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ISOMORPHISM TYPES OF THE HYPERARITHMETIC SETS H_a

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Introduction Historically, this paper originates from M. Davis' result [1] that for $|a| = |b| < \omega^2$ $(a, b \in O)$, H_a and H_b are recursively isomorphic. Spector, in [10], showed that H_a and H_b for $|a| = |b|$ have the same Turing degree. Y. Moschovakis in [6], had shown that these results are best possible in that the sets H_a for $|a| = \alpha$, $\omega^2 \leqslant \alpha$ a principal number for addition, have well-ordered sequences of type ω_1 under one-one reducibility and, also, incomparable one-one degrees. The author in his thesis [8] showed that any countable ordered set can be embedded in the one-one ordering of H_a, $|a| = \alpha$ as above and that there are incomparable one-one degrees below any H_a, if $|a| = \alpha \geqslant \omega^3$. Moschovakis also has shown that if $\beta = \xi + \alpha$, α principle for addition, that $\{\mathsf{H}_b\colon |b| = \beta\}$ has the same structure under one-one reducibility as does $\{\mathsf{H}_a\colon |a| = \alpha\}$. This carries over to this paper after Theorem 1.1 and we restrict ourselves to those H_a such that $|a|$ is principle for addition, i.e., $|a| = \omega^\beta$ for some $\beta \geqslant 1$.

In this paper we introduce a general notion of one-one reducibility applicable to the hyperarithmetic sets (since these sets are cylinders, [9], pp. 89-90, we need only to discuss one-one reducibility). The notion is simply the following; suppose a, $b \in O$ and $|a| = |b|$, when is there a one-one function $f(x)$ recursive in H_c such that $x \in \mathsf{H}_a$ iff $f(x) \in \mathsf{H}_b$? Since H_a and H_b have the same Turing degree, clearly any $c \in O$, $|c| \geqslant |a|$ is sufficient. The question we try to answer is how small can $|c|$ be chosen in general, so that H_a and H_b are one-one reducible to each other by functions recursive in H_c, i.e., H_a and H_b are isomorphic via a permutation of $\mathcal{N}$ recursive in H_c. Alternatively, for $|c| < |a|$, H_c can be viewed as a constructive subset of both H_a and H_b and using only an oracle for H_c can one show a question of membership in H_a is equivalent to a question of membership in H_b (this is similar to a "bounded truth-table" reduction except that the bound is H_c). We will give a necessary and sufficient condition on the size of $|c|$ in order to show H_a, H_b are isomorphic by a permutation of $\mathcal{N}$ recursive in H_c when $|a| < \omega^{\omega^2}$. That this condition is sufficient for all a, $b \in O$ is demonstrated. However, the necessity of this condition for $|a| \leqslant \varepsilon_0$ is not proven and

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contrary to as announced in [7] is an open question. The author hopes that the techniques introduced here will eventually demonstrate this necessity. Basically these results depend upon constructing ordinal notations $a \in O$ with very fast growth toward its limit $|a|$ as in [5] and [8]. Consequences about ordinal notations will follow immediately from these results. For example, there exist recursive well-orderings of order type ω^ω which are not isomorphic via any function recursive in $O^{(n)}$, i.e., via any arithmetic permutation of $\mathcal{N}$.

1 *One-one reducibility in* H_c The notation used in this paper will be that found in [2], [4], and [10]. Familiarity with the results and techniques of recursion theory are assumed as in [9]. Frequent use is made of Post's Theorem which is taken to refer to the results listed on pp. 314-15 of [9].

Definition 1.1 We say that a set A is *one-one reducible in* C to B and write $A \leqslant_1^C B$ if there is a function f (one-one) recursive in C such that $x \in A$ iff $f(x) \in B$.

These definitions are natural generalizations of the usual notion of one-one reducibility and become particularly relevant in the study of the one-one degrees in C of the hyperarithmetic sets $\mathsf{H}_{a'}$ where $a' = 3 \cdot 5^a \in O$. The following definition and theorem generalizes the notion and results introduced by Y. Moschovakis in [6] in studying the one-one degrees of $\mathsf{H}_{a'}$.

Definition 1.2 Let $a' = 3 \cdot 5^a$ and $b' = 3 \cdot 5^b$ be two Kleene notations for ordinals such that $|a'| = |b'|$. We say that a' *is recursively majorized in* C by b' and write $a' \prec^C b'$ if there is a function f recursive in C such that $|a_n| \leqslant |b_{f(n)}|$ for all n.

