ and $C, D, \neg D \vdash \bot$, but then $C, D, \neg C \vee \neg D \vdash \bot$.

$A = C \to D$. We need to show $C \to D, \neg(C \to D) \vdash \bot$, that is, by $\neg \to$ and the standard rules that $C \to D, C, \neg D \vdash \bot$. By modus ponens, $C \to D, C, \neg D \vdash D$, that is, $C \to D, C, \neg D \vdash D \wedge \neg D$ and so by cut and the induction hypothesis $C \to D, C, \neg D \vdash \bot$.

$A = \neg C$. We need to show $\neg C, \neg\neg C \vdash \bot$. But by the rule $\neg\neg$ this follows from $\neg C, C \vdash \bot$ which we have by the induction hypothesis. □

Definition 5.5 A standard inferential relation $\vdash$ (on $\mathcal{L}_\sim$) embodies conditional negation if it satisfies

($\sim\to$) $\sim(A \to B) \dashv\vdash A \to \sim B$;
($\sim\vee$) $\sim(A \vee B) \dashv\vdash \sim A \wedge \sim B$;
($\sim\wedge$) $\sim(A \wedge B) \dashv\vdash \sim A \vee \sim B$;
($\sim\sim$) $\sim\sim A \dashv\vdash A$.

We want to show that when $\vdash$ is the smallest standard inferential relation embodying conditional negation

$$\Gamma \vdash A \text{ if and only if } \Gamma \models_{\mathrm{CN}} A.$$

The soundness part (that $\Gamma \vdash A$ implies $\Gamma \models_{\mathrm{CN}} A$) is straightforward (one need only show that each inference satisfies the principle that it does not introduce falsity) and will not be proved in full. So, for instance, note that if I is a CN-valuation such that $I(B) = \mathrm{F}$, then $I(A \to B) \neq \mathrm{F}$ only if $I(A) = \mathrm{F}$, so $A, A \to B \models_{\mathrm{CN}} B$. Or note that if $\Gamma, A \models_{\mathrm{CN}} B$, then $\Gamma \models_{\mathrm{CN}} A \to B$. For if there is some CN-valuation I such that $I(A \to B) = \mathrm{F}$ but $I(C) \neq \mathrm{F}$ for each $C \in \Gamma$ (so that $\Gamma \not\models_{\mathrm{CN}} A \to B$), then $I(B) = \mathrm{F}$ and $I(A) \neq \mathrm{F}$, and so $\Gamma, A \not\models_{\mathrm{CN}} B$. Or note that for every valuation I,

$I(\sim(A \to B)) = \text{F}$ if and only if $I(A \to B) = \text{T}$ if and only if $I(A) \neq \text{F}$ and $I(B) = \text{T}$ if and only if $I(A) \neq \text{F}$ and $I(\sim B) = \text{F}$ if and only if $I(A \to \sim B) = \text{F}$; hence $A \to \sim B \equiv \sim(A \to B)$; thus the rule $\sim\to$ is sound.

A set Γ is $\vdash$-*consistent* if there is no finite subset Γ' of Γ such that $\Gamma' \vdash \bot$.

Lemma 5.6

1. *If $\vdash$ is a standard inference relation, then any $\vdash$-consistent set can be extended to a* maximal *$\vdash$-consistent set.*
2. *If Σ is a maximal $\vdash$-consistent set and there is some finite subset Σ' of Σ such that $\Sigma' \vdash A$, then $A \in \Sigma$.*
3. *Either $A \in \Sigma$ or $\sim A \in \Sigma$.*

Proof (1) Assume that Γ is a $\vdash$-consistent set. Let $\Sigma_0 = \Gamma$. Let $\Sigma_{i+1} = \Sigma_i \cup \{A_i\}$ if $\Sigma_i \cup \{A_i\}$ is a consistent set; otherwise $\Sigma_{i+1} = \Sigma_i$. Let $\Sigma_* = \bigcup \Sigma_i$. Clearly, every Σ_i is a $\vdash$-consistent set.

