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SUMS OF α-SPACES

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1 *Introduction** In [1] and [2], Dekker introduced and studied an $\aleph_0$-dimensional recursive vector space $\overline{U}_F$ over a countable field F. Briefly, it consists of an infinite recursive set ϵ_F of numbers (i.e., non-negative integers), an operation $+$ from $\epsilon_F \times \epsilon_F$ into ϵ_F and an operation $\cdot$ from $F \times \epsilon_F$ into ϵ_F. If the field F is identified with a recursive set, both $+$ and $\cdot$ are partial recursive functions. Let β be a subset of ϵ_F. We call β a *repère* if it is linearly independent; β is a *r.e. repère* if β is a r.e. set; and β is an *α-repère* if it is included in some r.e. repère. A subspace V of $\overline{U}_F$ is an *α-space* if it has at least one *α-basis*, i.e., at least one basis which is also an α-repère. A subspace V is *isolic* if it includes no r.e. repère; it is *r.e.* if it is r.e. as a set. The word "space" is used in the sense of "subspace of $\overline{U}_F$", and we denote "W is a subspace of V" by "$W \leqslant V$". We usually write (0) for $\{0\}$, and $\overline{U}$ for $\overline{U}_F$. Let $\alpha \subset \epsilon_F$. If $\alpha = \emptyset$, $\mathrm{L}(\alpha) = (0)$. If $\alpha \neq \emptyset$, $\mathrm{L}(\alpha)$ denotes the span of α, i.e., the set of all linear combinations (with coefficients in F) of finitely many elements of α. If $\alpha = \{a_0, \ldots\}$, we usually write $\mathrm{L}(a_0, \ldots)$ instead of $\mathrm{L}(\{a_0, \ldots\})$. We use $\mathfrak{c}$ to denote the cardinality of the continuum.

The reperes β and γ are *independent* if they are disjoint and their union is a repère. The spaces V and W are *independent* if $V \cap W = (0)$. The sets β and γ are *separable* (written: $\beta|\gamma$) if they can be separated by r.e. sets. The α-repères β and γ are *α-independent* (written: $\beta||\gamma$), if they can be separated by independent r.e. repères. The spaces V and W are *α-independent* (written: $V||W$), if there are independent r.e. spaces $\overline{V}$ and $\overline{W}$ such that $V \leqslant \overline{V}$ and $W \leqslant \overline{W}$. For spaces V, W, W is an *α-subspace* of V (written: $W \leqslant_\alpha V$) if there is an α-space S such that $W||S$ and $W \oplus S = V$.

In [3] we proved that the intersection of two α-spaces need not be an α-space. The same question naturally arises concerning the sum of two

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α-spaces. Dekker has shown ([1], Proposition P23) that the sum of two α-independent α-spaces is an α-space. We shall prove that the sum of two independent α-spaces need not be an α-space. For this purpose we shall define a family of spaces, $\mathfrak{c}$ of which are α-spaces and $\mathfrak{c}$ of which are not. As a side result we shall obtain a new proof of the existence of non-α-spaces. We shall need the following three propositions:

Proposition P1. (Dekker, [1], P30). *The $\leq_\alpha$-relation between α-spaces is reflexive, antisymmetric, and transitive.*

Proposition P2. ([3], L5). *Let $\Gamma = \{V_i | i \in \mathrm{I}\}$ be a non-empty family of distinct α-spaces, where $\mathrm{I} = \langle 0, \ldots, n-1\rangle$ if* card $\Gamma = n > 0$ *and $\mathrm{I} = \epsilon$ otherwise. Let $S = \bigcap \Gamma$. Then for all finite dimensional spaces B,*

$$S || B \Longleftrightarrow S \cap B = (0).$$

Proposition P3. (Dekker, [2], Theorem 5; see also [4]). *Let S, C, V be spaces. If $S||C$ and $S \oplus C = V$, and if V is an α-space and C is an isolic α-space, then S is an α-space.*

2 *Sums* Notations. In the following, a_n denotes a 1-1 function ranging over an infinite r.e. repère α, $\alpha_1 = \alpha \setminus \{a_0, a_1\}$ and $A = \mathrm{L}(\alpha)$. Moreover,

$$\sigma_0 = \{a_0 + x \,|\, x \in \alpha_1\}, \ \tau_0 = \{a_1 + x \,|\, x \in \alpha_1\}.$$

Let δ, β, $\gamma \subset \alpha_1$. Then

$$\begin{aligned} \sigma_\beta &= \{a_0 + x \,|\, x \in \beta\}, \ \tau_\gamma = \{a_1 + x \,|\, x \in \gamma\}, \\ \mathrm{E}_0(\beta) &= \mathrm{L}(\sigma_\beta), \ \mathrm{E}_1(\gamma) = \mathrm{L}(\tau_\gamma), \\ \mathrm{T}\langle\beta, \gamma\rangle &= \mathrm{L}(\sigma_\beta \cup \tau_\delta) = \mathrm{E}_0(\beta) + \mathrm{E}_1(\gamma), \\ Z_\delta &= \mathrm{T}\langle\delta, \delta\rangle. \end{aligned}$$

Proposition P4. (a) *σ_0 and τ_0 are disjoint repères such that neither of the two spaces $\mathrm{L}(\sigma_0)$, $\mathrm{L}(\tau_0)$ is a subspace of the other one,*