Theorem 1.1 *For* $c \in O$, $|c| < |a'| = |b'|$

$$\mathsf{H}_{a'} \leqslant_1^{\mathsf{H}_c} \mathsf{H}_{b'} \textit{ iff } a' \prec^{\mathsf{H}_c} b'.$$

Proof: The proof of this result is essentially as in [6] except that in Lemma 2b, p. 330, one asserts instead that there is a primitive recursive $\sigma_3(e)$ such that if $t = \sigma_3(e)$ and $(\mathrm{E}z)(T_1^P(e, t, z) \wedge U(z) = k)$, then $P'(t) \not\equiv P(k)$; which is just an effective way of saying P' is not many-one reducible in P to P.

By Myhill's Theorem [9], the following is evident:

Corollary 1.1 $\mathsf{H}_{a'}$ *and* $\mathsf{H}_{b'}$ *are isomorphic using a permutation recursive in* H_c $|c| < |a'| = |b'|$ *iff* $a' \prec^{\mathsf{H}_c} b'$ *and* $b' \prec^{\mathsf{H}_c} a'$.

It follows automatically from Moschovakis' work that we need only study one-one reducibility in H_c of $\mathsf{H}_{a'}$ such that $|a'|$ is a principle number for addition. The following definition and lemmas generalize the notion of "limit point of order n," see p. 51 of [5], to any order α (α constructive). We use the predicate $C(b)$ of [3], §12 and §13, in order to express $z <_0 x$ as an r.e. predicate noting that for $a, b \in O$, $a \in C(b)$ iff $a <_0 b$, and there is a primitive recursive predicate $V(a, b, x)$ such that for any numbers a, b, $a \in C(b)$ iff $(\mathrm{E}x)\, V(a, b, x)$.

Definition 1.3 We define predicates $L_b(x)$ for each $b \in O$, $b \neq 1$ inductively as follows:

$L_{2^1}(x)$ if $x = 3 \cdot 5^y$
$L_{2^b}(x)$ if $L_b(x) \wedge (z)(z \in C(x) \rightarrow (\mathrm{E}w)(z \in C(w) \wedge w \in C(x) \wedge L_b(w)))$
$L_{3 \cdot 5^y}(x)$ if $(n)(L_{y_n}(x))$.

Below $A^{(n)}$ refers to the jump operator applied n times to A.

Lemma 1.1 (a) *If* $b \in \dot{O}$, $|2^b| = k > 0$ *is finite, then* $L_{2^b}(x)$ *is one-one reducible to* $\overline{\mathrm{O}^{(2(k-1))}}$ (*for* $k = 1$, $L_{2^1}(x)$ *is recursive*).

(b) *There is a primitive recursive function* $f(a, x)$ *such that for* $a = 3 \cdot 5^y \in O$,

$$L_a(x) \text{ iff } f(a, x) \in \overline{\mathsf{H}'_a}.$$

(c) *If* $b = 3 \cdot 5^y +_0 c \in O$, $|c| = k \neq 0$, *then* $L_b(x)$ *is* 1-1 *reducible to* $\overline{\mathsf{H}^{(2k+1)}_{3 \cdot 5^y}}$.

Proof: Clearly, $L_{2^1}(x)$ is recursive. Consider $L_{2^2}(x) \equiv L_{2^1}(x) \wedge (z)((\mathrm{E}x_1)V(z, x, x_1) \rightarrow \mathrm{E}w(\mathrm{E}x_2\, V(z, w, x_2) \wedge \mathrm{E}x_3\, V(w, x, x_3) \wedge L_2(w))$ which is equivalent to a predicate of $\forall E$-form and consequently is 1-1 reducible to $\overline{\mathrm{O}''}$. Let e_0 be a primitive recursive index of this reduction of $L_{2^2}(x)$ to $\overline{\mathrm{O}''}$. Part (a) follows inductively by Post's Theorem as in [9].

We complete the proof by defining a primitive recursive function $f(y, x)$ such that for y in O, $y \neq 1, 2$, $L_y(x)$ iff $f(y, x) \in \overline{\mathsf{H}_{\gamma(y)}}$, where $\gamma(y)$ is as specified in the result, i.e., $\gamma(a) = b$ such that $|b| = 2(k - 1)$ if $|a| = k$ finite, $a \neq 1, 2$, and $\gamma(a) = 3 \cdot 5^y +_0 b$ if $a = 3 \cdot 5^y +_0 c$ for some finite c in O and $|b| = 2 \cdot |c| + 1$. By Post's Theorem let $g(u, x)$ be a 1-1 primitive recursive function such that uniformly for A, $g(u, x) \in \overline{A^{(3)}}$ iff $\{u\}(x) \notin A' \wedge (r)(r \in C(x) \rightarrow (\mathrm{E}w)(r \in C(w) \wedge w \in C(x) \wedge \{u\}(w) \notin A'))$. Now define $\phi(z, y, x)$ primitive recursively as follows:

$$\begin{aligned}
\phi(z, y, x) &= 0 \text{ if } y = 1 \vee y = 2 \vee (y \neq 2^{(y)_0} \wedge y \neq 3 \cdot 5^{(y)_2}).\\
\phi(z, 2^2, x) &= \{e_0\}(x) \text{ where } e_0 \text{ is obtained in Part (a).}\\
\phi(z, 2^b, x) &= g((\lambda x)\{z\}(b, x), x) \text{ if } b \neq 2.\\
\phi(z, 3 \cdot 5^y, x) &= \phi_1(z, y, x), \text{ where } \phi_1 \text{ is the primitive}\\
&\text{recursive function defined as follows:}
\end{aligned}$$

