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PARALLEL 1-FLATS IN 2-ARRANGEMENTS

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The terminology and numbering of propositions in [1] will be followed throughout this paper.

Suppose a topological space X with geometry G forms a 2-arrangement. The purpose of this paper is to answer the following questions:

I. If f is any 1-flat of X and y is any element of X, is there necessarily some 1-flat g which contains y and is parallel to f, that is, such that $g = f$, or $g \cap f = \emptyset$?

II. If the answer to I is affirmative, are there any "distinguished" 1-flats which contain y and are parallel to f?

Lemma 1. *The answer to* I *is affirmative if and only if whenever* $y \not\varepsilon f$, $X \neq \bigcup\{x \mid x \,\varepsilon\, f_1(w, y),\ w \,\varepsilon\, f\}$.

Proof: If $y \not\varepsilon f$ and g is any 1-flat parallel to f which contains y, then since $f \cap g = \emptyset$, any point of $g - \{y\}$ is not contained in $\bigcup\{x \mid x \,\varepsilon\, f_1(w, y),\ w \,\varepsilon\, f\}$. On the other hand, if $X \neq \bigcup\{x \mid x \,\varepsilon\, f_1(w, y),\ w \,\varepsilon\, f\}$, then choose $z \,\varepsilon\, X - \bigcup\{x \mid x \,\varepsilon\, f_1(w, y),\ w \,\varepsilon\, f\}$. Then $f_1(y, z)$ is a 1-flat which contains y and is parallel to f.

The discussion which follows concerns the following situation: X and G form a 2-arrangement; $y_0 \,\varepsilon\, \mathrm{Int}\, X$ and f is a 1-flat which does not contain y_0.

Let w_0 be a cut point of f. We can totally order f by $\leq$(**2.26**). Set $U = \{u \,\varepsilon\, f \mid w_0 \leq u\}$ and $V = \{v \,\varepsilon\, f \mid v \leq w_0\}$. Since $y_0 \,\varepsilon\, \mathrm{Int}\, X$, $y_0 \,\varepsilon\, \mathrm{Int}\, C(S)$ where $C(S)$ is a 2-simplex (**4.10.1** and

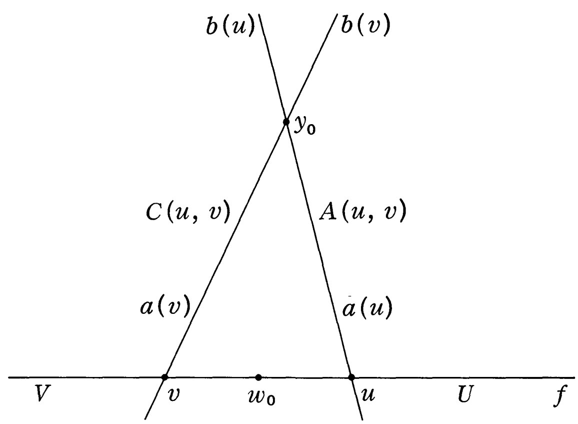

Figure 1

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4.6). For each $u \varepsilon U$ and $v \varepsilon V$, y_0 disconnects $f_1(y_0, u)$ and $f_1(y_0, v)$ each into two components (Fig. 1). That component of $f_1(y_0, u) - \{y_0\}$ which contains u will be denoted by $a(u)$. The analogous components of $f_1(y_0, v) - \{y_0\}$ will be denoted by $a(v)$ and $b(v)$ (*cf.* **2.22** and **2.23**).

Since $y_0 \varepsilon \operatorname{Int} C(S)$, if $u \neq v$, then $f_1(y_0, u) \cup f_1(y_0, v)$ disconnects X into four convex open components $A(u, v)$, $B(u, v)$, $C(u, v)$, and $D(u, v)$, where $\operatorname{Fr} A(u, v) = a(u) \cup b(v) \cup \{y_0\}$, $\operatorname{Fr} B(u, v) = a(u) \cup b(v) \cup \{y_0\}$, $\operatorname{Fr} C(u, v) = a(u) \cup a(v) \cup \{y_0\}$, and $\operatorname{Fr} D(u, v) = a(v) \cup a(u) \cup \{y_0\}$. This follows from **4.12**, **3.25**, and the following lemma.

Lemma 2. *If h is any 1-flat which disconnects X into components M and N, then* $\operatorname{Fr} M = \operatorname{Fr} N = h$.