(b) $\mathrm{L}(\sigma_0) \oplus \mathrm{L}(a_0, a_1) = A = \mathrm{L}(\tau_0) \oplus \mathrm{L}(a_0, a_1)$,

(c) $a_0 - a_1 \notin \mathrm{L}(\sigma_0) \cup \mathrm{L}(\tau_0)$, *but* $a_0 - a_1 \in \mathrm{L}(\sigma_0 \cup \tau_0)$,

(d) $\mathrm{L}(\sigma_0 \cup \tau_0) \oplus \mathrm{L}(a_0) = A = \mathrm{L}(\sigma_0 \cup \tau_0) \oplus \mathrm{L}(a_1)$,

(e) $\mathrm{L}(\sigma_0) \oplus \mathrm{L}(a_0 - a_1) = \mathrm{L}(\sigma_0 \cup \tau_0) = \mathrm{L}(\tau_0) \oplus \mathrm{L}(a_0 - a_1)$.

Proof: Left to the reader.

Proposition P5. (a) *The mapping $\delta \to Z_\delta$, for $\delta \subset \alpha_1$ has the following properties:*

(i) $\delta \neq \emptyset \Longleftrightarrow a_0 - a_1 \in Z_\delta$,

(ii) $Z_\delta \cap \sigma_0 = \sigma_\delta$, $Z_\delta \cap \tau_0 = \tau_\delta$,

(iii) *it is* 1-1.

(b) *The mappings $\beta \to \mathrm{E}_0(\beta)$, for $\beta \subset \alpha_1$, and $\gamma \to \mathrm{E}_1(\gamma)$, for $\gamma \subset \alpha_1$, have the following properties:*

(iv) $\mathrm{E}_0(\beta) \cap \sigma_0 = \sigma_\beta$, $\mathrm{E}_1(\gamma) \cap \tau_0 = \tau_\gamma$,

(v) *they are* 1-1.

Proof: Left to the reader.

Remarks. (a) If β and γ are known from the context, we often write $\sigma = \sigma_\beta$ and $\tau = \tau_\gamma$ and $V = \mathrm{T}_{\langle\beta,\gamma\rangle} = \mathrm{L}(\sigma \cup \tau)$. (b) For $\beta, \gamma \subset \alpha_1$, we have $\sigma \subset \sigma_0$, $\tau \subset \tau_0$. Hence σ is a basis for $\mathrm{L}(\sigma)$ and τ is a basis for $\mathrm{L}(\tau)$. We would like to know when $\sigma \cup \tau$ is a basis for $\mathrm{L}(\sigma \cup \tau)$, i.e., when $\mathrm{L}(\sigma) \cap \mathrm{L}(\tau) = (0)$.

Proposition P6. *Let α be an infinite r.e. repère and $a(n)$ a* 1-1 *recursive function ranging over α. Then for $\delta \subset \alpha_1$,*

(a) *Z_δ is a r.e. space $\Longleftrightarrow \delta$ is a r.e. set,*
(b) *Z_δ is an α-space for every $\delta \subset \alpha_1$.*

Proof: Assume the hypothesis. Then clearly σ_0 and τ_0 are r.e. sets. Part (a) follows directly from

$$\delta \text{ r.e.} \Longrightarrow \sigma_\delta,\ \tau_\delta \text{ r.e.} \Longrightarrow Z_\delta \text{ r.e.},$$
$$Z_\delta \text{ r.e.} \Longrightarrow Z_\delta \cap \sigma_0 = \sigma_\delta \text{ r.e.} \Longrightarrow \delta \text{ r.e.}$$

To prove part (b), note if δ is empty, $\mathrm{Z}_\delta = (0)$ and we are done. Now assume that $\delta \neq \emptyset$. Then $a_0 - a_1 \notin \mathrm{L}(\sigma_\delta)$, since $a_0 - a_1 \notin \mathrm{L}(\sigma_0)$ and $\sigma_\delta \subset \sigma_0$. Using P5 (i), it is readily proved that

$$\mathrm{L}(\sigma_\delta) \oplus \mathrm{L}(a_0 - a_1) = \mathrm{L}(\sigma_\delta \cup \tau_\delta) = \mathrm{Z}_\delta.$$

Hence Z_δ has as a basis the set $\sigma_\delta \cup \{a_0 - a_1\}$ which is included in the r.e. repère $\sigma_0 \cup \{a_0 - a_1\}$. Hence Z_δ is an α-space. Q.E.D.

Proposition P7. *Let $\beta, \gamma \subset \alpha_1$. Then*

$$\sigma,\ \tau \textit{ independent} \Longleftrightarrow \mathrm{card}(\beta \cap \gamma) \leqslant 1.$$

Proof: We will show:

(a) $\beta \cap \gamma = \emptyset \Longrightarrow \sigma \cup \tau$ is a repère,
(b) $\beta \cap \gamma = \{a_k\} \Longrightarrow \sigma \cup \tau$ is a repère,
(c) $\mathrm{card}\,(\beta \cap \gamma) \geqslant 2 \Longrightarrow \mathrm{L}(\sigma) \cap \mathrm{L}(\tau) \neq (0)$.

Consider the relations

(1) $r_2(a_0 + a_2) + \ldots + r_n(a_0 + a_n) + s_2(a_1 + a_2) + \ldots + s_n(a_1 + a_n) = 0$,
(2) $(r_2 + \ldots + r_n)a_0 + (s_2 + \ldots + s_n)a_1 + (r_2 + s_2)a_2 + \ldots + (r_n + s_n)a_n = 0$,
(3) at most one of r_i, s_i is $\neq 0$, for $2 \leqslant i \leqslant n$,
(4) at most one of r_i, s_i is $\neq 0$, for $3 \leqslant i \leqslant n$.