Consider $\neg(n)(\{z\}(y_n, x) \in \overline{\mathsf{H}_{\gamma(y_n)}}) \equiv (\mathrm{E}n)(\{z\}(y_n, x) \in \mathsf{H}_{\gamma(y_n)} \equiv (\mathrm{E}n)\, \rho(\gamma(y_n), 3 \cdot 5^y, \{z\}(y_n, x)) \in \mathsf{H}_{3 \cdot 5^y}$, where ρ is the partial recursive function of Lemma 3, p. 326 of [4], $\equiv \mathrm{E}n(\mathrm{T}_1^{\mathsf{H}_{3 \cdot 5^y}}(\phi_1(z, y, x), \phi_1(z, y, x), n)$, by Lemma 1, p. 325 of [4] where ϕ_1 is primitive recursive. Consequently,

$$(n)(\{z\}(y_n, x) \in \overline{\mathsf{H}_{\gamma(y_n)}}) \equiv (n)(\overline{\mathrm{T}}_1^{\mathsf{H}_{3 \cdot 5^y}}(\phi_1(z, y, x), \phi_1(z, y, x), n)).$$

By the Recursion Theorem, p. 352-3 of [2], there is an e such that $\phi_e(y, x) \cong \phi(e, y, x)$. Define $f(y, x) = \phi(e, y, x)$ and by the construction of f, it follows for all $y \in O$, $y \neq 1, 2$, $L_y(x)$ iff $f(y, x) \in \overline{\mathsf{H}_{\gamma(y)}}$, by induction on $|y|$ in O.
Q.E.D.

As is well known, every ordinal α has a unique Cantor Normal Form, i.e., $\alpha = \omega^{\beta_n} \cdot k_n + \ldots + \omega^{\beta_0} \cdot k_0$ such that $\beta_i > \beta_{i-1}$, for $1 \leqslant i \leqslant n$, and $0 \neq k_i < \omega$ for $0 \leqslant i \leqslant n$ with n, β_i, k_i uniquely determined by α.

Lemma 1.2 *For a', $b \epsilon O$, $b \neq 1$, $L_b(a')$ iff the Cantor Normal Form of $|a'| = \omega^{\beta_n} \cdot k_n + \ldots + \omega^{\beta_0} \cdot k_0$ is such that $\beta_0 \geqslant |b|$.*

Proof: By induction on $|b|$ for all a' in O, for $|b| = 1$ the result is clear. Suppose the result is true for all $b \epsilon O$ such that $\beta = |b| \geqslant 1$, consider $b = 2^{(b)_0}$ such that $|b| = \beta + 1$ and suppose $L_b(a')$. By definition, $L_b(a') \equiv (z)(\mathrm{E}w)(z <_0 a' \rightarrow z <_0 w <_0 a' \wedge L_{(b)_0}(w)) \wedge L_{(b)_0}(a')$. Thus, $\beta_0 \geqslant |(b)_0| = \beta \geqslant 1$ and, by the inductive hypothesis, $a' = 3 \cdot 5^a$. $|a'| = \lim |a_n|$ and, clearly, since $L_b(a')$ there exist a sequence $w_i <_0 w_{i+1} <_0 a'$ such that $L_{(b)_0}(w_i)$ and $\lim |w_i| = |a'|n$. For some k, $i > k$ implies $|w_i| = \omega^{\beta_0} \cdot k_n + \ldots + \omega^{\beta_1} \cdot k_1 + \omega^{|(b)_0| + \alpha_i} \cdot k_0^i$. Either for some j, $|(b)_0| + \alpha_i = |(b)_0|$ for all $i \geqslant j$ or $|(b)_0| + \alpha_i > |(b)_0|$ for all $i \geqslant j$ and, hence, $\omega^{\beta_0} = \lim \omega^{|(b)_0|} \cdot k_0^i = \omega^{|(b)_0|+1}$ or $\omega^{\beta_0} \geqslant \sup \omega^{|(b)_0| + \alpha_i} \geqslant \omega^{|(b)_0| + 1}$. Conversely, suppose $\beta_0 \geqslant |(b)_0| + 1$, if $\beta_0 = \gamma + 1$, then $\omega^{\beta_0} \cdot k_0 = \lim(\omega^{\beta_0} \cdot (k_0 - 1) + \omega^{\gamma} \cdot n)$ and there exists a sequence $w_i <_0 a'$ such that $|w_i| = \omega^{\beta_n} \cdot k_n + \ldots + (\omega^{\beta_0} \cdot (k_0 - 1) + \omega^{\gamma} \cdot i)$ and $\lim |w_i| = |a'|$. If β_0 is a limit, then there is a sequence γ_n such that $|(b)_0| < \gamma_n$ and $\lim \gamma_n = \beta_0$ and consequently we can find $w_i <_0 a'$ such that $\lim |w_i| = a'$ and $|w_i| = \omega^{\beta_n} \cdot k_n + \ldots + \omega^{\beta_0} \cdot (k_0 - 1) + \omega^{\gamma_i} \cdot k_0^i$. Thus, $L_b(a')$ follows.

