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A NEW FORMALIZATION OF NEWMAN ALGEBRA

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In [6][1] M. H. A. Newman constructed and investigated an algebraic system whose two basic binary operations are + and ×,[2] and which, as he has proved, is a direct join of a non-associative Boolean ring with unity element and a Boolean lattice, *i.e.* a Boolean algebra. In [7], p. 28, Newman calls this system a *complemented mixed algebra*, but in Birkhoff's [2] and [3], p. 48, it is called Newman algebra. The latter name will be used throughout this paper. Besides the property which is mentioned above, in [6] it has been proved that for all elements of the carrier set of any Newman algebra the additive operation + is commutative and associative, but not necessarily idempotent or nilpotent, and that the multiplicative operation × is idempotent and commutative, but not necessarily associative.

The main aim of this paper is to show that Newman algebra can be formalized as an equational system. For this end in section **1** below two definitions, (**A**) and (**B**), of two systems, $\mathfrak{A}$ and $\mathfrak{B}$ respectively, of the Newman algebras are given, and in section **2** it will be proved that these systems are inferentially equivalent, if their respective carrier sets A and B are the same, *i.e.* $A = B$, or these systems are inferentially equivalent up to isomorphism, if their carrier sets have only the same cardinality, *i.e.* $\mathrm{card}(A) = \mathrm{card}(B)$. Since definition (**A**) of $\mathfrak{A}$ is an obviously equivalent modification of a formalization of Newman algebra given in [1], p. 4, [2], p. 155, and [3], p. 49, and since (**B**) defines $\mathfrak{B}$ as an equational system, our claim will be justified. In section **3** it will be proved that in the field of $\mathfrak{B}$ the set of its proper algebraic postulates is inferentially equivalent to another set containing a very small number of axioms. Finally, in section **4** the mutual independence of the axioms belonging to the sets mentioned above will be established.

1. An acquaintance with the papers [6], [7] and one of [1], [2] or [3] is presupposed. *Cf.* also [8] and [4].

2. In the papers mentioned in note 1 "ab" is used instead of "$a \times b$."

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It should be remarked that throughout this paper the interesting and important properties of "even" and "odd" (in Birkhoff's terminology called "odd" and "even" respectively) elements belonging to the carrier set of the given Newman algebra will not be discussed, and that the axioms *A1-A9, B1, B2, A10* and *A11*, see section **1**, will be used mostly tacitly in the deductions which will be presented below.

1 We define the systems $\mathfrak{A}$ and $\mathfrak{B}$ as follows:

(**A**) *Any algebraic system*

$$\mathfrak{A} = \langle A, =, +, \times, 1, 0\rangle$$

with one binary relation =, *two binary operations* + *and* ×, *and two constant elements*, 1 *and* 0, *is a Newman algebra, if it satisfies the following postulates*

$A1$ $[a]:a\epsilon A. \supset . a = a$
$A2$ $[ab]: a,b\epsilon A . a = b . \supset . b = a$
$A3$ $[abc]: a,b,c\epsilon A . a = b . b = c . \supset . a = c$
$A4$ $[ab]: a,b\epsilon A . \supset . a + b\epsilon A$
$A5$ $[ab]: a,b\epsilon A. \supset . a\times b\epsilon A$
$A6$ $[abc]: a,b,c\epsilon A . a = c . \supset . a + b = c + b$
$A7$ $[abc]: a,b,c\epsilon A . b = c . \supset . a + b = a + c$
$A8$ $[abc]: a,b,c\epsilon A . a = c . \supset . a\times b = c\times b$
$A9$ $[abc]: a,b,c\epsilon A . b = c . \supset . a\times b = a\times c$

$B1$ $1\epsilon A$
$B2$ $0\epsilon A$

$C1$ $[abc]: a,b,c\epsilon A . \supset . a\times (b + c) = (a\times b) + (a\times c)$
$C2$ $[abc]: a,b,c\epsilon A . \supset . (a + b)\times c = (a\times c) + (b\times c)$
$C3$ $[a]: a\epsilon A . \supset . a = a\times 1$
$C4$ $[a]: a\epsilon A . \supset . a = a + 0$
$C5$ $[a]: a\epsilon A . \supset . [\exists b] . b\epsilon A . a + b = 1 . a\times b = 0$

and

(**B**) *Any algebraic system*

$$\mathfrak{B} = \langle B, =, +, \times, -\rangle$$

with one binary relation =, *with binary operations* + *and* ×, *and one unary operation* -, *is a Newman algebra, if it satisfies the postulates A1-A9, C1 and C2 given in* (**A**), *but adjusted to the carrier set B of* $\mathfrak{B}$, *and, additionally, the following axioms*