Assume that Σ_* is not a $\vdash$-consistent set. Then there is some finite set $\Sigma' \subseteq \Sigma$ such that $\Sigma' \vdash \bot$. As Σ' is finite there is some Σ_i such that $\Sigma' \subseteq \Sigma_i$. Due to monotonicity $\Sigma_i \vdash \bot$ which gives us a contradiction; hence Σ_* is a $\vdash$-consistent set.

Assume that there is some $A_i \notin \Sigma_*$ such that $\Sigma_* \cup \{A_i\}$ is a $\vdash$-consistent set. Due to monotonicity $\Sigma_i \cup \{A_i\}$ is a $\vdash$-consistent set, but then $A_i \in \Sigma_{i+1}$ and so $A_i \in \Sigma_*$, contrary to assumption. So, if $A_i \notin \Sigma_*$, then $\Sigma_* \cup \{A_i\}$ is not a $\vdash$-consistent set. So, due to monotonicity, any strict superset of Σ_* is not a $\vdash$-consistent set. Hence Σ_* is a maximal $\vdash$-consistent set.

(2) Assume that Σ' is a finite subset of Σ such that $\Sigma' \vdash A$. Assume for reductio that $\Sigma \cup \{A\}$ is $\vdash$-inconsistent set. Then there is a finite subset Σ'' of Σ such that $\Sigma'', A \vdash \bot$. Due to monotonicity and cut $\Sigma' \cup \Sigma'' \vdash \bot$, but then Σ is a $\vdash$-inconsistent set, contrary to assumption. $\Sigma \cup \{A\}$ is a $\vdash$-consistent set. As Σ is a maximal $\vdash$-consistent set, $A \in \Sigma$.

(3) Assume that $A \notin \Sigma$ and $\sim A \notin \Sigma$. Thus both $\Sigma \cup \{A\}$ and $\Sigma \cup \{\sim A\}$ are $\vdash$-inconsistent sets. Thus there are finite subsets Σ' and Σ'' of Σ such that $\Sigma', A \vdash \bot$ and $\Sigma'', \sim A \vdash \bot$. But then $\Sigma' \cup \Sigma'', A \vee \sim A \vdash \bot$. Thus, from $\Sigma' \cup \Sigma'' \vdash A \vee \sim A$ and cut, $\Sigma' \cup \Sigma'' \vdash \bot$, contradicting the assumption that Σ is a consistent set. □

Let Σ be a maximal $\vdash$-consistent set. Define

$I_\Sigma(A) = \text{T}$ if $A \in \Sigma$ and $\sim A \notin \Sigma$,
$I_\Sigma(A) = \text{F}$ if $\sim A \in \Sigma$ and $A \notin \Sigma$,
$I_\Sigma(A) = -$ if $A \in \Sigma$ and $\sim A \in \Sigma$.

Lemma 5.7 *If $\vdash$ is a standard inference relation embodying conditional negation, then I_Σ is a* CN-*valuation.*

Proof

($\sim$) Show: $I_\Sigma(\sim A) = \text{T}$ if and only if $I_\Sigma(A) = \text{F}$. By construction $I_\Sigma(\sim A) = \text{T}$ if and only if $\sim A \in \Sigma$ and $\sim\sim A \notin \Sigma$ if and only if (by ($\sim\sim$)) $\sim A \in \Sigma$ and $A \notin \Sigma$ if and only if (by construction) $I_\Sigma(A) = \text{F}$. Show: $I_\Sigma(\sim A) = \text{F}$ if and only if $I_\Sigma(A) = \text{T}$. By construction $I_\Sigma(\sim A) = \text{F}$ if and only if $\sim\sim A \in \Sigma$ and $\sim A \notin \Sigma$ if and only if (by ($\sim\sim$)) $A \in \Sigma$ and $\sim A \notin \Sigma$ if and only if (by construction) $I_\Sigma(A) = \text{T}$.