Suppose $b' = 3 \cdot 5^b$ and $L_{b'}(a')$ is true. Thus, $(n)(L_{b_n}(a'))$ is true. Consequently, by inductive hypothesis, $\beta_0 \geqslant |b_n|$, hence, $\beta_0 \geqslant \lim |b_n| = |b'|$. Conversely, suppose $\beta_0 \geqslant |b'|$, then $\beta_0 > |b_n|$ for every n and consequently $L_{b_n}(a')$ holds for all n.

Now we prove one of the main results of this paper.

Theorem 1.2 (a) *If $|a'| = |b'| = \omega^{\beta}$, β a limit, then $\mathrm{H}_{a'}$ is isomorphic to $\mathrm{H}_{b'}$ by a permutation of $\mathcal{N}$ recursive in H_c such that $|c| = \beta$.*

(b) *If $|a'| = |b'| = \omega^k$, $2 \leqslant k$ finite, then $\mathrm{H}_{a'}$ is isomorphic to $\mathrm{H}_{b'}$ by a permutation of $\mathcal{N}$ recursive in $\mathrm{O}^{(2 \cdot (k-2)+1)}$.*

(c) *If $|a'| = |b'| = \omega^{\gamma + k}$, $1 \leqslant k$ finite, and γ a limit ordinal, then $\mathrm{H}_{a'}$ is isomorphic to $\mathrm{H}_{b'}$ by a permutation of $\mathcal{N}$ recursive in H_c where $|c| = \gamma + 2k$.*

Proof: Suppose the hypothesis of part (a) and that $c \epsilon O$ and $|c| = \beta$. By Theorem 1.1, it is sufficient to show $a' \prec^{\mathrm{H}_c} b'$ and $b' \prec^{\mathrm{H}_c} a'$. Thus, we shall define a function g recursive in H_c such that $|a_i| \leqslant |b_{g(i)}|$ for all i. For each i, let n_i be the smallest number n such that $(z)(z \leqslant_0 a_i \rightarrow \neg L_{c_n}(z))$ where L_{c_n} is the predicate of the definition before Lemma 1.1. Consequently, by Lemma 1.2, the Cantor Normal Form for $|a_i|$ and any $\alpha < |a_i|$ is such that $\beta_0 < |c_{n_i}|$ and, hence, $|a_i| < \omega^{|c_{n_i}|}$. By Lemma 1.1, n_i is the smallest number n such that $(z)(z \leqslant_0 a_i \rightarrow f(c_n, z) \epsilon \overline{\mathrm{H}_{\gamma(c_n)}})$ for the primitive recursive $f(y, x)$ which is equivalent to $\phi(e, a_i, c_n) \epsilon \overline{\mathrm{H}'_{\gamma(c_n)}}$, where e is a g.n. of f and $\phi(e, a_i, c_n)$ is primitive recursive. Consequently, $n_i = \mu n \phi(e, a_i, c_n) \epsilon \overline{\mathrm{H}'_{\gamma(c_n)}}$, and clearly n_i as a function of i is recursive in H_c. Given n_i, start enumerating the elements $<_0 b'$ until one finds the first element z_i in this enumeration such that $L_{c_{n_i}}(z_i)$, i.e., $f(c_{n_i}, z_i) \epsilon \overline{\mathrm{H}_{\gamma(c_{n_i})}}$. Define $g(i) = \mu j z_i \leqslant_0 b_j$. Thus, for each i, $|a_i| < \omega^{|c_{n_i}|} \leqslant |z_i| \leqslant |b_{g(i)}|$, where g is recursive in H_c. Consequently, $a' \prec^{\mathrm{H}_c} b'$ and by the analogous argument, $b' \prec^{\mathrm{H}_c} a'$. Thus, (a) holds by Corollary 1.1.