$A10$ $[a]: a\epsilon B . \supset . \overline{a}\epsilon B$
$A11$ $[\exists a]: a\epsilon B$

$F1$ $[ab]: a,b\epsilon B . \supset . a = a + (b\times \overline{b})$
$F2$ $[ab]: a,b\epsilon B . \supset . a = a\times (b + \overline{b})$
$F3$ $[ab]: a,b\epsilon B . \supset . a = (b + \overline{b})\times a$

It is self-evident that system $\mathfrak{A}$, as defined in (**A**), is an equivalent reformularization of Birkhoff's formalization of Newman algebra, *cf*, e.g., [3], p. 48. The main differences existing between these two formalizations are the following:

1) The logical properties of the relation = are axiomatically assumed in $\mathfrak{A}$.
2) Formula *C7*, see section **2** below, which is a consequence of the axioms *A1*-*C5*, as it has been proved in [7], is not a postulate of $\mathfrak{A}$, while for some heuristic purposes it is assumed axiomatically in [1], [2], pp. 158-159, note 9, Ex. 7, and [3], p. 52, Ex. 9.
3) The constant elements 1 and 0 are assumed axiomatically in $\mathfrak{A}$, while Birkhoff introduces them by the way of the particular quantifiers with a proviso of the uniqueness of each of them.

Concerning system $\mathfrak{B}$ it should be noticed that the unary operation- is a kind of complementation peculiar to Newman algebra. Moreover, that in section **3** it will be proved that in its field the axioms $\{C1$; $C2$; $F1$; $F2$; $F3\}$ are inferentially equivalent to *F1*, *F2* and

$$G1 \quad [abcd]:a,b,c,d \in B\,.\supset.\,(a \times (b + c)) \times d = ((c \times a) \times d) + ((b \times a) \times d)$$

2 Theorem. *There is an equational formalization of Newman algebra.*

Proof: In the deductions which follow the proofs of several theorems will be omitted, since they are in the literature. In each such case a suitable reference will be given.

2.1 Let us assume system $\mathfrak{A}$. Hence, we have at our disposal axioms *A1*-*C5* and, moreover:

$C6 \quad [a]:a \in A\,.\supset.\,a = a \times a$
$C7 \quad [a]:a \in A\,.\supset.\,a = 0 + a$
$C8 \quad [a]:a \in A\,.\supset.\,a = 1 \times a$

For a proof of *C6* see [3], p. 49. Using methods analogous to those in [7] we easily obtain a proof of *C7* in the field of $\mathfrak{A}$.[3] Moreover, see the proof of *F15* in section **2.2** below. *C8* follows from *C1*-*C7* at once. Then:

$C9 \quad [ab]:a,b \in A\,.\,a + b = 1\,.\,a \times b = 0\,.\supset.\,b \times a = 0$
PR $\quad [ab]:\mathrm{Hp}(3)\,.\supset.$
$\quad [\exists d].$

$$\left.\begin{array}{ll} 4. & d \in A\,. \\ 5. & b + d = 1. \\ 6. & b \times d = 0. \end{array}\right\} \qquad [1; C5]$$

3. The main difference existing between the formalizations of Newman algebra given in [6] and in Birkhoff's papers is that instead of *C5* Newman accepts

$C5^* \quad [a]:\, a \in A\,.\supset.\,[\exists b]\cdot b \in A \cdot b + a = 1 \cdot b \times a = 0$

Consequently, instead of *C3* and *C4* he has to have *C8* and *C7* as the axioms.

7. $d = 1 \times d = (a + b) \times d = (a \times d) + (b \times d) = (a \times d) + 0$ [1; 4; *C8*; 2; *C2*; 6]
$= a \times d = 0 + (a \times d) = (a \times b) + (a \times d) = a \times (b + d)$ [*C4*; *C7*; 3; *C1*]
$= a \times 1 = a$ [5; *C3*]
$b \times a = 0$ [6; 7]

C10 $[abc]: a, b, c \epsilon A . a + b = 1 . a \times b = 0 . a + c = 1 . a \times c = 0 . \supset . b = c$

PR $[abc]: \mathrm{Hp}(5) . \supset .$
$b = b \times 1 = b \times (a + c) = (b \times a) + (b \times c) = 0 + (b \times c)$ [1; *C3*; 4; *C1*; *C9*; 2; 3]
$= (a \times c) + (b \times c) = (a + b) \times c = 1 \times c = c$ [5; *C2*; 2; *C8*]

Therefore, having *C5* and *C10* we can introduce into system a definition of complementation:

DI $[ab]: a, b \epsilon A . \supset . b = \overline{a} . \equiv . a + b = 1 . a \times b = 0$ [*C5*; *C10*]
C11 $[a]: a \epsilon A . \supset . \overline{a} \epsilon A$ [*C5*; *DI*]
C12 $[a]: a \epsilon A . \supset . 1 = a + \overline{a}$ [*C11*; *A1*; *DI*]
C13 $[a]: a \epsilon A . \supset . 0 = a \times \overline{a}$ [*C11*; *A1*; *DI*]
C14 $[ab]: a, b \epsilon A . \supset . a = a \times (b + \overline{b})$ [*C3*; *C12*]
C15 $[ab]: a, b \epsilon A . \supset . a = a + (b \times \overline{b})$ [*C4*; *C13*]
C16 $[ab]: a, b \epsilon A . \supset . a = (b + \overline{b}) \times a$ [*C8*; *C12*]
C17 $[a]: a \epsilon A . \supset . \overline{a} \times a = 0$ [*C9*; *C11*; *C12*; *C13*]
C18 $[ab]: a, b \epsilon A . a = b . \supset . \overline{a} = \overline{b}$

PR $[ab]: \mathrm{Hp}(2) . \supset .$
$\overline{a} = \overline{a} \times (b + \overline{b}) = (\overline{a} \times b) + (\overline{a} \times \overline{b}) = (\overline{a} \times a) + (\overline{a} \times \overline{b})$ [1; *C14*; *C11*; *C1*; 2]
$= 0 + (\overline{a} \times \overline{b}) = (b \times \overline{b}) + (\overline{a} \times \overline{b}) = (a \times \overline{b}) + (\overline{a} \times \overline{b})$ [*C17*; *C13*; 2]
$= (a + \overline{a}) \times \overline{b} = \overline{b}$ [*C2*; *C16*]

C19 $[\exists a] . a \epsilon A$ [*B1*]

2.2 Now, let us assume system $\mathfrak{B}$. Hence, we have at our disposal the axioms *A1-A9, C1* and *C2* adjusted to the carrier set *B* of $\mathfrak{B}$, and, moreover, the axioms *A10, A11, F1, F2* and *F3*. Then:

F4 $[ab]: a, b \epsilon B . \supset . a + \overline{a} = b + \overline{b}$ [*F2*; *F3*]

Therefore, having *A10* and *F4* we can introduce into system a definition of the constant element 1.

D1 $[a]: a \epsilon B . \supset . 1 = a + \overline{a}$ [*A10*; *A4*; *F4*]
F5 $[a]: a \epsilon B . \supset . a = a \times 1$ [*F2*; *D1*]
F6 $[a]: a \epsilon B . \supset . a = 1 \times a$ [*F3*; *D1*]
F7 $[a]: a \epsilon B . \supset . a = a \times a$

PR $[a]: \mathrm{Hp}(1) . \supset .$
$a = a \times (a + \overline{a}) = (a \times a) + (a \times \overline{a}) = a \times a$ [1; *F2*; *C1*; *F1*]

F8 $[a]: a \epsilon B . \supset . a + 1 = 1 + a$

PR $[a]: \mathrm{Hp}(1) . \supset .$
$a + 1 = (a + 1) \times (a + \overline{a}) = ((a + 1) \times a) + ((a + 1) \times \overline{a})$ [1; *A10*; *A4*; *F2*; *C1*]
$= ((a \times a) + (1 \times a)) + ((a \times \overline{a}) + (1 \times \overline{a}))$ [*C2*]

$= (a + a) + ((a \times \overline{a}) + \overline{a}) = ((1 \times a) + (a \times a)) + ((a \times \overline{a}) + (\overline{a} \times \overline{a}))$ [*F7*; *F6*]

$= ((1 + a) \times a) + ((a + \overline{a}) \times \overline{a}) = ((1 + a) \times a) + ((1 \times \overline{a}) + (a \times \overline{a}))$ [*C2*; *A5*; *A10*; *D1*; *F1*]

$= ((1 + a) \times a) + ((1 + a) \times \overline{a}) = (1 + a) \times (a + \overline{a}) = 1 + a$ [*C2*; *C1*; *F2*]

F9 $[ab] : a, b \in B . \supset . a \times \overline{a} = (b \times \overline{b}) \times (a \times \overline{a})$

PR $[ab] : \mathrm{Hp}(1) . \supset .$

$a \times \overline{a} = (a \times \overline{a}) \times (a \times \overline{a}) = ((a \times \overline{a}) + (b \times \overline{b})) \times (a \times \overline{a})$ [1; *F7*; *F1*]