($\wedge$) Show: $I_\Sigma(A \wedge B) = \mathrm{T}$ if and only if $I_\Sigma(A) = I_\Sigma(B) = \mathrm{T}$. By construction $I_\Sigma(A \wedge B) = \mathrm{T}$ if and only if $A \wedge B \in \Sigma$ and $\sim(A \wedge B) \notin \Sigma$ if and only if (by the standard rules and ($\sim\wedge$)) $A, B \in \Sigma$ and $\sim A \vee \sim B \notin \Sigma$ if and only if (by the standard rules) $A, B \in \Sigma$ and $\sim A, \sim B \notin \Sigma$ if and only if (by construction) $I_\Sigma(A) = I_\Sigma(B) = \mathrm{T}$. Show: $I_\Sigma(A \wedge B) = \mathrm{F}$ if and only if $I_\Sigma(A) = \mathrm{F}$ or $I_\Sigma(B) = \mathrm{F}$. By construction $I_\Sigma(A \wedge B) = \mathrm{F}$ if and only if $A \wedge B \notin \Sigma$ and $\sim(A \wedge B) \in \Sigma$ if and only if (by the standard rules and ($\sim\wedge$)) either $A \notin \Sigma$ or $B \notin \Sigma$ and $\sim A \vee \sim B \in \Sigma$ if and only if (by negation completeness and the standard rules) either $A \notin \Sigma$ and $\sim A \in \Sigma$ or $B \notin \Sigma$ and $\sim B \in \Sigma$ if and only if (by construction) either $I_\Sigma(A) = \mathrm{F}$ or $I_\Sigma(B) = \mathrm{F}$. The case for disjunction is similar.

($\rightarrow$) Show: $I_\Sigma(A \rightarrow B) = \mathrm{T}$ if and only if $I_\Sigma(A) \neq \mathrm{F}$ and $I_\Sigma(B) = \mathrm{T}$. By construction $I_\Sigma(A \rightarrow B) = \mathrm{T}$ if and only if $A \rightarrow B \in \Sigma$ and $\sim(A \rightarrow B) \notin \Sigma$ if and only if (by ($\sim\rightarrow$)) $A \rightarrow B \in \Sigma$ and $A \rightarrow \sim B \notin \Sigma$. Assume that $A \rightarrow B \in \Sigma$ and $A \rightarrow \sim B \notin \Sigma$. By the standard rules, $\sim B \notin \Sigma$ and as then $B \in \Sigma$, $I_\Sigma(B) = \mathrm{T}$. Assume for reductio that $A \notin \Sigma$. Then $A \rightarrow \perp \in \Sigma$ and so, by the standard rules, $A \rightarrow \sim B \in \Sigma$, contrary to assumption. Thus $A \in \Sigma$ and so $I_\Sigma(A) \neq \mathrm{F}$. Assume instead that $I_\Sigma(A) \neq \mathrm{F}$ and $I_\Sigma(B) = \mathrm{T}$. Then $A, B \in \Sigma$ and $\sim B \notin \Sigma$. By the standard rules $A \rightarrow B \in \Sigma$. Assume, for reductio, that $A \rightarrow \sim B \in \Sigma$. By the standard rules $\sim B \in \Sigma$ contradicting our previous assumption. So $A \rightarrow \sim B \notin \Sigma$. An analogous proof shows: $I_\Sigma(A \rightarrow B) = \mathrm{F}$ if and only if $I_\Sigma(A) \neq \mathrm{F}$ and $I_\Sigma(B) = \mathrm{F}$. □

Lemma 5.8 *If $\vdash$ is a standard inference relation embodying conditional negation then*

$$\vdash A \vee (A \rightarrow \perp).$$

Proof The proof proceeds by induction over the length of A. Begin by noting that due to the rules governing $\sim$, any sentence containing a conditional negation is provably equivalent to a sentence where conditional negation operates only on the propositional atoms (e.g., $\sim((\sim p \rightarrow q) \vee \sim r)$ is provably equivalent to $(\sim p \rightarrow \sim q) \wedge r)$. Thus it is enough that we can show the claim for all sentences where conditional negation operates only on the propositional atoms.