Suppose the hypothesis of (b) as in (a) we show that for arbitrary $a', b' \epsilon \bar{O}$, $|a'| = |b'| = \omega^k$ implies $a' \prec^{\mathsf{H}_c} b'$ where $c \epsilon O$ and $|c| = 2(k - 2) + 1$. Note that $k = 1$ implies $\mathsf{H}_{a'}$ and $\mathsf{H}_{b'}$ have the same one-one degree by [1]. Let $c \epsilon O$ such that $|c| = k - 1$, then by Lemma 1.1 $L_c(z)$ is one-one reducible to $\overline{\mathsf{H}_{\gamma(c)}} = \mathrm{O}^{(2(k-2))}$ (recursive if $|c| = 1$) and by Lemma 1.2, for $z <_0 a'$, $L_c(z)$ is true iff $|z| = \omega^{(k-1)} \cdot n$ for some $n \neq 0$, since $|a'| = \omega^k$. For each i, let n_i be the number of z's, $z \leqslant_0 a_i$ such that $L_c(z)$. We compute n_i recursively in $\mathrm{O}^{(2(k-2)+1)}$ as follows: First, consider $\mathrm{E}z(z <_0 a_i \wedge L_c(z))$ (this is equivalent to $\mathrm{E}z(z <_0 a_i \wedge f(c, z) \notin \mathrm{O}^{(2(k-2))}))$ which is equivalent to $\phi(c, a_i) \epsilon \mathrm{O}^{(2\cdot(k-2))+1}$ for a primitive recursive function ϕ. If the answer is $\phi(c, a_i) \notin \mathrm{O}^{(2\cdot(k-2)+1)}$, then $n_i = 0$ if $\neg L_c(a_i)$ and $n_i = 1$ if $L_c(a_i)$. If the answer is $\phi(c, a_i) \epsilon \mathrm{O}^{(2\cdot(k-2)+1)}$, then let z_1 be the first element in the enumeration of $z <_0 a_i$ such that $L_c(z_1)$. Suppose now we have defined $z_1, \ldots, z_k$ by this procedure $k \geqslant 1$. Consider $\mathrm{E}z(z \neq z_1 \wedge \ldots \wedge z \neq z_k \wedge z <_0 a_i \wedge L_c(z))$, if true, then take z_{k+1} to be the first element of the enumeration of all $z <_0 a_i$ different from $z_1, \ldots, z_k$ such that $L_c(z_{k+1})$, otherwise, $n_i = k$, if $\neg L_c(a_i)$, or $n_i = k + 1$ if $L_c(a_i)$. Clearly, n_i is defined recursively in $\mathrm{O}^{(2\cdot(k-2))+1}$ and $|a_i| \leqslant \omega^{n_i}$. Given n_i find the first $n_i + 1$ numbers $x_1, \ldots, x_{n_i+1}$ in the enumeration of $x <_0 b'$ such that $L_c(x_k)$, $k = 1, \ldots, n_i + 1$, (this computation is recursive in $\mathrm{O}^{2\cdot(k-2)}$ given that $|b'| = \omega^k$). Define $g(i) = \mu j$ such that $x_k \leqslant_0 b_j$ for $k = 1, \ldots, n_i + 1$. Clearly, g is recursive in $\mathrm{O}^{(2\cdot(k-2))+1}$ and $|a_i| \leqslant \omega^{n_i} < \omega^{n_i+1} \leqslant |b_{g(i)}|$. Thus, $a' \prec^{\mathsf{H}_d} b'$ where $|d| = 2\cdot(k - 2) + 1$, and analogously $b' \prec^{\mathsf{H}_d} a'$. Consequently, by Corollary 1.1, $\mathsf{H}_{a'}$ and $\mathsf{H}_{b'}$ are isomorphic by a permutation of $\mathcal{N}$ recursive in H_d, $|d| = 2\cdot(k - 2) + 1$.

The proof of (c) is completely analogous to the proof of (b). Noting that for $|c| = \omega^{\beta+(k-1)}$, $L_c(z)$ is 1-1 reducible to $\overline{\mathsf{H}_{\gamma(c)}}$, i.e., $\overline{\mathsf{H}_\beta^{2(k-1)+1}} = \overline{\mathsf{H}_\beta^{2k-1}}$.

Q.E.D.

Moschovakis notes in [6] that with respect to 1-1 reducibility there is a minimum one-one degree of the form $\mathsf{H}'_{a'}$, $|a'| = \omega^2$, in the Turing degree. The following shows this to be a rather general phenomenon.