$= ((a \times \overline{a}) \times (a \times \overline{a})) + ((b \times \overline{b}) \times (a \times \overline{a}))$ [*C2*]

$= (a \times \overline{a}) + ((b \times \overline{b}) \times (a \times \overline{a})) = (1 \times (a \times \overline{a})) + ((b \times \overline{b}) \times (a \times \overline{a}))$ [*F7*; *F6*]

$= (1 + (b \times \overline{b})) \times (a \times \overline{a}) = ((b \times \overline{b}) + 1) \times (a \times \overline{a})$ [*C2*; *F8*]

$= ((b \times \overline{b}) \times (a \times \overline{a})) + (1 \times (a \times \overline{a}))$ [*C2*]

$= ((b \times \overline{b}) \times (a \times \overline{a})) + (a \times \overline{a}) = (b \times \overline{b}) \times (a \times \overline{a})$ [*F6*; *F1*]

F10 $[ab] : a, b \in B . \supset . (a \times \overline{a}) = (a \times \overline{a}) \times (b \times \overline{b})$ [Similar proof; *F7*; *F1*; *C1*; *F5*; *F8*]

F11 $[ab] : a, b \in B . \supset . a \times \overline{a} = b \times \overline{b}$ [*F9*; *F10*]

Therefore, having *A10* and *F11* we can introduce into system a definition of the constant element 0.

D2 $[a] : a \in B . \supset . 0 = a \times \overline{a}$ [*A10*; *A5*; *F11*]

F12 $[a] : a \in B . \supset . a = a + 0$ [*F1*; *D2*]

F13 $[a] : a \in B . \supset . 0 + (0 + \overline{\overline{a}}) = a$

PR $[a] : \mathrm{Hp}(1) . \supset .$

$0 + (0 + \overline{\overline{a}}) = 0 + ((\overline{a} \times \overline{\overline{a}}) + (\overline{\overline{a}} \times \overline{\overline{a}})) = 0 + ((\overline{a} + \overline{\overline{a}}) \times \overline{\overline{a}})$ [1; *D2*; *F7*; *C2*]

$= 0 + ((a + \overline{a}) \times \overline{\overline{a}}) = 0 + ((a \times \overline{\overline{a}}) + (\overline{a} \times \overline{\overline{a}}))$ [*F4*; *C2*]

$= (a \times \overline{a}) + (a \times \overline{\overline{a}}) = a \times (\overline{a} + \overline{\overline{a}}) = a$ [*D2*; *F1*; *C1*; *F2*]

F14 $[a] : a \in B . \supset . a \times 0 = 0$

PR $[a] : \mathrm{Hp}(1) . \supset .$

$a \times 0 = (0 + (0 + \overline{\overline{a}})) \times 0 = (0 \times 0) + ((0 \times 0) + (\overline{\overline{a}} \times 0))$ [*C2*]

$= 0 + (0 + (\overline{\overline{a}} \times 0) = 0 + ((\overline{\overline{a}} \times \overline{\overline{\overline{a}}}) + (\overline{\overline{a}} \times 0))$ [*F7*; *D2*]

$= 0 + (\overline{\overline{a}} \times (\overline{\overline{\overline{a}}} + 0)) = 0 + (\overline{\overline{a}} \times \overline{\overline{\overline{a}}}) = 0$ [*C1*; *F12*; *F1*]

F15 $[a] : a \in B . \supset . 0 \times a = 0$ [Similar proof; *F13*; *C1*; *F7*; *D2*; *C2*; *F12*; *F1*]

F16 $0 + \overline{\overline{1}} = 1$

PR $0 + \overline{\overline{1}} = (0 + \overline{\overline{1}}) \times (0 + \overline{\overline{1}}) = ((0 + \overline{\overline{1}}) \times 0) + ((0 \times \overline{\overline{1}}) + (\overline{\overline{1}} \times \overline{\overline{1}}))$ [*F7*; *C1*; *C2*]

$= 0 + (0 + \overline{\overline{\overline{1}}}) = 1$ [*F14*; *F15*; *F7*; *F13*]

F17 $[a] : a \in B . \supset . a = 0 + a$

PR $[a] : \mathrm{Hp}(1) . \supset .$

$a = a \times 1 = a \times (0 + (0 + \overline{\overline{1}})) = a \times (0 + 1)$ [*F5*; *F13*; *F16*]

$= (a \times 0) + (a \times 1) = 0 + a$ [*C1*; *F14*; *F5*]

The proof of *F17* is patterned after the deductions given by Newman in [7], but due to the axiom-system assumed here it is a little shorter than Newman's proof.