Proof: If $x \varepsilon h$ and W is any neighborhood of x, then if W does not intersect both M and N, then $h - \{x\}$ still disconnects X. But h is a minimal disconnecting subset of X(**2.12**). We continue our discussion with the following lemmas.

Lemma 3. $dC(S)$ *is compact and closed.*

Proof: If $S = \{x_0, x_1, x_2\}$, then $dC(S) = \overline{x_0x_1} \cup \overline{x_1x_2} \cup \overline{x_2x_0}$. Each segment is compact (**2.29**) and closed; hence $dC(S)$ is compact and closed.

Lemma 4. *If $u' > u$, then* $\operatorname{Cl} A(u', v)$ *is properly contained in* $\operatorname{Cl} A(u, v)$.

Proof: The 1-flat $f_1(y_0, u)$ disconnects X into components $M(u)$, which contains v, and $N(u)$. The analogous components of $X - f_1(y_0, v)$ and $X - f_1(y_0, u')$ will be $M(v)$ and $N(v)$, and $M(u')$ and $N(u')$, respectively (Fig. 2). Then $\operatorname{Cl} A(u, v) = (M(v) \cap N(u)) \cup a(u) \cup b(v) \cup \{y_0\}$ and $\operatorname{Cl} A(u', v) = (M(v) \cap N(u')) \cup a(u') \cup b(v) \cup \{y_0\}$. A simple argument shows that $a(u') \subset M(v)$; hence $\operatorname{Cl} A(u', v) \subset \operatorname{Cl} A(u, v)$.

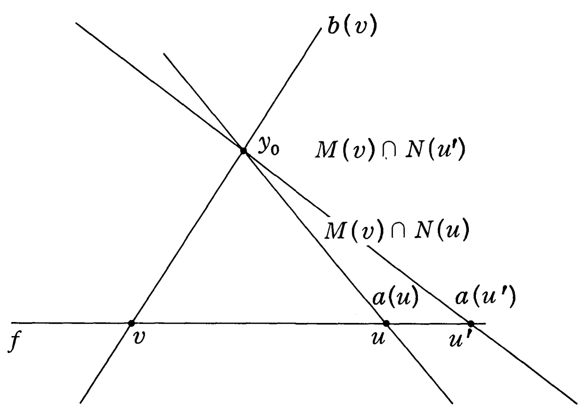


Figure 2

Since $u \varepsilon \operatorname{Cl} A(u, v) - \operatorname{Cl} A(u', v)$, the containment is proper. Similarly, if $v > v'$, then $\operatorname{Cl} A(u, v')$ is properly contained in $\operatorname{Cl} A(u, v)$; moreover, corresponding statements can be proved in like manner about $\operatorname{Cl} C(u, v)$. We therefore have:

Lemma 5. *If $u \leq u'$ and $v \geq v'$, then* $\operatorname{Cl} A(u', v') \subset \operatorname{Cl} A(u, v)$ *and* $\operatorname{Cl} C(u', v') \subset \operatorname{Cl} C(u, v)$. *If one of the first inequalities is strict, then the containment in both instances is proper.*

Lemma 6. *If* $u \varepsilon U$ *and* $v \varepsilon V$, *then* $\mathrm{Cl}\, A(u, v) \cap \mathrm{d}C(S)$ *and* $\mathrm{Cl}\, C(u, v) \cap \mathrm{d}C(S)$ *are both non-empty.*

Proof: Both $b(v) \cap \mathrm{d}C(S)$ and $a(u) \cap \mathrm{d}C(S)$ are non-empty (**3.24.1** and a straightforward argument).

Partially order $U \times V$ by $\leq'$ where $(u, v) \leq' (u', v')$ if $u \leq u'$ and $v \geq v'$. Take a maximal chain W in $U \times V$. Then since $\mathrm{d}C(S)$ is compact, using Lemmas 5 and 6 we have $\bigcap_W \mathrm{Cl}\, A(u, v) \cap \mathrm{d}C(S)$ and $\bigcap_W \mathrm{Cl}\, C(u, v) \cap \mathrm{d}C(S)$ are both non-empty. Choose $z \varepsilon \bigcap_W \mathrm{Cl}\, A(u, v) \cap \mathrm{d}C(S)$ and $z' \varepsilon \bigcap_W \mathrm{Cl}\, C(u, v) \cap \mathrm{d}C(S)$. Now if $X = \bigcup \{x \mid x \varepsilon f_1(y_0, w), w \varepsilon f\}$, then $f_1(y_0, z) \cap f$ and $f_1(y_0, z') \cap f$ must each consist of a single point e and e', respectively. We will see later that we can have $e = e'$. If $e \neq e'$, then e and e' must both be end points of f, or else we could get a contradiction to the maximality of W. Assume $e \neq e'$, but $X = \bigcup \{x \mid x \varepsilon f_1(y_0, w), w \varepsilon f\}$. Then $f = \overline{ee'}$; hence $X = \bigcup \{x \mid x \varepsilon f_1(y_0, w), w \varepsilon \overline{ee'}\}$. Now y_0 is a cut point of each $f_1(y_0, w)$. Choose $p \varepsilon b(e)$ (Fig. 3). Then it follows that $f_1(p, e') \cap f_1(p, e)$ must consist of at least p and y_0, a contradiction. Consequently, if $e \neq e'$. then $X \neq \bigcup \{x \mid x \varepsilon f_1(y_0, w), w \varepsilon \overline{ee'}\}$. We have therefore established:

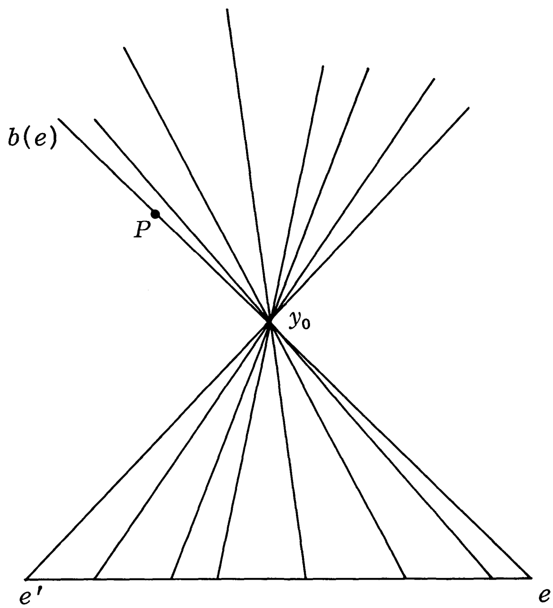


Figure 3

Theorem 1. *If* $e \neq e'$, *then* $X \neq \bigcup \{x \mid x \varepsilon f_1(y_0, w), w \varepsilon f\}$. *Consequently, there is some* 1-*flat* g *which contains* y_0 *and is parallel to* f.

Relative to Question I posed at the beginning of this paper, we may say: If f is any 1-flat of X and y is any element of $\mathrm{Int}\, X$, then if f has either two end points or no end points, then there is a 1-flat g which contains y and is parallel to f.

Since no 1-flat in an open 2-arrangement can have any end points, we also have:

Corollary. *If* X *and* G *form an open* 2-*arrangement, then for any* 1-*flat* f *and* $y \varepsilon x$, *there is a* 1-*flat* g *which contains* y *and is parallel to* f.

If $y \varepsilon \mathrm{Bd}\, X$ and f is any 1-flat in $X - \{y\}$, there may not be any 1-flat which contains y and is parallel to f. For example, if $S = \{x_0, x_1, x_2\}$ is a linearly independent subset of X, then $C(S)$ with geometry $G_{C(S)}$ forms a 2-arrangement. There is no 1-flat which contains x_0 which is parallel to any other 1-flat of $G_{C(S)}$.

We now continue the discussion which led to Theorem 1 and its Corollary. In particular we will examine the non-empty set $\bigcap_W \mathrm{Cl}\, A(u, v) \cap \mathrm{d}C(S)$; analogous results will hold for $\bigcap_W \mathrm{Cl}\, C(u, v) \cap \mathrm{d}C(S)$. Recall that $y_0 \varepsilon \operatorname{Int} C(S)$, where $C(S)$ is a 2-simplex. If $S = \{x_0, x_1, x_2\}$, then $\bigcap_W \mathrm{Cl}\, A(u, v) \cap \mathrm{d}C(S) = \bigcap_W \mathrm{Cl}\, A(u,v) \cap (\overline{x_0x_1} \cup \overline{x_1x_2} \cup \overline{x_2x_0}) = (\bigcap_W \mathrm{Cl}\, A(u,v) \cap \overline{x_0x_1}) \cup (\bigcap_W \mathrm{Cl}\, A(u, v) \cap \overline{x_1x_2}) \cup (\bigcap_W \mathrm{Cl}\, A(u, v) \cap \overline{x_2x_0})$. Each set in this latter union is a closed convex subset of a segment.