Note that (1) $\Longrightarrow$ (2). Under the hypothesis of (a), we work with (2) and (3). Thus $r_i = s_i = 0$, for $2 \leqslant i \leqslant n$ and hence $\sigma \cup \tau$ is a repère. To prove (b), we may assume w.l.g. that $k = 2$; thus we work with (2) and (4). Then $r_i = s_i = 0$, for $3 \leqslant i \leqslant n$, and (2) implies

$$r_2 a_0 + s_2 a_1 + (r_2 + s_2)a_2 = 0.$$

Hence $r_2 = s_2 = 0$ and $\sigma \cup \tau$ is a repère. We now prove (c). Let $p, q \in \beta \cap \gamma$, where $p \neq q$. Then

$$p - q = (a_0 + p) - (a_0 + q) \in \mathrm{L}(\sigma),$$
$$p - q = (a_1 + p) - (a_1 + q) \in \mathrm{L}(\tau).$$

However, $p - q \neq 0$, hence $\mathrm{L}(\sigma) \cap \mathrm{L}(\tau) \neq (0)$. Q.E.D.

Proposition P8. *The restriction of the mapping* $\langle\beta, \gamma\rangle \to \mathrm{T}\langle\beta, \gamma\rangle$ *to the family of all ordered pairs of disjoint subsets of* α_1 *has the following properties:*

(a) $\mathrm{T}\langle\beta, \gamma\rangle \cap \sigma_0 = \sigma$, $\mathrm{T}\langle\beta, \gamma\rangle \cap \tau_0 = \tau$,

(b) $\mathrm{T}\langle\beta, \gamma\rangle \cap (\sigma_0 \cup \tau_0) = \sigma \cup \tau$,

(c) *it is* 1-1.

Proof: Let $\langle\beta, \gamma\rangle$ be an ordered pair of disjoint subsets of α_1: Clearly, $\sigma \subset \mathrm{T}\langle\beta, \gamma\rangle \cap \sigma_0$. Now assume $x \in \mathrm{T}\langle\beta, \gamma\rangle \cap \sigma_0$, say

$$x = r_2(a_0 + p_2) + \ldots + r_n(a_0 + p_n) + s_2(a_1 + q_2) + \ldots + s_m(a_1 + q_m),$$

i.e.,

$$x = (r_2 + \ldots + r_n)a_0 + (s_2 + \ldots + s_m)a_1 + r_2p_2 + \ldots + r_np_n + s_2q_2 + \ldots + s_mq_m,$$

where $\{p_2, \ldots, p_n\} \subset \beta$, $\{q_2, \ldots, q_m\} \subset \gamma$. Since $x \in \sigma_0$, it can also be written in the form $x = a_0 + u$, where $u \in \alpha_1$. Thus, $u \in \beta \cup \gamma$ since α_1 is a repère. We see that the hypothesis $u \in \gamma$ leads to the contradiction

$$r_2 + \ldots + r_n = 1,\ r_2 = \ldots = r_n = 0.$$

Hence $u \in \beta$ and $x = a_0 + u \in \sigma_\beta = \sigma$, and we have proved that $\mathrm{T}\langle\beta, \gamma\rangle \cap \sigma_0 = \sigma$. The second part of (a) can be proved similarly. Clearly, (a) implies (b), while according to (a), $\mathrm{T}\langle\beta, \gamma\rangle$ uniquely determines $\langle\sigma, \tau\rangle$. Since $\sigma = \sigma_\beta$ and $\tau = \tau_\gamma$, we see that $\langle\sigma, \tau\rangle$ uniquely determines $\langle\beta, \gamma\rangle$. Thus the mapping $\langle\beta, \gamma\rangle \to \mathrm{T}\langle\beta, \gamma\rangle$ is 1-1. Q.E.D.

Proposition P9. *If* β *and* γ *are disjoint and non-empty,* $a_0 - a_1 \notin \mathrm{L}(\sigma \cup \tau)$ *and* $\mathrm{L}(\sigma \cup \tau) \oplus \mathrm{L}(a_0 - a_1) = \mathrm{Z}_{\beta\cup\gamma}$.

Proof: Left to the reader.

Proposition P10. *For disjoint subsets* β, γ *of* α_1,

(a) $\dim \mathrm{L}(\sigma \cup \tau) = \mathrm{card}(\beta) + \mathrm{card}(\gamma)$,

(b) $\mathrm{codim}_A \mathrm{L}(\sigma \cup \tau) = 2 + \mathrm{card}[a_1 \backslash (\beta \cup \gamma)]$.

