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A NOTE ON UNIVERSALLY FREE DESCRIPTION THEORY

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1 In this note a universally free first order logic—(**UFL**) for short—in the sense of Meyer and Lambert [5], incorporating the rubrics required for a treatment of "descriptions" will be sketched. (**UFL**) is intended to satisfy the following conditions: (a) the self-identity of terms should be assertable in (**UFL**) without qualifications or restrictions; (b) i) $(\forall_{x_i})\phi_i \rightarrow (\exists_{x_i})\phi_i$, and ii) $(\forall_{x_i})\phi_i \rightarrow (S_{x_i})\phi_i$—where (S_{x_i}) is the singular quantifier to be read as 'there is only one . . . such that'—should not be assertable in (**UFL**); (c) Quine's criterion of ontological commitment, as interpreted in Rao [11], should be applicable to theories incorporating (**UFL**).

2 Let (**SFL**) be the standard system of first order logic, as presented by, for instance, Mendelson [6]. To have (**UFL**) we shall (1) augment the primitive base of (**SFL**) by i) some specially introduced *monadic predicate constants* A^i, $i = 1, 2, \ldots$, as in Rao [9], such that A^i_J, $i \leqslant J$ is a wff of (**UFL**), and ii) the singular quantifier (S . . .), where the blank is to be filled by an individual variable, such that if ϕ_i is a wff of (**UFL**) containing free occurrences of an individual variable x_i, $(S_{x_i})\phi_i$ is a wff of (**UFL**); and (2) delete from the primitive base of (**SFL**) the equality sign =. (This deletion is motivated by considerations shown in Rao [10].) To pick up the theorems of (**UFL**), we shall replace the meta-axioms, and rules of (**SFL**) by the following:

Ax1. *If* ϕ_i *is a tautology by two-valued truth tables then* $\vdash\phi_i$.
Ax2. $\vdash\ulcorner(\forall_{x_i})(\phi_i \rightarrow \phi_J) \rightarrow ((\forall_{x_i})\phi_i \rightarrow (\forall_{x_i})\phi_J)\urcorner$.
Ax3. $\vdash\ulcorner\phi_i \rightarrow (\forall_{x_i})\phi_i$ *provided* x_i *does not occur free in* ϕ_i.
Ax4. $\vdash\ulcorner(\forall_{x_i})\phi_i \rightarrow \phi_i\urcorner$ *provided* x_i *does occur free in* ϕ_i.
Ax5. $\vdash(\forall_{x_i})A^J_{x_i}$ *provided* $J = i$.
Ax6. $\vdash\ulcorner(A^i_{x_J} \rightarrow A^J_{x_i}) \rightarrow (\phi_i \rightarrow \phi_J)$ *provided* ϕ_i *and* ϕ_J *are alike except that* ϕ_J *contains* x_i *wherever* ϕ_i *contains* x_J.
Ax7. $\vdash\ulcorner(\exists_{x_i})\phi_i \rightarrow ((\forall_{x_J})\phi_J \rightarrow (S_{x_i})\phi_i)\urcorner$ *where* ϕ_i *and* ϕ_J *are alike except that* ϕ_i *contains* x_i *wherever* ϕ_J *contains* x_J, *and* $i \leqslant J$.
Ax8. $\vdash\ulcorner(S_{x_i})\phi_i \rightarrow ((\exists_{x_i})\phi_i \rightarrow ((\forall_{x_i})\phi_J \rightarrow (A^i_{x_J} \rightarrow A^J_{x_i})))\urcorner$ *provided* ϕ_i *and* ϕ_J *are as in* Ax7.
Ax9. *If* $\vdash\phi_i$ *and* $\vdash\ulcorner\phi_i \rightarrow \phi_J\urcorner$ *then* $\vdash\phi_J$.

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Here $(\exists \ldots)$ is used as an abbreviation of $\sim(\forall \ldots)\sim$, and the quasi-quotes are used as in Quine [7], i.e., to refer to the universal closures of what occurs within the quotes.

3 In order to set up a semantic interpretation of (**UFL**), let i) $\mathscr{L} = \langle \mathscr{D}, \mathscr{R}, \mathscr{F} \rangle$ be a structure, where $\mathscr{D}$ is a possibly empty set of individuals, $\mathscr{R}$ and $\mathscr{F}$ are sets of relations and operations, respectively, defined over $\mathscr{D}$; ii) μ be a unary mapping function with individual variables, individual constants, function letters, and monadic predicate constants specially introduced into (**UFL**) as arguments, and values in $\mathscr{L}$; iii) ν be a binary function with wffs of (**UFL**) as its first arguments, with second arguments in $\mathscr{L}$, and values in $\mathbf{V} = \{\mathbf{T}, \mathbf{F}\}$. Now, when $\mathscr{D}$ is non-empty, the interpretations will be similar of those of (**SFL**) in Mendelson [6], except that there will be additional clauses in the inductive definition of ν, one to the effect that $\nu(\mathrm{S}_{x_i})P^i_{J}{}_{x_1}, \ldots, x_i\, \mu(P^i_J)\mu(x_1) \ldots \mu(x_i) = \mathbf{T}$ if and only if i) there exists an n-tuple n_i of Σ such that x_i satisfies $P^i_{J}{}_{x_1} \ldots x_i$; ii) for each J, n_J of Σ satisfies $P^i_{J}{}_{x_1} \ldots x_i$ if and only if the i'th components of n_i and n_J are the same, and other clauses covering the wffs of (**UFL**) in which sentential connectives occur and are quantified by the singular quantifier; and iii) $\nu\mu(A^i)\mu(x_J) = \mathbf{T}$ if and only if $\mu(x_i) = \mu(x_J)$. Obviously $\mu(A^i)$ is x_i.