Lemma 7. *A closed convex subset W of a segment $\overline{xy}$ is either a segment, a point, or the empty set.*

Proof: Suppose W does not consist of a single point and $W \neq \emptyset$. Totally order $\overline{xy}$ by $\leq$ with $x \leq y$. Let $u =$ 1.u.b.W and $v =$ g.1.b.W(**2.28**). Since W is closed and connected, $W = \{z \varepsilon \overline{xy} \mid v \leq z \leq u\}$; hence $W = \overline{uv}$(**2.27**).

Suppose that h is a 1-flat, $z_0 \varepsilon h$, and p is any point of $h - \{z_0\}$. Then we define $\mathrm{ray}(z_0, p)$ to be the component of $h - \{z_0\}$ which contains p together with the point z_0.

Lemma 8. *If $p \varepsilon \mathrm{Cl}\, A(u, v) - \{y_0\}$, then $\mathrm{ray}(y_0, p) \subset \mathrm{Cl}\, A(u, v)$.*

Proof: Since $f_1(y_0, p) \cap \mathrm{Cl}\, A(u, v)$ is connected (since it is the intersection of two convex sets), this intersection is contained in $\mathrm{ray}(y_0, p)$. But if t is a point of $f_1(y_0, p)$ not in this intersection, then $y_0 \varepsilon \overline{tp}$; hence t cannot be in the same component of $f_1(y_0, p) - \{y_0\}$ as p.

Lemma 9. *If $x \varepsilon \bigcap_W \mathrm{Cl}\, A(u, v) \cap \overline{x_0x_1}$ and $y \varepsilon \bigcap_W \mathrm{Cl}\, A(u, v) \cap \overline{x_1x_2}$, then either $\overline{yx_2} \cup \overline{x_2x_1} \cup \overline{x_0x}$ or $\overline{yx_1} \cup \overline{x_1x}$ is a subset of $\bigcap_W \mathrm{Cl}\, A(u, v) \cap \mathrm{d}C(S)$.*

Proof: The detailed proof is quite lengthy and involves a number of different cases. It uses Lemma 8 and is essentially contained in Figs. 4a, b, and c.

It follows then that $\bigcap_W \mathrm{Cl}\, A(u, v) \cap \mathrm{d}C(S)$ is the union of at most three segments S_1, S_2, and S_3 which form a simple (non-closed) polygonal path joining two points a and a' of $\mathrm{d}C(S)$. Moreover, we may suppose that $f_1(y_0, a)$ is the limiting posi-

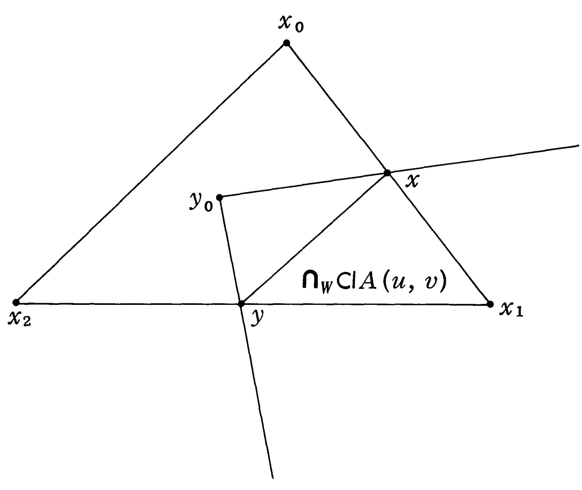


Figure 4a.

tion of the $f_1(y_0, u)$ and $f_1(y_0, a')$ is the limiting position of the $f_1(y_0, v)$ (Fig. 5). (It may be, of course, that $a = a'$.) It would, in fact, be easy to show that if F^1 of G is given Topology II as described in [2], then the nets $\{f_1(y_0, u)\}, u \varepsilon U$, and $\{f_1(y_0, v)\}$, $v \varepsilon V$, converge to $f_1(y_0, a)$ and $f_1(y_0, a')$, respectively.