Proof: Under the hypothesis, σ and τ are disjoint and $\sigma \cup \tau$ is a basis of $\mathrm{L}(\sigma \cup \tau)$. Thus

$$\dim \mathrm{L}(\sigma \cup \tau) = \mathrm{card}\ \sigma + \mathrm{card}\ \tau = \mathrm{card}\ \sigma_\beta + \mathrm{card}\ \tau_\gamma = \mathrm{card}\ \beta + \mathrm{card}\ \gamma.$$

This proves (a). Now let $\delta = \alpha_1 \backslash (\beta \cup \gamma)$ and $\rho = \tau_\delta$, then

$$\mathrm{L}(\sigma \cup \tau \cup \rho) \oplus \mathrm{L}(a_0 - a_1) = \mathrm{Z}_{\beta\cup\gamma\cup\delta} = \mathrm{Z}_{\alpha_1},$$
$$\mathrm{L}(\sigma \cup \tau) \oplus \mathrm{L}(\rho) \oplus \mathrm{L}(a_0 - a_1) = \mathrm{L}(\sigma_0 \cup \tau_0).$$

According to P4 (d), $\mathrm{L}(\sigma_0 \cup \tau_0)$ has codimension 1 with respect to A. Hence,

$$\operatorname{codim}_A L(\sigma \cup \tau) = 2 + \operatorname{card} \rho = 2 + \operatorname{card} \delta.$$

Q.E.D.

Corollary C11. *If $\langle\beta, \gamma\rangle$ is a decomposition of α_1, then* $\operatorname{codim}_A L(\sigma \cup \tau) = 2$.

Proposition P12. *Let α be an infinite r.e. repère, a_n a 1-1 recursive function ranging over α, and $\langle\beta, \gamma\rangle$ an ordered pair of disjoint non-empty subsets of α_1. Then σ and τ are α-repères which are separated by the r.e. repères σ_0 and τ_0.*

Proof: Note that $\sigma \subset \sigma_0$, $\tau \subset \tau_0$, where σ_0, τ_0 are disjoint repères by P4. Since a_n is a recursive function, σ_0 and τ_0 are r.e. repères and σ, τ are α-repères. Q.E.D.

Agreement. We recall that α is an infinite repère and a_n 1-1 function ranging over α. In the special case that α is an α-repère, there is a r.e. repère $\overline{\alpha}$ such that $\alpha \subset \overline{\alpha}$. With $\overline{\alpha}$ we associate a 1-1 recursive function $\overline{a}_n$ ranging over $\overline{\alpha}$, and we agree to choose $\overline{a}_n$ in such a way that $\overline{a}_0 = a_0$, $\overline{a}_1 = a_1$, and put $\overline{\alpha}_1 = \overline{\alpha}\backslash\{a_0, a_1\}$, resulting in $\alpha_1 = \alpha \cap \overline{\alpha}_1$. We define

$$\overline{\sigma}_0 = \{\overline{a}_0 + x \mid x \in \overline{\alpha}_1\},\ \overline{\tau}_0 = \{\overline{a}_1 + x \mid x \in \overline{\alpha}_1\}.$$

Corollary C13. *Let α be an infinite α-repère, and $\langle\beta, \gamma\rangle$ an ordered pair of disjoint non-empty subsets of α_1. Then σ and τ are α-repères, and furthermore, there are r.e. disjoint r.e. repères $\overline{\sigma}_0$, $\overline{\tau}_0$ such that*

$$\sigma \subset \sigma_0 \subset \overline{\sigma}_0 \textit{ and } \tau \subset \tau_0 \subset \overline{\tau}_0.$$

Proof: Since α is an infinite α-repère, $\alpha \subset \overline{\alpha}$, an infinite r.e. repère. Then apply P12 to $\overline{\alpha} = \rho a_n$, where, by the above agreement, we are assuming that $\overline{a}_1 = a_1$ and $\overline{a}_0 = a_0$, hence $\sigma \subset \sigma_0 \subset \overline{\sigma}_0$ and $\tau \subset \tau_0 \subset \overline{\tau}_0$. Q.E.D.

Remark. Under the hypothesis of C13, $L(\sigma) \cap L(\tau) = (0)$, $L(\sigma) \oplus L(\tau) = L(\sigma \cup \tau)$. Here $L(\sigma)$, $L(\tau)$ are α-spaces. Since $\beta \simeq \sigma$ and $\gamma \simeq \tau$, we have $\dim_\alpha L(\sigma) = \operatorname{Req}(\beta)$ and $\dim_\alpha L(\tau) = \operatorname{Req}(\gamma)$. We wish to solve the following two problems:

(I) "When is $\sigma \cup \tau$ an α-repère?"
(II) "When is $L(\sigma \cup \tau)$ an α-space?"

Proposition P14. *For separable, independent α-repères δ and θ,*

$$\delta \cup \theta \textit{ is an } \alpha\textit{-repère} \Leftrightarrow \delta \,||\, \theta.$$

Proof: Let δ, θ be independent α-repères, $\delta \subset \overline{\delta}$, $\theta \subset \overline{\theta}$, where $\overline{\delta}$ and $\overline{\theta}$ are disjoint r.e. repères.

(a) Assume that $\delta \cup \theta$ is an α-repère, say $\delta \cup \theta \subset \overline{\lambda}$, where $\overline{\lambda}$ is an r.e. repère. Then $\delta \subset \overline{\delta}$ and $\delta \subset \overline{\lambda}$ imply $\delta \subset \overline{\delta} \cap \overline{\lambda}$. Similarly, $\theta \subset \overline{\theta} \cap \overline{\lambda}$. Note that $\overline{\delta} \cap \overline{\lambda}$ and $\overline{\theta} \cap \overline{\lambda}$ are r.e. repères which are disjoint, since $\overline{\theta} \cap \overline{\delta} = \emptyset$. Since $\overline{\delta} \cap \overline{\lambda}$ and $\overline{\theta} \cap \overline{\lambda}$ are both included in the r.e. repère $\overline{\lambda}$, they are also independent. Thus $\delta \,||\, \theta$.

