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DECISION FOR K4

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It was asked in [1] whether K4 contained K5. We show that it does, and give a decision procedure for the system, which has the third degree of completeness. To this end we establish a system SR which turns out to be an alternative version of K4. As a basis we take propositional calculus, **PC**, with substitution and C-detachment, and the axioms:

1. $RCpRp$
2. $CRNpNRp$
3. $CNRpRNp$
4. $CRCpqCRpRq$

with the rule to infer $R\alpha$ from α ($\mathcal{R}$).
Having **PC**, 2-4, we obviously have the meta-rule:
To infer $\phi\beta$ from $E\alpha\beta$ and $\phi\alpha$ (**EXT**).

5.	$ENRpRNp$	[2, 3
6.	$CRpRRp$	[4 q/Rp, 1
7.	$CRRpRp$	[6 p/Np, 5, **EXT**, **PC**
8.	$ERpRRp$	[6, 7
9.	$CNRCpqNCRpRq$	
	Dem. (1) $CNRCpqRNCpq$	[**PC**, 5
	(2) $CRNCpqRp$	[**PC**, $\mathcal{R}$, 4
	(3) $CRNCpqRNq$	[**PC**, $\mathcal{R}$, 4
	(4) $CRNCpqNRq$	[(3), 5
	(5) $CRNCpqNCRpRq$	[(2), (4)
	Prop.	[(1), (5)
10.	$ERCpqCRpRq$	[4, 9

With 5, 8, 10 and **EXT** we can reduce every expression to an inferentially equivalent set of forms

$$\text{(I)}\ C\alpha_1, \ldots\ldots C\alpha_n\beta$$

with each α_i an elementary variable or such negated, or either of those preceded by R, and β a variable not appearing as a component in any α_i.

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Forms (I) are provable if there are antecedents π and $N\pi$ or $R\pi$ and $RN\pi$. Otherwise they are inferentially equivalent to one of $CRpCNpq$, $CRpCpq$. If the latter was provable SR would be inconsistent; if the former was provable SR would be two-valued. The following theorem shows that neither is provable.

Definition. $W\alpha$: α is reducible to a substitution in a tautology by finite replacements of

$$\begin{array}{lll} RC\beta\gamma & \text{by} & CR\beta R\gamma \\ RN\beta & \text{by} & NR\beta \\ RR\beta & \text{by} & R\beta \end{array}$$

Theorem. *All theses have the property W.*

Proof. All tautologies and 1-4 have W, and W is hereditary under the rules.

Now $CRpCNpq$ and $CRpCpq$ do not have W, for they are not substitutions in tautologies and the replacements are inapplicable. Thus we see that W is a defining property of theses. In future proofs we shall often simply state the proposition to be proved and the tautology in which its reduction is a substitution.

Def. L $L\alpha = K\alpha R\alpha$

Def. M $M\alpha = A\alpha R\alpha$

11. $CLpp$	[$CKpqp$, q/Rp, Def. L
12. $CLCpqCLpLq$	[$CKCpqCrsCKprKqs$, r/Rp, s/Rq
13. $CLpLLp$	[$CKpqKKpqKqq$, q/Rp
14. $CpCMLpLp$	[$CpCAKpqKqqKpq$, q/Rp
15. $CLMpMLp$	[$CKApqAqqAKpqKqq$, q/Rp
16. *From α we can infer $K\alpha R\alpha$ by* PC *and* $\mathcal{R}$, *and so $L\alpha$*	[by Def. L.
17. $ALpALCpqLCpNq$	[$AKprAKCpqCrsKCpNqCrNs$, r/Rp, s/Rq

With 11-16 we have K4, and with 11-17 we have K5. But if we define $R\alpha$ as $LM\alpha$ in K4, then 1-4 and $\mathcal{R}$ are provable, as are $ELpKpRp$, $EMpApRp$. Thus K4 contains K5 and SR $\longleftrightarrow$ K4.

REFERENCE

[1] Sobociński, B.: Family $\mathcal{K}$ of the non-Lewis modal systems. *Notre Dame Journal of Formal Logic*, vol. 5 (1964), pp. 313-318.

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