If $f_1(y_0, a)$ is parallel to f, then we call $f_1(y_0, a)$ the *upper parallel* to f through y_0; if $f_1(y_0, a')$ is parallel to f, it will be called the *lower parallel* to f through y_0. Straightforward arguments show that the same flats $f_1(y_0, a)$ and $f_1(y_0, a')$ are obtained regardless of which cut point w_0 of f and which 2-simplex $C(S)$ is used, that is, $f_1(y_0, a)$ and $f_1(y_0, a')$ are independent of w_0 and $C(S)$. If $f_1(y_0, a)$ is not parallel to f, then $f_1(y_0, a) \cap f$ is an end point of f; a similar conclusion applies to $f_1(y_0, a')$. Thus, we can say:

Theorem 2. *If X and G form a 2-arrangement, $y \varepsilon \operatorname{Int} X$ and f is a 1-flat of X with two end points, then there is neither an upper or lower parallel to f through y. If X and G form an open 2-arrangement, f is a 1-flat*

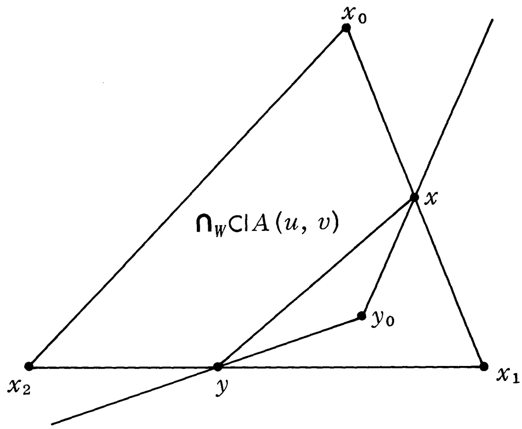
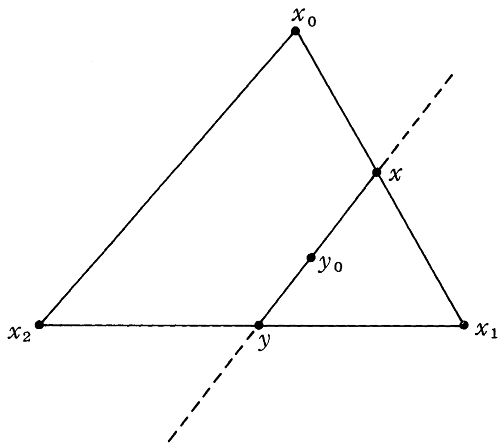


Figure 4b



$\cap_W \mathrm{Cl} A(u, v)$ on one side of $f_1(x, y)$

Figure 4c

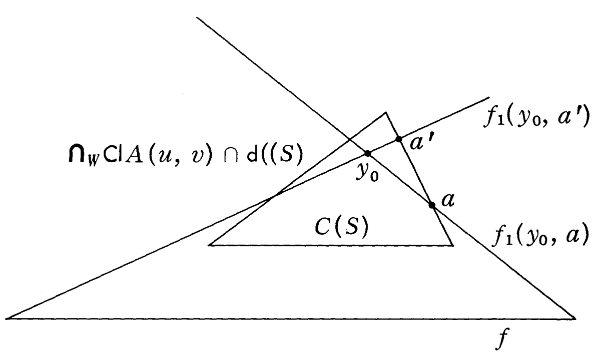


Figure 5

of X, and $y \notin f$, then f has both an upper and lower parallel through y (though these parallels might be the same 1-flat.

The following example shows that it is possible to have $e = e'$ (cf. the discussion preceding Theorem 1); thus we may have $X = \bigcup \{x \mid x \varepsilon f_1(y_0, w), w \varepsilon f\}$. We can thus conclude that a 2-arrangement need not satisfy either a hyperbolic or euclidean parallel postulate.

Example: Let P be a point not in R^2, the usual coordinate plane. Let $X = R^2 \cup \{P\}$. As a subbasis for a topology on X we take the open sets of R^2 and all sets of the form $\{(x, y) \mid a < x\} \cup \{P\}$, where a is a real number. Suppose z and z' are points of X. We define $f_1(z, z')$ as follows: If $z, z' \varepsilon R^2$, we let $f_1(z, z')$ be the usual line in R^2 if this line is not parallel to the x-axis, and this usual line together with P if that line is parallel to the x-axis. If $z = P$ and $z' \varepsilon R^2$, we let $f_1(z, z')$ be the line in R^2 which contains z' and is parallel to the x-axis together with P. Then $X = \bigcup \{z \mid z \varepsilon f_1((0, 1), w), w \varepsilon f_1((0,0), P)\}$ even though the structure defined is a 2-arrangement.

REFERENCES

[1] Gemignani, M., "Topological geometries and a new characterization of R^m," *The Notre Dame Journal of Formal Logic*, vol. VII (1966), pp. 57-100.

[2] Gemignani, M., "On topologies for F^i," *Fundamenta Mathematicae*, vol. LIX (1966), pp. 153-157.

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