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CRITICAL POINTS OF NORMAL FUNCTIONS. II

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Unless the contrary is made explicit, the notation and terminology of this present note will follow that in [1]. Perhaps the main difference lies in our concept of function; we are now more restrictive, and adopt the convention that all functions mentioned have domain ON. In the results that we are about to present,* the number 0 has the annoying habit of appearing as a special case to be considered with a good deal of frequency. We cannot eradicate this entirely, but can expedite matters somewhat by admitting 0 to the domain of the cofinality function cf, with the definition "$\mathrm{cf}(0) = 0$" (we do *not*, however, admit 0 to the class of regular ordinals). Thus we have $\mathrm{cf}(\alpha) \leqslant 1$ if and only if $\alpha = 0$ or $\alpha = \beta + 1$ for some β. By the prime component representation of an ordinal $\alpha > 0$, we mean the unique representation $\alpha = \rho_0 + \rho_1 + \ldots + \rho_n$, where each ρ_m is a prime component, and $\rho_m \geqslant \rho_{m+1}$ for $m < n$.

Let X be a class. We shall often enumerate X as (x_ξ), where the subscripts range over some ordinal (if X is a set) or over ON (if X is a proper class); in each case the subscript range will be clear from the context. Whenever such an enumeration is given, it will be assumed to be increasing.

Definition 1 A proper class $X = (x_\xi)$ is called "appropriate" if the following conditions are satisfied.

(1) If $x_0 \neq 0$, then $x_0 = \omega^\gamma$ for some γ such that $\mathrm{cf}(\omega^\gamma) = \omega$:
(2) For each ξ, $x_{\xi+1} - x_\xi = \omega^\gamma$ for some γ such that $\mathrm{cf}(\omega^\gamma) \leqslant \omega$.
(3) For each $\lambda \in \mathrm{LIM}^*$, $x_\lambda = \lim_{\xi<\lambda} x_\xi$.

We wish to show that for any class X, $X = \mathbf{CR}_f$ for some normal function f if and only if X is appropriate.

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Theorem 1 *Let f be a normal function, and let $\alpha \in \mathbf{CR}_f - \{0\}$ have prime component representation $\rho_0 + \ldots + \rho_n$, where $n > 0$. Then $\rho_0 + \ldots + \rho_{n-1} \in \mathbf{CR}_f$.*

Proof: Put $\beta = \rho_0 = \ldots + \rho_{n-1}$, and suppose that $f(\beta) > \beta$. Consider the set $Y = \{\delta < \beta;\ f(\delta) \geqslant \beta\}$; then $Y \neq \emptyset$, and Y is a final segment of β (considered of course as a well-ordered set). From the properties of prime components, it follows that if τ is the order-type of Y, then $\tau \geqslant \rho_{n-1}$, and hence $\tau \geqslant \rho_n$. But $f''Y$ is an initial segment of $Z = \{\delta < \alpha;\ \delta \geqslant \beta\}$, of order-type ρ_n, and so $f''Y = Z$. But then $\alpha \leqslant f(\beta) < f(\alpha)$, contradicting $\alpha \in \mathbf{CR}_f$. Thus we must have $f(\beta) = \beta$.

Theorem 2 *For any $\alpha \in \mathsf{ON}$, there is a normal function f with $\alpha = \mathsf{C}_f(0)$ if and only if either $\alpha = 0$ or $\alpha = \omega^\gamma$ for some γ with $\mathsf{cf}(\omega^\gamma) = \omega$.*

Proof: The case $\alpha = 0$ is trivial, since the identity function is obviously normal. Thus we assume henceforth that $\alpha \neq 0$.