For the induction base, show that the claim holds when A is a propositional atom, that is, that $\vdash p \vee (p \rightarrow \perp)$. By LEM $\vdash p \vee \sim p$. As $p \vdash p \vee (p \rightarrow \perp)$, it is enough to show that $\sim p \vdash p \vee (p \rightarrow \perp)$. But $\sim p, p \vdash \perp$; thus $\sim p \vdash p \rightarrow \perp$ and so $\sim p \vdash p \vee (p \rightarrow \perp)$.

Next we show that the claim of the theorem holds when A is a negated atom, that is, that $\vdash \sim p \vee (\sim p \rightarrow \perp)$. By LEM $\vdash \sim p \vee \sim\sim p$. As $\sim p \vdash \sim p \vee (\sim p \rightarrow \perp)$, it is enough to show that $\sim\sim p \vdash \sim p \vee (\sim p \rightarrow \perp)$. But $\sim\sim p, \sim p \vdash \perp$ (due to the rule for $\perp$ and the rule $\sim\sim$); thus $\sim\sim p \vdash \sim p \rightarrow \perp$ and so $\sim\sim p \vdash \sim p \vee (\sim p \rightarrow \perp)$.

Assume that it has been shown that $\vdash A \vee (A \rightarrow \perp)$ and $\vdash B \vee (B \rightarrow \perp)$. Show that $\vdash (A \vee B) \vee ((A \vee B) \rightarrow \perp)$. (In the following let $D = (A \vee B) \vee ((A \vee B) \rightarrow \perp)$). We know that $\vdash A \vee (A \rightarrow \perp)$ and that $A \vdash D$. So we need to show that $A \rightarrow \perp \vdash D$. We know that $A \rightarrow \perp \vdash B \vee (B \rightarrow \perp)$ and we know that $A \rightarrow \perp, B \vdash D$. Thus all we need to show is that $A \rightarrow \perp, B \rightarrow \perp \vdash D$. As $A \rightarrow \perp, B \rightarrow \perp, A \vdash \perp$ and $A \rightarrow \perp, B \rightarrow \perp, B \vdash \perp$, it follows that $A \rightarrow \perp, B \rightarrow \perp, (A \vee B) \vdash \perp$; that is, $A \rightarrow \perp, B \rightarrow \perp \vdash (A \vee B) \rightarrow \perp$; that is, $A \rightarrow \perp, B \rightarrow \perp \vdash (A \vee B) \vee ((A \vee B) \rightarrow \perp)$, and we are done.

Show that $\vdash (A \wedge B) \vee ((A \wedge B) \to \perp)$. (In the following let $D = (A \wedge B) \vee ((A \wedge B) \to \perp)$). We know that $\vdash A \vee A \to \perp$ and that $A \to \perp \vdash D$ ($A \to \perp, A \wedge B \vdash \perp$ so $A \to \perp \vdash (A \wedge B) \to \perp$). So we need to show that $A \vdash D$. We know that $A, B \vdash D$ so (as we know that $\vdash B \vee B \to \perp$), all we need to show is that $A, B \to \perp \vdash D$. But as $B \to \perp, A \wedge B \vdash \perp$ so $B \to \perp \vdash (A \wedge B) \to \perp$, and so $A, B \to \perp \vdash D$ and we are done.