(b) Assume $\delta \,||\, \theta$, say $\delta \subset \delta_0$, $\theta \subset \theta_0$, where θ_0, δ_0 are independent r.e.

reperes. Then $\delta \cup \theta \subset \delta_0 \cup \theta_0$, where $\delta_0 \cup \theta_0$ is an r.e. repere. Hence $\delta \cup \theta$ is an α-repere. Q.E.D.

Proposition P15. *Let α be an infinite α-repere. Let $\langle \beta, \gamma\rangle$ be an ordered pair of disjoint non-empty subsets of α_1. Then*

(a) $\mathrm{L}(\sigma)$ *r.e. space* $\Leftrightarrow \beta$ *r.e. set,* $\mathrm{L}(\tau)$ *r.e. space* $\Leftrightarrow \gamma$ *r.e. set,*
(b) $\mathrm{L}(\sigma \cup \tau)$ *r.e. space* $\Leftrightarrow \beta$ *and* γ *are r.e. sets,*
(c) $\mathrm{Z}_{\beta\cup\gamma}$ *r.e. space* $\Leftrightarrow \beta \cup \gamma$ *r.e. set.*

Proof: Let $\alpha \subset \overline{\alpha}$, where $\overline{\alpha}$ is an r.e. repere.

(a) If β is an r.e. set, so is $\sigma_\beta = \sigma$, hence $\mathrm{L}(\sigma)$ is an r.e. space. Now assume that $\mathrm{L}(\sigma)$ is an r.e. space. Since $\beta \subset \overline{\alpha}_1$, we see by P5 (b) that $\mathrm{L}(\sigma) \cap \overline{\sigma}_0 = \sigma$. Then σ is r.e. since $\overline{\sigma}_0$ is r.e.; then β is also r.e. Similarly, one proves the second part of (a).
(b) If β, γ are r.e. sets, so are σ, τ, hence $\mathrm{L}(\sigma \cup \tau)$ is an r.e. space. Now assume that $\mathrm{L}(\sigma \cup \tau)$ is an r.e. space. Using P8, we see that $\mathrm{L}(\sigma \cup \tau) \cap \overline{\sigma}_0 = \sigma$, $\mathrm{L}(\sigma \cup \tau) \cap \overline{\tau}_0 = \tau$. Hence σ and τ are r.e. and so are β and γ.
(c) By P6 (a). Q.E.D.

Proposition P16. *Let α be an infinite α-repere. Let $\langle \beta, \gamma\rangle$ be an ordered pair of disjoint non-empty subsets of α_1. Then the three following conditions are mutually equivalent:*

$$\text{(a) } \beta|\gamma, \quad \text{(b) } \sigma||\tau, \quad \text{(c) } \sigma \cup \tau \textit{ is an } \alpha\textit{-repere.}$$

Proof: Let $\alpha \subset \overline{\alpha}$, where $\overline{\alpha}$ is an infinite r.e. repere. Suppose that $\langle \beta, \gamma\rangle$ is an ordered pair of disjoint non-empty subsets of α_1. Then the sets σ and τ are independent α-reperes separated by the r.e. reperes $\overline{\sigma}_0$ and $\overline{\tau}_0$. Thus (b)$\Leftrightarrow$(c) by P14. Thus all we need to show is (a)$\Leftrightarrow$(b).

(a)$\Rightarrow$(b). Suppose that $\beta|\gamma$, say $\beta \subset \overline{\beta}'$, $\gamma \subset \overline{\gamma}'$ where $\overline{\beta}'$, $\overline{\gamma}'$ are disjoint r.e. sets. Then set $\overline{\beta} = \overline{\beta}' \cap \overline{\alpha}_1$, $\overline{\gamma} = \overline{\gamma}' \cap \overline{\alpha}_1$. Then $\beta \subset \overline{\beta}$, $\gamma \subset \overline{\gamma}$ where $\overline{\beta}$, $\overline{\gamma}$ are independent disjoint r.e. subsets of $\overline{\alpha}_1$. Let $\overline{\sigma} = \overline{\sigma}_{\overline{\beta}}$, $\overline{\tau} = \overline{\tau}_{\overline{\gamma}}$, then $\overline{\sigma}$, $\overline{\tau}$ are independent r.e. reperes, since $\overline{\beta}$, $\overline{\gamma}$ are disjoint r.e. sets. Moreover, $\sigma \subset \overline{\sigma}$, $\tau \subset \overline{\tau}$ and hence $\sigma||\tau$.
(b)$\Rightarrow$(a). Assume $\sigma||\tau$, say $\sigma \subset \sigma'$, $\tau \subset \tau'$, where σ', τ' are independent r.e. reperes. Put $\overline{\sigma} = \sigma' \cap \overline{\sigma}_0$, $\overline{\tau} = \tau' \cap \overline{\tau}_0$. Then $\overline{\sigma}$, $\overline{\tau}$ are independent r.e. reperes. Let

$$\overline{\beta} = \{y \in \overline{\alpha}_1 | a_0 + y \in \overline{\sigma}\},$$
$$\overline{\gamma} = \{y \in \overline{\alpha}_1 | a_1 + y \in \overline{\tau}\}.$$