F18 $[a]:a\epsilon B.\supset.[\exists b].b\epsilon B.a+b=1.a\times b=0$ [*A10*; *D1*; *D2*]

F19 $[a]:a\epsilon B.\supset.a=\overline{\overline{a}}$ [*F13*; *A10*; *F17*]

F20 $[a]:a\epsilon B.\supset.0=\overline{a}\times a$ [*D2*; *A10*; *F19*]

F21 $[ab]:a,b\epsilon B.a+b=1.a\times b=0.\supset.b=\overline{a}$

PR $[ab]:\mathrm{Hp}(3).\supset.$

$b=(a+\overline{a})\times b=(a\times b)+(\overline{a}\times b)=0+(\overline{a}\times b)$ [1; *F3*; *C2*; 3]

$=(\overline{a}\times a)+(\overline{a}\times b)=\overline{a}\times(a+b)=\overline{a}\times 1=\overline{a}$ [*F20*; *C1*; 2; *F5*]

F22 $[ab]:a,b\epsilon B.\supset.b=\overline{a}.\equiv.a+b=1.a\times b=0$ [*D1*; *D2*; *F21*]

F23 $[ab]:a,b\epsilon B.a=b.\supset.\overline{a}=\overline{b}$

PR $[ab]:\mathrm{Hp}(2).\supset.$

$\overline{a}=\overline{a}\times(b+\overline{b})=(\overline{a}\times b)+(\overline{a}\times\overline{b})=(\overline{a}\times a)+(\overline{a}\times\overline{b})$ [1; *F2*; *C1*; 2]

$=0+(\overline{a}\times\overline{b})=(b\times\overline{b})+(\overline{a}\times\overline{b})=(b+\overline{a})\times\overline{b}$ [*F20*; *D2*; *C2*]

$=(a+\overline{a})\times\overline{b}=\overline{b}$ [2; *F3*]

F24 $1\epsilon B$ [*A11*; *A10*; *A4*; *D1*]

F25 $0\epsilon B$ [*A11*; *A10*; *A5*; *D2*]

2.3 An inspection of the deductions presented above in **2.1** and **2.2** shows that:

(i) The theses *A1*-*A9*, *B1*, *B2*, *C1*, *C2*, *C3*, *C4*, *C5*, *C11*, *C19*, *C15*, *C14* and *C16* of $\mathfrak{A}$ correspond synonymously and respectively to the theses *A1*-*A9*, *F24*, *F25*, *C1*, *C2*, *F5*, *F12*, *A10*, *A11*, *F1*, *F2* and *F3* of $\mathfrak{B}$

and, moreover, that:

(ii) The theses *DI*, *C12* and *C13* of $\mathfrak{A}$ correspond in the same manner to the theses *F22*, *D1* and *D2* of $\mathfrak{B}$.

Therefore, due to (ii) it follows from (i) immediately that the system $\mathfrak{B}$ is a Newman algebra. Thus, the theorem is proved. Furthermore, it should be noticed that:

(iii) Since (i) and (ii) establish that the systems $\mathfrak{A}$ and $\mathfrak{B}$ are inferentially equivalent, if their respective carrier sets are equal, or they are inferentially equivalent up to isomorphism, if their carrier sets have only the same cardinality, any theorem provable in the field of one of these systems, is also provable in the field of the other

and that:

(iv) Since the theses *C18* and *F23* are provable in $\mathfrak{A}$ and $\mathfrak{B}$ respectively, the acceptance of complementation, as a primitive notion in Newman algebra, does not require necessarily an assumption of a special postulate concerning an extensionality of the relation = with respect to this unary operation.

3 In this section it will be proved that in the field of the remaining axioms of $\mathfrak{B}$ the axioms *C1*, *C2*, *F1*, *F2* and *F3* are inferentially equivalent to the following set of formulas: *F1*, *F2* and

G1 $[abcd]:a,b,c,d\epsilon B.\supset.(a\times(b+c))\times d=((c\times a)\times d)+((b\times a)\times d)$

3.1 Let us assume the axiom-system of $\mathfrak{B}$. Hence, we have at our disposal not only all formulas which are proved in section **2.2**, but also the theorems which Newman has proved in [6]. Thus, we can accept without proof:

F26 $[ab]:a,b\in B\,.\supset.\,a+b=b+a$ [*Cf.* **P17** in [6], p. 260]

F27 $[ab]:a,b\in B\,.\,a+a=a\,.\,b+b=b\,.\supset.\,a\times b=b\times a$ [*Cf.* **P31** in [6], p. 263]