Corollary 1.2 *If* $|a'| = |b'| = \omega^{\beta+k}$, β *a limit and* $k \neq 0$ *or* $\beta = 0$ *and* $k \geqslant 2$, *then there is a minimal* 1-1 *degree* $\mathsf{H}_{a'}$ *in* H_c *in the Turing degree of* $\{\mathsf{H}_{b'}: |b'| = \omega^{\beta+k}\}$ *where* $|c| = 2\cdot(k - 2)$ *if* $\beta = 0$ *and* $k \geqslant 2$ *and where* $|c| = \beta + 2k - 1$ *if* $\beta \neq 0$.

Proof: As noted in the proof of Theorem 1.2 (b), once one had found n_i for each i, the rest of the computation can be carried out at the next lower level. Consequently, let $d \epsilon O$ such that $|d| = \omega^{\beta+(k-1)}$ and $a_i = (d +_0 d) +_0 \ldots +_0 d)$ with i summands d and consequently for any $e \epsilon O$, $|e| = \beta + (k - 1)$, the number n_i of elements $z \leqslant_0 a_i$ such that $L_e(z)$ equals i. Consequently, $a' \prec b'$ via a function recursive in $L_e(z)$ and the result follows by Theorem 1.1 and Lemma 1.1.

2 *Natural well-orderings* In this section we define from predicates $R(x_1, y_1, \ldots, x_k, y_k)$ recursive in a set A sets e_k^A of k-tuples of $\mathcal{N}$ recursive in A ordered by first difference such that the size of these well-orderings depends upon whether or not $(x_1)(\mathrm{E}y_1) \ldots (x_k)(\mathrm{E}y_k) R(x_1, y_1, \ldots, x_k, y_k)$ is true or false. Instead of working with the quantifier (Uz) of [5], we require uniqueness on existential quantifiers which will lead to analogous results. However, Lemma 4 of [5], which is quite sufficient to obtain the results in section **3** for arithmetic predicates up through $\mathbf{H}_\omega$, we know of no way of generalizing to obtain all the results of section **3** below.

Lemma 2.1 *Given a predicate $R(y, z)$ we can effectively find predicates $S_1(y, z)$ and $S_2(y, z)$ recursive uniformly in $R(y, z)$ such that*

(a) $(\mathrm{E}y)(z)R(y, z)$ *iff* $(\mathrm{E}!y)(z)S_1(y, z)$

and

(b) $(y)(\mathrm{E}z)R(y, z)$ *iff* $(y)(\mathrm{E}!z)S_2(y, z)$.

Proof: Define $S_1(y, z)$ to be $y = 2^{(y)_0}\cdot 3^{(y)_1} \wedge (w)(w \leqslant \max((y)_1, z) \rightarrow R((y)_0, w)) \wedge (t)(\mathrm{E}w)(t < (y)_0 \rightarrow w \leqslant (y)_1 \wedge \neg R(t, w)) \wedge \neg(t)(\mathrm{E}w)(t < (y)_0 \rightarrow w < (y)_1 \wedge \neg R(t, w))$. Define $S_2(y, z)$ to be $R(y, z) \wedge (w)(w < z \rightarrow \neg R(y, w))$.

Lemma 2.2 *Given a predicate of the form $R(x_1, y_1, \ldots, x_n, y_n)$ we can effectively find a predicate $S(x_1, y_1, \ldots, x_n, y_n)$ recursive uniformly in R such that*

$$(x_1)(\mathrm{E}y_1)\ldots(x_n)(\mathrm{E}y_n)R \textit{ iff } (x_1)(\mathrm{E}!y_1)\ldots(x_n)(\mathrm{E}!y_n)S.$$

Proof: By induction on $n \geqslant 1$ with $n = 1$ by Lemma 2.1 (b), consider $((\mathrm{E}y_2)(x_3)\ldots(x_n)(\mathrm{E}y_n)R(x_1, y_1, \ldots, x_n, y_n))$. By Lemma 2.1 (a), there is predicate $S_1(x_1, y_1, x_2)$ recursive uniformly in $(\mathrm{E}y_2)(x_3)\ldots(x_n)(\mathrm{E}y_n)R(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$ such that $\mathrm{E}!y_1(x_2)S_1(x_1, y_1, x_2)$ iff $(\mathrm{E}y_1)(x_2)\ldots(x_n)(\mathrm{E}y_n)$ $R(x_1, y_1, \ldots, x_n, y_n)$ and, hence, $(x_1)(\mathrm{E}!y_1)(x_2)S_1(x_1, y_1, x_2)$ iff $(x_1)(\mathrm{E}y_1)\ldots$ $(x_n)(\mathrm{E}y_n)R(x_1, y_1, \ldots, x_n, y_n)$. Since $S_1(x_1, y_1, x_2)$ is recursive uniformly in $(\mathrm{E}y_2)(x_3)\ldots(x_n)(\mathrm{E}y_n)R$, then by Post's Theorem,