Then $\overline{\beta}$, $\overline{\gamma}$ are r.e. subsets of $\overline{\alpha}_1$ such that $\beta \subset \overline{\beta}$, $\gamma \subset \overline{\gamma}$, $\overline{\sigma}_{\overline{\beta}} = \overline{\sigma}$, $\overline{\tau}_{\overline{\gamma}} = \overline{\tau}$, $\sigma \subset \overline{\sigma}$, and $\tau \subset \overline{\tau}$. According to P7, the relation $\overline{\sigma}||\overline{\tau}$ implies $\mathrm{card}(\overline{\beta} \cap \overline{\gamma}) \leq 1$. If $\overline{\beta} \cap \overline{\gamma} = \emptyset$, β and γ are separated by the r.e. sets $\overline{\beta}$ and $\overline{\gamma}$, hence $\beta|\gamma$. Now suppose $\overline{\beta} \cap \overline{\gamma} = (k)$. Since β and γ are disjoint, we have $k \notin \beta$ or $k \notin \gamma$. We may assume w.l.g. that $k \notin \beta$. Then, β and γ are separated by the r.e. sets $\overline{\beta}\setminus\{k\}$ and $\overline{\gamma}$, hence again $\beta|\gamma$. Q.E.D.

Remark. P16 answers question (I) in the remark following the proof of C13.

We have not yet answered the second question. The relevant problem is whether

$$\mathrm{L}(\sigma \cup \tau) \text{ an } \alpha\text{-space} \Rightarrow \sigma \cup \tau \text{ an } \alpha\text{-repère}.$$

We shall see that this is indeed the case.

Notation. Let W be any space. Then

$$\theta_0(W) = \{x \in \alpha_1 | a_0 + x \in W\},\ \theta_1(W) = \{x \in \alpha_1 | a_1 + x \in W\}.$$

Now assume $\alpha \subset \overline{\alpha}$, where $\overline{\alpha}$ is an r.e. repère. Then

$$\overline{\theta}_0(W) = \{x \in \overline{\alpha}_1 | a_0 + x \in W\},\ \overline{\theta}_1(W) = \{x \in \overline{\alpha}_1 | a_1 + x \in W\}.$$

Remark. Clearly (a) $\theta_1(W) = \overline{\theta}_1(W) \cap \alpha_1$, $\theta_2(W) = \overline{\theta}_2(W) \cap \alpha_1$; (b) $\theta_0[\mathrm{L}(\sigma)] = \beta$, and $\theta_1[\mathrm{L}(\tau)] = \gamma$; (c) if W is an r.e. space, then $\overline{\theta}_0(W)$ and $\overline{\theta}_1(W)$ are r.e. sets.

Proposition P17. *Let $\langle \beta, \gamma\rangle$ be an ordered pair of disjoint non-empty subsets of α_1 and $\mathrm{L}(\sigma \cup \tau) \leqslant W$. Then exactly one of the following is true:*

(a) $\theta_0(W) \cap \theta_1(W) = \emptyset$ *and* $a_0 - a_1 \notin W$,
(b) $\beta \cup \gamma \subset \theta_0(W) \cap \theta_1(W)$ *and* $a_0 - a_1 \in W$.

Proof: It is readily seen that $\theta_0(W) \cap \theta_1(W) \neq \emptyset$ if and only if $a_0 - a_1 \in W$. Thus, either (a) holds or $a_0 - a_1 \in W$. We now prove

$$a_0 - a_1 \in W \Rightarrow \beta \cup \gamma \subset \theta_0(W) \cap \theta_1(W).$$

Assume the hypothesis. Trivially, $\beta \subset \theta_0(W)$. Also,

$$p \in \gamma \Rightarrow a_1 + p,\ a_0 - a_1 \in W \Rightarrow a_0 + p \in W \Rightarrow p \in \theta_0(W).$$

Thus $\beta \cup \gamma \subset \theta_0(W)$. Similarly one shows that $\beta \cup \gamma \subset \theta_1(W)$. Hence $\beta \cup \gamma \subset \theta_0(W) \cap \theta_1(W)$. Q.E.D.

Proposition P18. *Let α be an infinite α-repère. Let $\langle \beta, \gamma\rangle$ be an ordered pair of disjoint non-empty subsets of α_1. Then the following five conditions are mutually equivalent:*

(a) $\beta | \gamma$, (b) $\sigma || \tau$, (c) $\sigma \cup \tau$ *an α-repère*,
(d) $\mathrm{L}(\sigma \cup \tau)$ *an α-space*, (e) $\mathrm{L}(\sigma \cup \tau) || \mathrm{L}(a_0 - a_1)$.

Proof: Let $\alpha \subset \overline{\alpha}$. In view of P16, it suffices to show that (c)$\Rightarrow$(d)$\Rightarrow$(e)$\Rightarrow$(a). The first conditional is obvious. According to P9, $a_0 - a_1 \notin \mathrm{L}(\sigma \cup \tau)$. If (d) holds, we obtain (e) by P2. Finally, assume (e). Then there is a r.e. space $\overline{W}$ such that $\mathrm{L}(\sigma \cup \tau) \leqslant \overline{W}$ and $a_0 - a_1 \notin \overline{W}$. We conclude by P17 that $\overline{\theta}_0(\overline{W})$ and $\overline{\theta}_1(\overline{W})$ are disjoint r.e. sets. Moreover, $\beta \subset \overline{\theta}_0(\overline{W})$ and $\gamma \subset \overline{\theta}_1(\overline{W})$, hence $\beta | \gamma$ and (a) holds. Q.E.D.