F28 $[ab]:a,b\in B\,.\,a+a=0\,.\,b+b=0\,.\supset.\,a\times b=b\times a$ [*Cf.* **P34** in [6], p. 264]

Then[4]:

F29 $[a]:a\in B\,.\supset.\,(a+a)+(a+a)=a+a$

PR $[a]:\mathrm{Hp}(1)\,.\supset.$

$$(a+a)+(a+a)=((a\times a)+(a\times a))+((a\times a)+(a\times a)) \quad [1;\,F7]$$
$$=((a+a)\times a)+((a+a)\times a) \quad [C2]$$
$$=(a+a)\times(a+a)=a+a \quad [C1;\,F7]$$

F30 $[a]:a\in B\,.\supset.\,((\overline{a+a})\times a)+((\overline{a+a})\times a)=0$ [*C1*; *F20*]

F31 $[a]:a\in B\,.\supset.\,[\exists bc]\,.\,b,c\in B\,.\,b+b=b\,.\,c+c=0\,.\,a=b+c$

PR $[a]:a\in B\,.\supset.$

2\. $a=((a+a)+(\overline{a+a}))\times a=((a+a)\times a)+((\overline{a+a})\times a)$ [1; *A4*; *F3*; *C2*]

$=((a\times a)+(a\times a))+((\overline{a+a})\times a)=(a+a)+((\overline{a+a})\times a)$ [*C2*; *F7*]

$[\exists bc]\,.\,b,c\in B\,.\,b+b=b\,.\,c+c=0\,.\,a=b+c$ [1; *A4*; *A10*; *A5*; *F29*; *F30*; 2]

F32 $[abcd]:a,b,c,d\in B\,.\,a+a=a\,.\,b+b=b\,.\,c+c=0\,.\,d+d=0\,.\supset.\,(a+c)\times(b+d)=(a\times b)+(c\times d)$

PR $[abcd]:\mathrm{Hp}(5)\,.\supset.$

$$(a+c)\times(b+d)=((a\times b)+(c\times b))+((a\times d)+(c\times d)) \quad [1;\,C1;\,C2]$$
$$=((a\times b)+(c\times(b+b)))+(((a+a)\times d)+(c\times d)) \quad [3;\,2]$$
$$=((a\times b)+((c+c)\times b))+((a\times(d+d))+(c\times d)) \quad [C1;\,C2]$$
$$=((a\times b)+(0\times b))+((a\times 0)+(c\times d)) \quad [4;\,5]$$
$$=(a\times b)+(c\times d) \quad [F15;\,F14;\,F12;\,F17]$$

F33 $[ab]:a,b\in B\,.\supset.\,a\times b=b\times a$

PR $[ab]:\mathrm{Hp}(1)\,.\supset.$

$[\exists cdmn]\,.$

$$\left.\begin{array}{ll} 2. & c,d,m,n\in B\,. \\ 3. & c+c=c\,. \\ 4. & d+d=0\,. \\ 5. & m+m=m. \\ 6. & n+n=0\,. \\ 7. & a=c+d\,. \\ 8. & b=m+n\,. \end{array}\right\} \quad [1;\,F31]$$

4\. The deductions presented below excluding *G1* are also due to Newman, *cf.* [6], but often they are given in a very compact way, or even verbally. For this reason I decided to present them more formally. Concerning the formulas *F31* and *F32*, *cf.* **P25** in [6], p. 262.

9. $a \times b = (c + d) \times (m + n) = (c \times m) + (d \times n)$ [1; $A5$; 2; 7; 8; $F32$; 3; 5; 4; 6]

$= (m \times c) + (n \times d)$ [$F27$; 3; 5; $F28$; 4; 6]

$= (m + n) \times (c + d) = b \times a$ [$F32$; 5; 3; 6; 4; 7; 8]

$a \times b = b \times a$ [9]

G1 $[abcd]: a, b, c, d \epsilon B . \supset . (a \times (b + c)) \times d = ((c \times a) \times d) + ((b \times a) \times d)$

PR $[abcd]: \mathrm{Hp}(1) . \supset .$

$(a \times (b + c)) \times d = ((a \times b) + (a \times c)) \times d$ [1; $C1$]

$= ((c \times a) + (b \times a)) \times d$ [$F33$, $F26$]

$= ((c \times a) \times d) + ((b \times a) \times d)$ [$C2$]

Thus, in $\mathfrak{B}$ $\{A1\text{-}A11; C1; C2; F1; F2; F3\} \rightarrow \{A1\text{-}A11; F1; F2; G1\}$