Show that $\vdash (A \to B) \vee ((A \to B) \to \perp)$. (In the following let $D = (A \to B) \vee ((A \to B) \to \perp)$). We know that $\vdash B \vee B \to \perp$ and that $B \vdash D$ ($B, A \vdash B$ so $B \vdash A \to B$). So we need to show that $B \to \perp \vdash D$. First note that $A \to \perp \vdash D$, for $A \to \perp, A \vdash \perp$ and so $A \to \perp, A \vdash B$ and so $A \to \perp \vdash A \to B$, but then $\ominus A \vdash D$. So, as $A \to \perp \vee A \to \perp$ all we need to show is that $B \to \perp, A \vdash D$. We know that $A, B \to \perp, A \to B \vdash B$ and so $A, B \to \perp, A \to B \vdash \perp$. Thus $A, B \to \perp \vdash (A \to B) \to \perp$. But then $A, B \to \perp \vdash D$ and we are done. □

Theorem 5.9 *If $\vdash$ is a standard inference relation embodying conditional negation and $\Gamma \models_{\mathrm{CN}} B$, then $\Gamma \vdash B$.*

Proof Assume that $A_1, \ldots, A_n \nvdash B$. Let $C = (A_1 \wedge \cdots \wedge A_n) \to B$. It follows that $\nvdash C$.

Assume for reductio that $\{C \to \perp\}$ is not a $\vdash$-consistent set, that is, that $C \to \perp \vdash \perp$. From Lemma 5.8 we know that $\vdash C \vee (C \to \perp)$. As $C \to \perp \vdash \perp$ it follows that $C \to \perp \vdash C$, and as $C \vdash C$, it follows that $\vdash C$, contrary to assumption. Thus $\{C \to \perp\}$ is a $\vdash$-consistent set and can, by Lemma 5.6, be extended to a maximal $\vdash$-consistent set Σ. Note that $C \to \perp \in \Sigma$ and so $\sim C \in \Sigma$. As $C \to \perp \in \Sigma$ and as Σ is consistent, $C \notin \Sigma$. Thus, by Lemma 5.7, there is a CN-valuation I, namely, I_Σ, such that $I_\Sigma(C) = \mathrm{F}$. So $\not\models_{\mathrm{CN}} C$. But then $A_1, \ldots, A_n \not\models_{\mathrm{CN}} B$, and we are done. □

5.1 Discussion The inference rules for classical logic and the logic of conditional negation have been presented so as to maximize ease of comparison. The rule that sets them apart is the rule for how negation governs implication $\to$. Note that the inference from $\neg(A \to B)$ to $A \wedge \neg B$ is classically valid but the corresponding inference from $\sim(A \to B)$ to $A \wedge \sim B$ is not valid in the logic of conditional negation. Note also that the inference from $A \to \sim B$ to $\sim(A \to B)$ is valid in the logic of conditional negation, but the corresponding inference for $\neg$ is not classically valid. Thus the logic of conditional negation is "orthogonal" to classical logic: it is neither weaker nor stronger.

Note that in the logic of conditional negation it is not in general the case that A and $\sim A$ are mutually inconsistent. For instance, from the pair $p \to q$ and $\sim(p \to q)$ one cannot derive the contradictory sentence $\perp$, only $\sim p$. For $\sim(p \to q)$ is logically equivalent to $p \to \sim q$ which does not logically contradict $p \to q$. Thus the pair $p \to q$ and $\sim(p \to q)$ expresses a *conditional* contradiction: a contradiction conditional upon p being true.

6 Expressiveness

The languages $\mathcal{L}_\sim$ and $\mathcal{L}_\neg$ are equally expressive *assuming that the propositional atoms are bivalent*. This follows from the two properties of the following theorem.

Theorem 6.1

1. *For any $\mathcal{L}_{\sim}$-sentence A there is a semantically equivalent (with respect to* CN-*valuations) $\mathcal{L}_{\sim}$-sentence B where $\sim$ operates only on the propositional atoms.*
2. *For any $\mathcal{L}_{\neg}$-sentence A there is a semantically equivalent (with respect to* NBC-*valuations) sentence $\mathcal{L}_{\neg}$-sentence B where $\neg$ operates only on the propositional atoms.*

Proof (1) As the pairs $(\sim(A \rightarrow B), A \rightarrow \sim B)$, $(\sim(A \wedge B), \sim A \vee \sim B)$, $(\sim(A \vee B), \sim A \wedge \sim B)$, and $(\sim\sim A, A)$ are all semantically equivalent (with respect to CN-valuations), we can see that any sentence A can be transformed into a semantically equivalent sentence B where $\sim$ governs the propositional atoms only.