The "if" part was actually proved in [1], but since it is short, we give the proof again for the sake of completeness. Thus let us suppose that $\alpha = \mathsf{C}_f(0)$ for some normal f. From Theorem 1 we obtain immediately that $\alpha = \omega^\gamma$ for some γ, and the case $\gamma = 0$ is ruled out because we are assuming $f(0) \geqslant 1$. This same assumption tells us that the ω-sequence $(f^n(0))$ is increasing, and since $\alpha = \mathsf{C}_f(0) = \mathsf{lim}_{n<\omega} f^n(0)$, we see that $\mathsf{cf}(\alpha) = \omega$. Let us now assume that $\alpha = \omega^\gamma$ for some γ with $\mathsf{cf}(\omega^\gamma) = \omega$, and construct a normal function f such that $\alpha = \mathsf{C}_f(0)$. If γ is successor, say $\gamma = \delta + 1$, the situation is easy; we simply define f by $f(\beta) = \omega^\delta + \beta$. Hence we may suppose that $\gamma \in \mathsf{LIM}^*$, whence from $\mathsf{cf}(\omega^\gamma) = \omega$ we deduce that $\mathsf{cf}(\gamma) = \omega$. Let (γ_n) be an increasing ω-sequence of successor ordinals such that $\gamma = \mathsf{lim}_{n<\omega}\gamma_n$, and for each n put $\alpha_n = \omega^{\gamma_n}$. Then (α_n) is an increasing ω-sequence of prime components such that $\mathsf{lim}_{n<\omega}\alpha_n = \alpha$ and $\mathsf{cf}(\alpha_n) = \omega$ for each n. From these properties of the α_n it follows that for each n there is an increasing α_n-sequence $(\theta^\circ_{\xi n})$ of ordinals such that $\alpha_n < \theta^\circ_{0n}$ and $\mathsf{lim}_{\xi<\alpha_n}\theta^\circ_{\xi n} = \alpha_{n+1}$. From this α_n-sequence we obtain a second one, $(\theta_{\xi n})$, by $\theta_{\zeta n} = \theta^\circ_{\zeta n}$ if $\mathsf{cf}(\zeta) \leqslant 1$, and $\theta_{\zeta n} = \mathsf{lim}_{\xi<\zeta}\theta_{\xi n}$ otherwise. It is clear that this α_n-sequence is increasing and continuous on its domain, and is such that $\alpha_n < \theta_{\zeta n}$ and $\mathsf{lim}_{\xi<\alpha_n}\theta_{\xi n} = \alpha_{n+1}$.

We now define our desired function f as follows. For $\beta \geqslant \alpha$ we put $f(\beta) = \beta$; we also put $f(\alpha_m) = \alpha_{m+1}$ for each m. Now take $\beta < \alpha$, and suppose that for no m do we have $\beta = \alpha_m$. Setting $p = \min\{n;\ \alpha_n > \beta\}$, we now put $f(\beta) = \theta_{\beta p}$. It is routine to check that f is normal and that $\mathsf{C}_f(0) = \alpha$.

Theorem 3 *Let X be a proper class of ordinals, $X = (x_\xi)$. Then $X = \mathbf{CR}_f$ for some normal function f if and only if X is appropriate.*

Proof: Suppose that $X = \mathbf{CR}_f$ for some normal function f. Theorem 2 tells us that Definition 1 (1) is satisfied. Take any ξ, and suppose that $x_{\xi+1} > x_\xi + 1$. Then we have $x_{\xi+1} = \mathsf{C}_f(x_\xi + 1) > x_\xi + 1$, and so $\mathsf{cf}(x_{\xi+1}) = \omega$. Let $x_{\xi+1}$ have prime component representation $\rho_0 + \ldots + \rho_n$; if $n = 0$, then we have $x_{\xi+1} - x_\xi = x_{\xi+1}$, and Definition 1 (2) is satisfied. On the other hand, if $n > 0$, then by Theorem 1 we have $x_\xi \geqslant \rho_0 + \ldots + \rho_{n-1}$, and so $x_{\xi+1} - x_\xi = \rho_n$. Since from $\mathrm{cf}(x_{\xi+1}) = \omega$ we conclude that $\mathrm{cf}(\rho_n) = \omega$, we see that Definition 1 (2) is still satisfied. Finally, take $\lambda \in \mathrm{LIM}^*$, and put $\alpha = \lim_{\xi<\lambda} x_\xi$. Continuity of f gives $\alpha \in \mathbf{CR}_f$, and so, as $\alpha > x_\xi$ for $\xi < \lambda$, $x_\lambda \leqslant \alpha$. But clearly $\alpha \leqslant x_\lambda$. Thus $\alpha = x_\lambda$, and Definition 1 (3) is satisfied. Hence X is appropriate.

We now suppose that X is appropriate, and at this stage we find it easier to break the convention stated on p. 20 and consider functions f_ξ with ordinals as domains. By Theorem 2 there is a normal function h with $\mathrm{C}_h(0) = x_0$; we put $f_0 = h \mid (x_0 + 1)$. Now take $\xi > 0$, and assume that for each $\zeta < \xi$, we have defined a normal function f_ζ such that:

(i) $\mathrm{dom}(f_\zeta) = x_\zeta + 1$;
(ii) $\mathbf{CR}_{f_\zeta} = \{x_\psi \in X;\ \psi \leqslant \zeta\}$;
(iii) $f_\zeta \subset f_\psi$ for $\zeta < \psi < \xi$.