3.2 Now, let us assume, as the axioms of $\mathfrak{B}$, *A1-A11, F1, F2* and *G1*. Then:

G2 $[abc]: a, b, c \epsilon B . \supset . a \times (b + c) = (c \times a) + (b \times a)$ [$F2$; $G1$]

G3 $[abc]: a, b, c \epsilon B . \supset . (a + \overline{a}) \times (b + c) = c + b$ [$G2$; $F2$]

C2 $[abc]: a, b, c \epsilon B . \supset . (a + b) \times c = (a \times c) + (b \times c)$

PR $[abc]: \mathrm{Hp}(1) . \supset .$

$(a + b) \times c = ((a + \overline{a}) \times (b + a)) \times c$ [1; $G3$]

$= ((a \times (a + \overline{a})) \times c) + ((b \times (a + \overline{a})) \times c)$ [$G1$]

$= (a \times c) + (b \times c)$ [$F2$]

G4 $[abc]: a, b, c \epsilon B . \supset . a \times (b + c) = (c + b) \times a$

PR $[abc]: \mathrm{Hp}(1) . \supset .$

$a \times (b + c) = (c \times a) + (b \times a) = (c + b) \times a$ [1; $G2$; $C2$]

G5 $[ab]: a, b \epsilon B . \supset . a + \overline{a} = \overline{b} + b$

PR $[ab]: \mathrm{Hp}(1) . \supset .$

$a + \overline{a} = (a + \overline{a}) \times (b + \overline{b}) = (\overline{b} + b) \times (a + \overline{a}) = \overline{b} + b$

[1; $A10$; $A4$; $F2$; $G4$; $F2$]

F3 $[ab]: a, b \epsilon B . \supset . a = (b + \overline{b}) \times a$

PR $[ab]: \mathrm{Hp}(1) . \supset .$

$a = a \times (b + \overline{b}) = (\overline{b} + b) \times a = (b + \overline{b}) \times a$ [1; $F2$; $G4$; $G5$]

F26 $[ab]: a, b \epsilon B . \supset . a + b = b + a$ [$F2$; $G4$; $F3$]

F33 $[ab]: a, b \epsilon B . \supset . a \times b = b \times a$

PR $[ab]: \mathrm{Hp}(1) . \supset .$

$a \times b = a \times (b + (b \times \overline{b})) = ((b \times \overline{b}) + b) \times a$ [1; $A5$; $F1$; $G4$]

$= (b + (b \times \overline{b})) \times a = b \times a$ [$F26$; $F1$]

C1 $[abc]: a, b, c \epsilon B . \supset . a \times (b + c) = (a \times b) + (a \times c)$ [$G2$; $F33$; $F26$]

Thus, $\{A1\text{-}A11; F1; F2; G1\} \rightarrow \{A1\text{-}A11; C1; C2; F1; F2; F3\}$

3.3 It follows from **3.1** and **3.2** that in the field of the remaining axioms of $\mathfrak{B}$ $\{C1; C2; F1; F2; F3\} \rightleftarrows \{F1; F2; G1\}$ and, therefore, the proof is complete.

4 The mutual independence of the axioms *C1, C2, F1, F2, F3* and of the axioms *F1, F2, G1* is established by using the following algebraic tables (matrices):