(2) Define recursively the following three functions from sentences of $\mathcal{L}_{\neg}$ to sentences of $\mathcal{L}_{\neg}$.

1. $F(p) = \neg p$; $T(p) = p$; $LTV(p) = \bot$.
2. $F(A \wedge B) = F(A) \vee F(B)$; $T(A \wedge B) = T(A) \wedge T(B)$;
   $LTV(A \wedge B) = (LTV(A) \wedge (LTV(B) \vee T(B)))$
   $\vee((LTV(A) \vee T(A)) \wedge LTV(B))$.
3. $F(A \vee B) = F(A) \wedge F(B)$; $T(A \vee B) = T(A) \vee T(B)$;
   $LTV(A \vee B) = (LTV(A) \wedge (F(B) \vee LTV(B)))$
   $\vee((F(A) \vee LTV(A)) \wedge LTV(B)$.
4. $F(A \rightarrow B) = (T(A) \vee LTV(A)) \wedge F(B)$;
   $T(A \rightarrow B) = (T(A) \vee LTV(A)) \wedge T(B)$;
   $LTV(A \rightarrow B) = F(A) \vee LTV(B)$.
5. $F(\neg A) = T(A) \vee LTV(A)$; $T(\neg A) = F(A)$; $LTV(\neg A) = \bot$.

An inspection of the clauses shows that

(i) $F(A)$ is true if and only if A is false, and $F(A)$ is false if and only if A is not false,
(ii) $T(A)$ is true if and only if A is true, and $T(A)$ is false if and only if A is not true,
(iii) $LTV(A)$ is true if and only if A lacks truth value, and $LTV(A)$ is false if and only if A does not lack truth value.

Note also that for any sentence A every occurence of $\neg$ in $F(A)$, $T(A)$ and $LTV(A)$ operates only on a propositional atom. Note finally that A is semantically equivalent (with respect to the class of NBC-valuations) to $(T(A) \vee F(A)) \rightarrow T(A)$. Thus every sentence A is semantically equivalent (with respect to the class of NBC-valuations) to a sentence where $\neg$ only operates on the propositional atoms. □

As $\sim$ and $\neg$ are semantically equivalent when operating on the propositional atoms any sentence of $\mathcal{L}_{\sim}$ can be expressed by a sentence of $\mathcal{L}_{\neg}$ and vice versa. Thus a language containing one kind of negation can express the other kind of negation (classical negation can by a recursive procedure be defined in terms of conditional negation, and vice versa).

There is a direct way of expressing a form of negation that behaves precisely like classical negation in the logic of suppositional reasoning. Define

$$\ominus A =_{\text{def}} A \rightarrow \bot\ .$$

This gives us the following truth table:

A	$\ominus A$
T	F
F	–
–	F

Note that $\ominus A$ differs from $\neg A$ in that when A is false $\ominus A$ lacks truth value, while $\neg A$ is true. However, the distinctive inference rules of classical negation (1–3 below) are derivable for $\ominus$.

Theorem 6.2 *If $\vdash$ is a standard inference relation embodying conditional negation, then*

1. *if* $\Gamma, A \vdash \bot$*, then* $\Gamma \vdash \ominus A$*,*
2. $A, \ominus A \vdash B$*,*
3. $\vdash A \vee \ominus A$*,*
4. $\ominus A \vdash \sim A$*.*

Proof (1) If $\Gamma, A \vdash \bot$, then $\Gamma \vdash A \to \bot$; that is, $\Gamma \vdash \ominus A$. (2) As $\ominus A = A \to \bot$ it follows that $A, \ominus A \vdash \bot$ and so $A, \ominus A \vdash B$. (3) This is just Lemma 5.8. (4) We have $\vdash A \vee \sim A$. As $\ominus A, \sim A \vdash \sim A$ we only need to show that $\ominus A, A \vdash \sim A$, that is, that $A \to \bot, A \vdash \sim A$. But $A \to \bot, A \vdash \bot$, so $A \to \bot, A \vdash \sim A$. □

Clearly, $\ominus$ behaves like $\neg$ in the logic of suppositional reasoning (and is, in addition, logically stronger than $\sim$).