Suppose that $\xi = \delta + 1$. If $x_\xi = x_\delta + 1$, then we put $f_\xi = f_\delta \cup \{(x_\xi, x_\xi)\}$. If $x_\xi > x_\delta + 1$, then we have $x_\xi - x_\delta = \omega^\gamma$ for some γ with $\mathrm{cf}(\omega^\gamma) = \omega$, and by Theorem 2 there is a normal function h with $\mathrm{C}_h(0) = \omega^\gamma$. We now put $f_\xi = f_\delta \cup \{(x_\delta + \alpha, x_\delta + h(\alpha));\ \alpha \leqslant \omega^\gamma\}$. It is easily seen that f_ξ is normal and extends our induction hypotheses. Secondly, suppose that $\xi \in \mathrm{LIM}^*$: this time we simply put $f_\xi = \bigcup\{f_\zeta;\ \zeta < \xi\} \cup \{(x_\xi, x_\xi)\}$, and once again we see that f has the required properties.

We define the desired function f by $f(\alpha) = f_\alpha(\alpha)$. It is routine to check that this definition is valid, that f is normal, and that $\mathbf{CR}_f = X$.

In [1] we considered the collection $\mathbf{CR}$ of all classes $\mathbf{CR}_f$ (f normal and with domain ON), and showed that the structure $\langle \mathbf{CR}, \cup, \cap \rangle$ is a lattice. We also exhibited an ω-sequence (X_n) of elements of $\mathbf{CR}$ such that $\bigcup\{X_n;\ n < \omega\} \notin \mathbf{CR}$, and an ω_1-sequence (Y_α) of elements of $\mathbf{CR}$ such that $\bigcap\{Y_\alpha;\ \alpha < \omega_1\} \notin \mathbf{CR}$. Despite this, however, it does turn out that the lattice $\mathbf{CR}$ is complete (in the ordinary sense; that is, we consider sups and infs of arbitrary *sets* of elements), and it is this result that we now wish to present.

Before so doing, however, we wish to recall Theorem 12 of [1]:

Let $\{A_i\}_{i\in I}$ *be a nonempty indexed set of elements of* $\mathbf{CR}$. *Then there is* $B \in \mathbf{CR}$ *such that* $B \subseteq \bigcap\{A_i;\ i \in I\}$.

Theorem 4 *The lattice* $\mathbf{CR} = \langle \mathbf{CR}, \cup, \cap \rangle$ *is complete.*

Proof: Let $\{X_i\}_{i\in I}$ be a nonempty indexed set of elements of $\mathbf{CR}$, and put $X^* = \bigcup\{X_i;\ i \in I\}$. As we have seen in [1], it is not necessarily true that $X^* \in \mathbf{CR}$; we claim, however, that $X \in \mathbf{CR}$ and $X = \sup\{X_i;\ i \in I\}$, where we define the class X by $X = \{\lim_{\xi<\zeta} x_\xi^*;\ \zeta \in \mathrm{ON} - \{0\}\}$; here of course (x_ξ^*) is the enumeration of X^*.

We must show first of all that X is a proper class and is appropriate. Since $x_\zeta^* = \lim_{\xi<\zeta+1} x_\xi^*$ for each ζ, we see at once that $X \supseteq X^*$, and is thus a proper class. As well, it is obvious that, where (x_ξ) is the enumeration of X, $x_0 = x_0^*$, and so X satisfies Definition 1 (1).

Now take any ξ; it is clear from the definition of X that $x_{\xi+1} \in X^*$ and thus $x_{\xi+1} \in X_j$ for some $j \in I$. Let $x_{\xi+1}$ have prime component representation $\rho_0 + \ldots + \rho_n$. If $n = 0$, then of course $x_{\xi+1} - x_\xi = \rho_0$. If $n > 0$, then $\rho_0 + \ldots + \rho_{n-1} \in X_j$, whence $x_\xi \geqslant \rho_0 + \ldots + \rho_{n-1}$, and so $x_{\xi+1} - x_\xi = \rho_n$. Thus we see that $x_{\xi+1} - x_\xi = \omega^\gamma$ for some γ. Suppose that $\mathrm{cf}(\omega^\gamma) > \omega$. Then we must have $x_{\xi+1} = \lim_{\zeta<\alpha} y_\zeta$ for some increasing α-sequence (y_ξ) of elements of X_j and some $\alpha \in \mathrm{LIM}^*$. Since this implies $y_\zeta > x_\xi$ for some $\zeta < \alpha$, a contradiction, we must have $\mathrm{cf}(\omega^\gamma) \leqslant \omega$. Thus Definition 1 (2) is satisfied.