$$S_1(x_1, y_1, x_2) \equiv (x_2')(\mathrm{E}y_2)\ldots(x_n)(\mathrm{E}y_n)R_1(x_1, y_1, x_2, x_2', y_2, \ldots, x_n, y_n)$$

with R_1 recursive uniformly in R. By inductive hypothesis, there is a predicate $S_2(x_1, y_1, x_2, x_2', y_2, \ldots, x_n, y_n)$ recursive uniformly in R_1 and, hence, in R such that

$$S_1(x_1, y_1, x_2) \equiv (x_2')(\mathrm{E}!y_2)\ldots(x_n)(\mathrm{E}!y_n)S_2(x_1, y_1, x_2, x_2', y_2, \ldots, x_n, y_n).$$

Define $S(x_1, y_1, \ldots, x_n, y_n)$ to be $S_2(x_1, y_1, (x_2)_0, (x_2)_1, y_2, \ldots, x_n, y_n)$ and, consequently,

$$(x_2)S_1(x_1, y_1, x_2) \equiv (x_2)(\mathrm{E}!y_2)\ldots(x_n)(\mathrm{E}!y_n)S(x_1, y_1, x_2, y_2, \ldots, x_n, y_n).$$

Thus,

$$\begin{aligned}(\mathrm{E}y_1)(x_2)\ldots(\mathrm{E}y_n)(x_n)R(x_1, y_1, \ldots, x_n) &\equiv \mathrm{E}!y_1(x_2)S_1(x_1, y_1, x_2)\\ &\equiv (\mathrm{E}!y_1)(x_2)(\mathrm{E}!y_2)\ldots(x_n)(\mathrm{E}!y_n)S.\end{aligned}$$

Thus, the result follows easily.

Definition 2.1 A subset e of $\mathcal{N}^k$ belongs to W_k^R if e is recursive in R and the following four conditions hold:

(i) $(z_1, \ldots, z_k) \in e$ implies for all $i \leqslant k$, $z_i = 2^x \cdot 3^y$ for some x and y.

(ii) For every x, there is at most one y such that $|\{(2^x \cdot 3^y, z_2, \ldots, z_k) \in e\}| = \omega^{k-1}$, here $||$ means order type by first difference.

(iii) For every x and y $\{(z_2, \ldots, z_k)\colon (2^x \cdot 3^y, z_2, \ldots, z_k) \in e\} \in W^R_{k-1}$.

(iv) If for x and y $|\{(2^x \cdot 3^y, z_2, \ldots, z_k) \in e\}| = \omega^{k-1}$, then for every $z < x$, there is a $y_z < y$ such that

$$|\{(2^z \cdot 3^{y_z}, z_2, \ldots, z_k) \in e\}| = \omega^{k-1}.$$

Definition 2.2 For $e_1, e_2, \ldots, e_k \in W^R_n$, define $\circledast(e_1, \ldots, e_k) = \{(2^{x_1} \cdot 3^{y_1}, \ldots, 2^{x_n} \cdot 3^{y_n})\colon \mathsf{seq}(y_i) \wedge \mathsf{lh}(y_i) = k \wedge$ for all $i \leqslant k - 1\, (2^{x_1} \cdot 3^{(y_1)_i}, \ldots, 2^{x_n} \cdot 3^{(y_n)_i}) \in e_{i+1}\}$.

Lemma 2.3 (1) *For any* $k \geqslant 1$ *and* $e_1, \ldots, e_k \in W^R_n \circledast(e_1, \ldots, e_k) \in W^R_n$.

(2) *If for some* $i \leqslant k$ $|e_i| < \omega^n$, *then* $|\circledast(e_1, \ldots, e_k)| \leqslant \min(|e_1|, \ldots, |e_k|) + \omega^{n-1}$.

(3) $|\circledast(e_1, \ldots, e_k)| = \omega^n$ *iff for all* $i \leqslant k$, $|e_i| = \omega^n$.

Proof: By induction on n. For $n = 1$, the result is clear. Consider $e_1, \ldots, e_k \in W^R_n$, to show $\circledast(e_1, \ldots, e_k) \in W^R_n$. (i) is immediate. For a fixed x and y such that $\mathsf{seq}(y) \wedge \mathsf{lh}(y) = k$, then

$$\begin{aligned}(*)\ &\{(2^x \cdot 3^y, 2^{x_2} \cdot 3^{y_2}, \ldots, 2^{x_n} \cdot 3^{y_n}) \in \circledast(e_1, \ldots, e_k)\}\\ &= \{(2^x \cdot 3^y, 2^{x_2} \cdot 3^{y_2}, \ldots, 2^{x_n} \cdot 3^{y_n})\colon (2^{x_2} \cdot 3^{y_2}, \ldots, 2^{x_n} \cdot 3^{y_n})\\ &\in \circledast(\{(z_2, \ldots, z_n)\colon (2^x \cdot 3^{(y)_0}, z_2, \ldots, z_n) \in e_1\}, \ldots,\\ &\{(z_2, \ldots, z_n)\colon (2^x \cdot 3^{(y)_{k-1}}, z_2, \ldots, z_n) \in e_k\})\}.\end{aligned}$$