Proposition P19. *Let $W \leqslant \overline{V}$, where $\overline{V}$ is an r.e. space, and* $\mathrm{codim}_{\overline{V}}(W)$ *finite. Then*

$$W \text{ is an } \alpha\text{-space} \Leftrightarrow W \text{ is an r.e. space}.$$

Proof: Only the $\Rightarrow$ conditional needs a proof. Let W be an α-space. Since $W \leqslant \overline{V}$, there is an α-basis β of W and an r.e. repère $\overline{\beta}$ such that $\beta \subset \overline{\beta} \subset \overline{V}$.

Then the fact that $\mathrm{codim}_{\overline{V}}(W)$ is finite implies that $\overline{\beta}\backslash\beta$ finite. This and the fact that $\overline{\beta}$ is r.e. implies β is r.e. Hence $W = \mathrm{L}(\beta)$ is an r.e. space. Q.E.D.

We present a new proof of the existence of non-α-spaces. Let Δ denote the family of all ordered pairs of disjoint non-empty subsets of α_1. For $\delta \subset \alpha_1$, δ infinite, we define

$$\Delta_\delta = \{\langle\beta, \gamma\rangle \in \Delta \mid \beta \cup \gamma = \delta\}.$$

Consider the mapping

$$\langle\beta, \gamma\rangle \to \mathrm{T}\langle\beta, \gamma\rangle = V, \text{ for } \langle\beta, \gamma\rangle \in \Delta_{\alpha_1}.$$

This mapping is 1-1 by P8. Since its domain has cardinality $\mathfrak{c}$, so has its range. By P10 the range consists of spaces of codimension 1 w.r.t. the space $\mathrm{Z}_{\alpha(1)}$. We now make the additional assumption that α_1 is r.e. Then $\mathrm{Z}_{\alpha(1)}$ is an r.e. space. Thus, for $V \in \mathrm{T}(\Delta_{\alpha(1)})$, V is r.e. if and only if V is an α-space by P19. However, there are only $\aleph_0$ r.e. spaces, and hence $\mathrm{T}(\Delta_{\alpha(1)})$ contains exactly $\mathfrak{c}$ non-α-spaces.

While this proof uses a cardinality argument, P18 enables us to characterize those $\langle\beta, \gamma\rangle \in \Delta_{\alpha_1}$ for which $\mathrm{T}\langle\beta, \gamma\rangle$ is an α-space. There are the ordered pairs $\langle\beta, \gamma\rangle$ such that $\beta|\gamma$; since α_1 is r.e., these are the ordered pairs $\langle\beta, \gamma\rangle$ such that β is r.e. and $\gamma = \alpha_1\backslash\beta$ is r.e. Thus, even in the simple case that β is r.e., but γ is not, $\mathrm{T}\langle\beta, \gamma\rangle$ is a non-α-space. In that case,

$$\mathrm{T}(\beta, \gamma) = \mathrm{L}(\sigma \cup \tau) = \mathrm{L}(\sigma) \oplus \mathrm{L}(\tau)$$

while $\mathrm{L}(\sigma)$ is r.e., but $\mathrm{L}(\tau)$ is not r.e.

More generally, consider the mapping $\langle\beta, \gamma\rangle \to \mathrm{T}\langle\beta, \gamma\rangle = V$, for $\langle\beta, \gamma\rangle \in \Delta_\delta$, where $\delta = \beta \cup \gamma$. For each of the $\mathfrak{c}$ choices of the infinite subset δ of α_1, we obtain a class of $\mathfrak{c}$ spaces V of the form $\mathrm{T}\langle\beta, \gamma\rangle$, with $\beta \cup \gamma = \delta$. Among these, $\aleph_0$ are α-spaces, namely those for which $\beta|\gamma$, and $\mathfrak{c}$ are non-α-spaces, namely those for which not $(\beta|\gamma)$. Let V_1, V_2 correspond to different choices of $\langle\beta, \gamma\rangle$ for a fixed $\delta = \beta \cup \gamma$. Then $V_1 \oplus \mathrm{L}(a_0 - a_1) = \mathrm{Z}_\delta$, and $V_2 \oplus \mathrm{L}(a_0 - a_1) = \mathrm{Z}_\delta$. Since $V_1 \neq V_2$, we have $V_1 + V_2 = \mathrm{Z}_\delta$. Thus, we have a family of $\mathfrak{c}$ distinct non-α-spaces all of which are subspaces of Z_δ with codum 1 w.r.t. Z_δ. According to P6, Z_δ is an r.e. space if and only if δ is an r.e. set.

Proposition P20. (a) *There are* $\mathfrak{c}$ *ordered pairs* $\langle\sigma, \tau\rangle$ *of independent* α-*repères which are separable by r.e. repères, but not by independent r.e. repères.*
(b) *There are* $\mathfrak{c}$ *ordered pairs* $\langle S, T\rangle$ *of independent* α-*spaces whose sum is not an* α-*space.*

Proof: Let $\langle\beta, \gamma\rangle$ be an ordered pair of non-empty subsets of α_1, which are disjoint but not separable, $\sigma = \sigma_\beta$, $\tau = \tau_\gamma$, $S = \mathrm{L}(\sigma)$, $T = \mathrm{L}(\tau)$. Then $\langle\sigma, \tau\rangle$ and $\langle S, T\rangle$ satisfy the requirements. Moreover, $\langle\beta, \gamma\rangle$ can be chosen in $\mathfrak{c}$ ways and the mappings $\langle\beta, \gamma\rangle \to \langle\sigma, \tau\rangle$, $\langle\beta, \gamma\rangle \to \langle S, T\rangle$ are 1-1. Q.E.D.