When $\mathscr{D}$ is empty, making use of the well-known result about first order theories, namely that if the sequence $\phi_i, \phi_{i+1}, \ldots$ of wffs is countable, then a conjunction with countable number of conjuncts ϕ_i & ϕ_{i+1} & ... is also a wff, and that if for each $J = i, i+1, \ldots, \phi_J$ can be mapped onto $\mathscr{L}$, then for each n-tuple n_i of Σ, n_i satisfies ϕ_J if and only if n_i satisfies ϕ_i & ϕ_{i+1} & .... We shall treat all wffs of (**UFL**) closed by the universal quantifier as conjunctions of zero conjuncts, and all wffs of (**UFL**) closed by singular quantifier as disjunctions of zero disjuncts *à la* Hailperin [3]. This is possible as the set of wffs of (**UFL**) is a countable one. When $\mathscr{D}$ is empty, μ will be taken as an empty function, as in van Fraassen [1], and van Fraassen and Lambert [2], and ν is taken as a unary function with arguments in the set of wffs of (**UFL**) and values in $\{\mathbf{T}, \mathbf{F}\}$. For each i, if ϕ_i is a wff of (**UFL**) i) $\nu\phi_i = \mathbf{T}$ if ϕ_i is $(\forall_{x_i})\phi_J$; ii) $\nu\phi_i = \mathbf{F}$ if ϕ_i is either $(\exists_{x_i})\phi_J$ or $(\mathrm{S}_{x_i})\phi_J$; and iii) $\nu\phi_i = \nu\phi_i'$ if ϕ_i is ϕ_J such that a term t_i occurs free in ϕ_J, where ϕ_i' is the result of deleting t_i and reducing by one the superscript of all those predicates having t_i as one of their arguments, such that ϕ_i' is a wff of (**UFL**).

4 That (**UFL**) is an adequate logic of "descriptions" follows from the fact that

Th1. $\vdash \ulcorner(\mathrm{S}_{x_i})\phi_i \leftrightarrow ((\exists_{x_i})\phi_i \rightarrow ((\forall_{x_J})\phi_J \rightarrow (A^i_{x_J} \rightarrow A^J_{x_i})))\urcorner$ *where* ϕ_i *and* ϕ_J *are as in* Ax8,

can be proved as a theorem in (**UFL**). Ax8 and its converse together, by virtue of the definitional identity of $\phi_i \leftrightarrow \phi_J$ with $(\phi_i \rightarrow \phi_J)$ & $(\phi_J \rightarrow \phi_i)$, which holds for (**UFL**), yield Th1. The converse of Ax8 can be had from Ax8, the tautology

$$(\phi_i \rightarrow (\phi_J \rightarrow \phi_k)) \rightarrow (\phi_i \rightarrow ((\phi_J \rightarrow \phi_i) \rightarrow \phi_k))$$

and Ax9. Next, that (**UFL**) has within its framework the necessary apparatus for an adequate treatment of identity follows from the fact that the set $\mathcal{J}$ of wffs of (**SFL**) containing, for each $i \leqslant J$, i) $x_J = x_i$, ii) the closures of (i) as subformulae can be put to one-to-one correspondence with the set $\mathcal{J}'$ of wffs of (**UFL**) containing for each $i \leqslant J$, a) $A^i_{x_J}$, b) $A^J_{x_i} \rightarrow A^i_{x_J}$, and iii) the closures of (i) and (ii) as subformulae, such that for each i, if $\phi_i \in \mathcal{J}$ then there is a wff ϕ'_i of $\mathcal{J}'$ satisfying the following conditions: i) $\mu(\phi_i) = \mu(\phi'_i)$ and ii) $\nu(\phi_i)\mu(\phi_i) = \nu(\phi'_i)\mu(\phi'_i) = \mathbf{T}$ or else $\nu(\phi_i)\mu(\phi_i) = \nu(\phi'_i)\mu(\phi'_i) = \mathbf{F}$.

5 A formula A is said to be *universally valid* if and only if it is valid in all domains, including the empty one. In order to show that (**UFL**) is universally free all that needs to be shown is that each theorem (**UFL**) is universally valid. The task of showing that the theorems of (**UFL**) are valid in non-empty domains can be dispensed with here, as the set of those theorems is a subset of the theorems of (**SFL**). When the domain is empty, by virtue of the interpretation suggested above, Ax1-Ax8 will be valid, and Ax9 is truth preserving. Hence, (**UFL**) is universally free.

It was shown in Quine [8] that the fragment (**UFL1**) of (**UFL**), based on Ax1-Ax4 and Ax9, is complete. (**UFL**), being a consistent extension of (**UFL1**), is complete. Further, following Quine, it can be shown that an extension (**UFL2**) of (**UFL1**), which is the result of augmenting (**UFL1**) by Ax5-Ax6, and Ax0

$$(\exists_{x_i})A^i_{x_i}$$

is isomorphic with the unextended part of (**SFL**). Thus (**UFL**) is isomorphic with (**SFL1**), where (**SFL1**) is the result of extending (**SFL**) by adding to it Ax7-Ax8. Then, whether (**UFL**) is an adequate theory of "descriptions" hinges on the question whether Ax7-Ax8 capture the formal structure of descriptions. A positive answer can be given to this because of Th1.

6 (**UFL**) preserves the precise distinction between "terms" and "predicates" as does (**SFL**), unlike the systems of Russell and Whitehead [12]—as has been pointed out by Solon and Wertz [13]—Lambert [4], and van Fraassen and Lambert [2].

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