As it is clear from the definition that X satisfies Definition 1 (3), we see that X is appropriate and hence $X \in \mathbf{CR}$.

Take $Y \in \mathbf{CR}$ with $X^* \subseteq Y$. It is easily seen that then we have $X \subseteq Y$. Thus $X = \sup\{X_i;\ i \in I\}$. We note at this stage that although we have made (implicit) use of the fact that I is nonempty, we have not used any "set" properties of it. Thus we may conclude that $\sup\{X_i;\ i \in I\}$ exists even if I is a proper class.

In this second part of our proof, however, we make essential use of the fact that I is a (nonempty) set. Put $A = \{Y \in \mathbf{CR};\ Y \subseteq X_i \text{ for all } i \in I\}$. By Theorem 12 of [1], $A \neq \emptyset$, and by our preceding remarks, $B = \sup A$ exists. Clearly if $Y \in \mathbf{CR}$ and $Y \subseteq X_i$ for all i, then $Y \subseteq B$. On the other hand, since $A \subseteq X_i \in \mathbf{CR}$ for all i, we must have $B \subseteq X_i$ for all i. Thus $B = \inf\{X_i;\ i \in I\}$.

Although $\mathbf{CR}$ is complete, results near the end of [1] show that we do not have $\bigcup\{X_i;\ i \in I\} = \sup\{X_i;\ i \in I\}$ or $\bigcap\{X_i;\ i \in I\} = \inf\{X_i;\ i \in I\}$ for every infinite I. Our one positive result occurs in the latter equality when I is countable. We conclude this paper with its proof.

Theorem 5 *Let (X_n) be an ω-sequence of elements of* $\mathbf{CR}$. *Then* $\bigcap\{X_n;\ n < \omega\} \in \mathbf{CR}$.

Proof: Put $X = \bigcap\{X_n;\ n < \omega\}$; since $X \supseteq \inf\{X_n;\ n < \omega\}$, X is a proper class. We show that $X = (x_\xi)$ is appropriate.

Clearly X satisfies Definition 1 (3), and the by-now-familiar argument shows that $x_0 = 0$ or ω^γ for some γ and that $x_{\xi+1} - x_\xi = \omega^\gamma$ for some γ: indeed, these facts in no way depend upon (X_n) being a countable sequence.

It thus suffices to verify the sections of Definition 1 that refer to cofinality, and so we show that if $x_{\xi+1} - x_\xi = \omega^\gamma$, then $\mathrm{cf}(\omega^\gamma) \leqslant \omega$. The case of x_0 will be seen to be a particular instance of this.

Suppose that $\gamma \neq 0$; we must show that $\mathrm{cf}(\omega^\gamma) = \omega$. For each n, let f_n be a normal function with $X_n = \mathbf{CR}_{f_n}$, and put $\alpha = x_\xi + 1$. We define the set Y of ordinals by $Y = \{f_{m_0} f_{m_1} \ldots f_{m_p}(\alpha);\ p, m_0, m_1, \ldots, m_p < \omega\}$. Clearly $\beta < x_{\xi+1}$ for all $\beta \in Y$, and so $\tau = \sup Y \leqslant x_{\xi+1}$. We claim that $\tau \in X$. This is obvious if $\tau \in Y$, and so we may assume $\tau \notin Y$, whence there is an increasing ω-sequence (τ_n) of elements of Y with $\tau = \lim_{n<\omega} \tau_n$. Now take any m; then $f_m(\tau_n) \in Y$ and so $f_m(\tau_n) < \tau$. Thus by normality $f_m(\tau) \leqslant \tau$, whence $f_m(\tau) = \tau$. Therefore $\tau \in X$. This shows, however, that $\tau \notin Y$. Also, as $x_\xi < \tau \leqslant x_{\xi+1}$, we must have $\tau = x_{\xi+1}$. Thus $x_{\xi+1} = \lim_{n<\omega} \tau_n$, and so $\mathrm{cf}(x_{\xi+1}) = \omega$. This of course implies that $\mathrm{cf}(\omega^\gamma) = \omega$, which completes the proof of our result.

We note in passing that the lattice $\mathbf{CR}$ is atomless; the proof is almost trivial.

REFERENCE

[1] Hickman, J. L., "Critical points of normal functions, I," *Notre Dame Journal of Formal Logic*, vol. XVIII (1977), pp. 527-534.

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