7 Conclusion

The logic of conditional negation is orthogonal to both intuitionistic and classical logic: the latter logics validate inferences that the logic of conditional negation does not, and the logic of conditional negation validates inferences that are neither intuitionistically nor classically valid. Still, the logic of conditional negation presented here has a "classical feel" (validating all the standard rules for $\to$, $\wedge$, and $\vee$, as well as LEM and the rule of double-negation and the de Morgan equivalences) and its distinctive rule of inference carries an obvious linguistic plausibility.

For all this it should be noted that many classically valid derived inference rules are not valid in the logic of conditional negation. For instance, we don't have unrestricted contraposition (consider the case where A is true and B lacks truth value):

$$A \to B \not\models_{\mathrm{CN}} \sim B \to \sim A.$$

Nor do we have the unrestricted disjunctive syllogism (consider the case where A is false and B lacks truth value):

$$A \vee B, \sim B \not\models_{\mathrm{CN}} A.$$

Of course, restricted versions of these inference rules still remain valid (in the above cases, introduce the restriction that B does not contain a conditional).

Note

1. The following conventions are used: $A \vdash B$ is shorthand for $\{A\} \vdash B$, and $\Gamma, A \vdash B$ is shorthand for $\Gamma \cup \{A\} \vdash B$.

References

[1] Adams, E. W., *The Logic of Conditionals. An Application of Probability to Deductive Logic*, Synthese Library, D. Reidel Publishing Co., Dordrecht, 1975. Zbl 0324.02002. MR 0485189. 250

[2] Belnap, N. D., Jr., "Conditional assertion and restricted quantification," *Noûs*, vol. 4 (1970), pp. 1–13. Zbl 0278.02010. MR 0414327. 246

[3] Bradley, R., "A representation theorem for a decision theory with conditionals," *Synthese*, vol. 116 (1998), pp. 187–229. Zbl 0963.03047. MR 1672703. 250

[4] Bradley, R., "Indicative conditionals," *Erkenntnis*, vol. 56 (2002), pp. 345–78. Zbl 1011.03014. MR 1920096. 246

[5] Cantwell, J., "The Ramsey tests derived," manuscript, 2007. 246, 250

[6] Cantwell, J., "The laws of non-bivalent probability," *Logic and Logical Philosophy*, vol. 15 (2006), pp. 163–71. Zbl 05230348. MR 2244823. 250

[7] Cantwell, J., "Indicative conditionals: The logic of assertion," *Hommage à Wlodek*, (2007). http://www.fil.lu.se/HommageaWlodek/. 246

[8] Cantwell, J., "Indicative conditionals: Factual or epistemic?" *Studia Logica*, vol. 88 (2008), pp. 157–94. 246, 249, 250

[9] Edgington, D., "On conditionals," *Mind*, vol. 104 (1995), pp. 235–329. 250

[10] Levi, I., *For the Sake of the Argument*, Cambridge University Press, Cambridge, 1996. 249

[11] Lewis, D., "Probabilities of conditionals and conditional probabilities," *Philosophical Review*, vol. 85 (1976), pp. 297–315. 251

[12] McDermott, M., "On the truth conditions of certain 'if'-sentences," *Philosophical Review*, vol. 105 (1996), pp. 1–37. 246

[13] McGee, V., "Conditional probabilities and compounds of conditionals," *Philosophical Review*, vol. 98 (1989), pp. 485–542. 246

[14] Quine, W. V. O., *Methods of Logic*, Henry Holt & Company, New York, 1950. Zbl 0038.14811. MR 0037255. 246

Department of Philosophy
Royal Institute of Technology
Teknikringen 78B
100 44 Stockholm
SWEDEN
cantwell@infra.kth.se