$\mathfrak{M}1$

$+$	0	α	β	γ	1
0	0	α	β	γ	1
α	α	0	1	1	β
β	β	1	0	0	α
γ	γ	1	0	0	α
1	1	β	α	α	0

$\times$	0	α	β	γ	1
0	0	0	0	0	0
α	0	α	0	0	α
β	0	0	β	γ	β
γ	0	0	β	γ	γ
1	0	α	β	γ	1

x	$\overline{x}$
0	1
α	β
β	α
γ	α
1	0

$\mathfrak{M}2$

$+$	0	α	β	γ	1
0	0	α	β	γ	1
α	α	0	1	1	β
β	β	1	0	0	α
γ	γ	1	0	0	α
1	1	β	α	α	0

$\times$	0	α	β	γ	1
0	0	0	0	0	0
α	0	α	0	0	α
β	0	0	β	β	β
γ	0	0	γ	γ	γ
1	0	α	β	γ	1

x	$\overline{x}$
0	1
α	β
β	α
γ	α
1	0

$\mathfrak{M}3$

$+$	α	β	γ
α	β	α	γ
β	α	γ	β
γ	γ	β	α

$\times$	α	β	γ
α	β	γ	α
β	γ	α	β
γ	α	β	γ

x	$\overline{x}$
α	γ
β	β
γ	α

$\mathfrak{M}4$

$+$	α	β
α	α	β
β	β	α

$\times$	α	β
α	α	α
β	α	α

x	$\overline{x}$
α	α
β	β

$\mathfrak{M}5$

$+$	α	1	0
α	α	1	α
1	1	1	1
0	α	1	0

$\times$	α	1	0
α	α	1	0
1	α	1	0
0	0	0	0

x	$\overline{x}$
α	0
1	0
0	1

$\mathfrak{M}6$

$+$	α	1	0
α	α	1	α
1	1	1	1
0	α	1	0

$\times$	α	1	0
α	α	α	0
1	1	1	0
0	0	0	0

x	$\overline{x}$
α	0
1	0
0	1

Matrices $\mathfrak{M}1$, $\mathfrak{M}2$, $\mathfrak{M}3$ and $\mathfrak{M}4$ are Newman's examples E7, E6, E2 and E8, *cf.* [6], pp. 269-270, respectively, which are adjusted to the system $\mathfrak{B}$. Matrices $\mathfrak{M}5$ and $\mathfrak{M}6$ are the modifications of Croisot's examples $E_{3\alpha}$ and $E_{3\beta}$, *cf.* [5], p. 26.

Since:

(a) matrix $\mathfrak{M}1$ verifies *C2*, *F1*, *F2*, *F3*, but falsifies *C1* for a/γ, b/α, c/β: (i) $\gamma \times (\alpha + \beta) = \gamma \times 1 = \gamma$, (ii) $(\gamma \times \alpha) + (\gamma \times \beta) = 0 + \beta = \beta$, and *G1* for a/β, $b/0$, c/γ, $d/1$: (i) $(\beta \times (0 + \gamma) \times 1) = \beta \times \gamma = \gamma$, (ii) $((\gamma \times \beta) \times 1) + ((0 \times \beta) \times 1) = \beta + 0 = \beta$,

(b) matrix $\mathfrak{M}2$ verifies *C1*, *F1*, *F2* and *F3*, but falsifies *C2* for $a/\alpha \,.\, b/\beta$, c/γ: (i) $(\alpha + \beta) \times \gamma = 1 \times \gamma = \gamma$, (ii) $(\alpha \times \gamma) + (\beta \times \gamma) = 0 + \beta = \beta$,

(c) matrix $\mathfrak{M}3$ verifies $C1$, $C2$, $F2$, $F3$ and $G1$, but falsifies $F1$ for a/α, b/α: (i) $\alpha = \alpha$, (ii) $\alpha + (\alpha \times \overline{\alpha}) = \alpha + (\alpha \times \gamma) = \alpha + \alpha = \beta$,
(d) matrix $\mathfrak{M}4$ verifies $F1$ and $G1$, but falsifies $F2$ for a/β, b/α: (i) $\beta = \beta$, (ii) $\beta \times (\alpha + \overline{\alpha}) = \beta \times (\alpha + \alpha) = \alpha$,
(e) matrix $\mathfrak{M}5$ verifies $C1$, $C2$, $F1$, $F3$, but falsifies $F2$ for a/α, $b/1$: (i) $\alpha = \alpha$, (ii) $\alpha \times (1 + \overline{1}) = \alpha \times (1 + 0) = \alpha \times 1 = 1$, and
(f) matrix $\mathfrak{M}6$ verifies $C1$, $C2$, $F1$, $F2$, but falsifies $F3$ for a/α, $b/1$: (i) $\alpha = \alpha$, (ii) $(1 + \overline{1}) \times \alpha = (1 + 0) \times \alpha = 1 \times \alpha = 1$,

it is established by matrices $\mathfrak{M}1$, $\mathfrak{M}2$, $\mathfrak{M}3$, $\mathfrak{M}5$ and $\mathfrak{M}6$ that the axioms $C1$, $C2$, $F1$, $F2$ and $F3$ are mutually independent, and by matrices $\mathfrak{M}1$, $\mathfrak{M}3$ and $\mathfrak{M}4$ that the axioms $F1$, $F2$ and $G1$ are also mutually independent.

5 Final remark. A characteristic feature of the formalizations of Newman algebra given in [6] and, e.g., in [3] is that in both of them two laws of distribution ($C1$, $C2$) are accepted as, say, the basic postulates to which some axioms peculiar to this theory are added, *cf.* a remark of Newman in [6], p. 257, note 4. In some degree this structure is preserved in the first axiom system ($C1$, $C2$, $F1$, $F2$, $F3$) presented above of system $\mathfrak{B}$. On the other hand, the second axiom-system ($F1$, $F2$, $G1$) of $\mathfrak{B}$ is very compact, and for this reason the essential structure of the theory is much more hidden and cannot be recognized without some deductions.

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