Proposition P21. *Let* α *be an infinite* α-*repère, and* $A = \mathrm{L}(\alpha)$. *Then*

(a) $Z_{\alpha(1)} \oplus L(a_0) = Z_{\alpha(1)} \oplus L(a_1) = A$,
(b) $Z_{\alpha(1)} || L(a_0)$ *and* $Z_{\alpha(1)} || L(a_1)$,
(c) $Z_{\alpha(1)} \leqslant_\alpha A$ *and* $Z_{\alpha(1)}$ *is an* α-*space.*

Proof: Let $\alpha \subset \bar{\alpha}$, where $\bar{\alpha}$ is an infinite r.e. repère. Recall that the 1-1 function a_n ranging over α, and the 1-1 recursive function $\bar{a}_n$ ranging over $\bar{\alpha}$ are chosen so that $\bar{a}_0 = a_0$ and $\bar{a}_1 = a_1$. Put $\bar{A} = L(\bar{\alpha})$. Suppose that σ_0, τ_0, $Z_{\alpha(1)}$ are defined w.r.t. α_1 and $\bar{\sigma}_0$, $\bar{\tau}_0$, $Z_{\bar{\alpha}(1)}$ are defined w.r.t. $\bar{\alpha}_1$. Then we have by P4 and the definitions of $Z_{\alpha(1)}$, $Z_{\bar{\alpha}(1)}$,

$$L(\sigma_0 \cup \tau_0) = Z_{\alpha(1)},\quad L(\bar{\sigma}_0 \cup \bar{\tau}_0) = Z_{\bar{\alpha}(1)},$$
$$L(\bar{\sigma}_0 \cup \bar{\tau}_0) \oplus L(a_0) = L(\bar{\sigma}_0 \cup \bar{\tau}_0) \oplus L(a_1) = \bar{A},$$
$$L(\sigma_0 \cup \tau_0) \oplus L(a_0) = L(\sigma_0 \cup \tau_0) \oplus L(a_1) = A.$$

Moreover, $L(\bar{\sigma}_0 \cup \bar{\tau}_0)$ is an r.e. space, while $Z_{\alpha(1)}$ is an α-space by P6. Note that the first part of (c) follows from (a) and (b). Thus it suffices to prove $L(\sigma_0 \cup \tau_0) \leqslant L(\bar{\sigma}_0 \cup \bar{\tau}_0)$. Since $\bar{a}_0 = a_0$, $\bar{a}_1 = a_1$, we have $\sigma_0 \subset \bar{\sigma}_0$, $\tau_0 \subset \bar{\tau}_0$, hence $L(\sigma_0 \cup \tau_0) \leqslant L(\bar{\sigma}_0 \cup \bar{\tau}_0)$. Q.E.D.

Proposition P22. *Let* α *be an infinite* α-*repère. Consider the* 1-1 *mapping* $\langle\beta, \gamma\rangle \to T\langle\beta, \gamma\rangle = V$, *with as domain the family of all decompositions* $\langle\beta, \gamma\rangle$ *of* α_1 *into non-empty sets. Then we have for each such ordered pair* $\langle\beta, \gamma\rangle$,

(a) $V \oplus L(a_0 - a_1) = Z_{\alpha(1)}$, $V \oplus L(a_0, a_1) = A$,
(b) *the following five conditions are mutually equivalent:*

$$\text{(i) } V || L(a_0, a_1),\quad \text{(ii) } V || L(a_0 - a_1),$$
$$\text{(iii) } V \leqslant_\alpha Z_{\alpha(1)},\quad \text{(iv) } V \leqslant_\alpha A,\quad \text{(v) } V \text{ is an } \alpha\text{-space.}$$

Proof: Part (a) follows from P9 and P4 (d). We prove part (b) by showing that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i). The first two conditionals are immediate. By P21 (c) we know that $Z_{\alpha(1)} \leqslant_\alpha A$. Thus, assuming (iii),

$$V \leqslant_\alpha Z_{\alpha(1)} \text{ and } Z_{\alpha(1)} \leqslant_\alpha A \Rightarrow V \leqslant_\alpha A$$

by P1. This proves (iii) $\Rightarrow$ (iv). According to P3,

$$V \leqslant_\alpha A \text{ and } A\ \alpha\text{-space and } \operatorname{codim}_A(V) \text{ finite} \Rightarrow V\ \alpha\text{-space.}$$

Thus (iv) $\Rightarrow$ (v) follows from the fact that A is an α-space and $\operatorname{codim}_A(V) = 2$. Finally, assume (v). By part (a) and P2, we have $V || L(a_0, a_1)$. Q.E.D.

Corollary C23. *For every* $\aleph_0$-*dimensional* α-*space* A, *there are* $\mathfrak{c}$ *non-*α-*spaces* V *such that* V *is a subspace of* A *of codimension* 2 *w.r.t.* A.

Remark. If in C23 we choose an isolic $\aleph_0$-dimensional α-space for A, we obtain Hamilton's result: there are exactly $\mathfrak{c}$ isolic non-α-spaces ([5], Theorem 5, p. 93).

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