By $(*)$ and our inductive hypothesis (1), it follows that $\circledast(e_1, \ldots, e_k)$ satisfies (iii). (ii) holds for $\circledast(e_1, \ldots, e_k)$ since (ii) holds for $e_1, \ldots, e_k$ and apply the inductive hypothesis (3) to the right-hand side of $(*)$. Suppose for fixed x and y,

$$|\{(2^x \cdot 3^y, 2^{x_2} \cdot 3^{y_2}, \ldots, 2^{x_n} \cdot 3^{y_n}) \in \circledast(e_1, \ldots, e_k)\}| = \omega^{n-1},$$

then by $(*)$ and (3) for $i \leqslant k - 1$,

$$|\{(2^x \cdot 3^{(y)_i}, z_2, \ldots, z_n) \in e_{i+1}\}| = \omega^{n-1}.$$

Since $e_{i+1} \in W^R_n$ and by (iv), for each $z < x$ there is a $y_{z,i+1} < (y)_i$ such that $|\{(2^z \cdot 3^{y_{z,i+1}}, z_2, \ldots, z_n) \in e_{i+1}\}| = \omega^{n-1}$. Thus,

$$|\{(2^z \cdot 3^{2^{y_{z,1}}} \ldots p_{k-1}^{y_{z,k}}, z_2, \ldots, z_n) \in \circledast(e_1, \ldots, e_k)\}| = \omega^{n-1}$$

by $(*)$ and (3). Clearly, $2^{y_{z,1}} \ldots p_{k-1}^{y_{z,k}} < y$ and thus (iv) holds for $\circledast(e_1, \ldots, e_k)$. Thus, $\circledast(e_1, \ldots, e_k) \in W^R_n$.

Suppose $e_1 \in W^R_n$ and $|e_1| < \omega^n$. By (ii) and (iv), there is a unique x such that for all $z < x$ there is exactly one y_z such that $|\{(2^z \cdot 3^{y_z}, z_2, \ldots, z_n) \in e_1\}| = \omega^{n-1}$ and for any $z \geqslant x$ and any y, $|\{2^z \cdot 3^y, z_2, \ldots, z_n) \in e_1\}| < \omega^{n-1}$. Consequently, by $(*)$, (2), and (3), noting the choice of y_z for $e_2, \ldots, e_n$ given $z < x$ must also be correct, we have at most x numbers of the form $2^z \cdot 3^y$ such that $|\{(2^z \cdot 3^y, z_2, \ldots, z_n) \in \circledast(e_1, \ldots, e_n)\}| = \omega^{n-1}$. From this (2) follows readily.

By (2), $|\circledast(e_1, \ldots, e_k)| = \omega^n$ implies for all $i \leqslant k$, $|e_i| = \omega^n$. Conversely, suppose $|e_{i+1}| = \omega^n$ for $i + 1 \leqslant k$. It is easy to verify that for every x, there is a unique $y_{x,i}$ such that for all $i \leqslant k - 1$, $|\{(2^x \cdot 3^{y_{x,i+1}}, z_2, \ldots, z_n) \in e_{i+1}| = \omega^{n-1}$. Hence,

$$|\{(2^x \cdot 3^{2^{y_{x,1}}} \ldots p_{k-1}^{y_{x,k}}, z_2, \ldots, z_n) \in \circledast(e_1, \ldots, e_k)\}| = \omega^{n-1}$$

by (*) and inductive hypothesis (3). Clearly, then $|\circledast(e_1, \ldots, e_k)| = \omega^n$.
Q.E.D.

We define by induction on n a well-ordering $e_{S_n} \in W_n^R$ recursive in R for each predicate of the form $(x_1)(\mathrm{E}y_1) \ldots (x_n)(\mathrm{E}y_n) S_n(x_1, y_1, \ldots, x_n, y_n)$ with S_n recursive in R. Moreover, for each n the above operator $\circledast$ maps any $k + 1$-tuple of well-orderings of the form $e_{S_{i,n}}$, $S_{i,n}$ recursive in R for $i \leqslant k$, to $\circledast(e_{S_{0,n}}, \ldots, e_{S_{k,n}})$ a well-ordering recursive in R. We can always assume by the above Lemma 2.2 that if $(\mathrm{E}y_1) \ldots (x_n)(\mathrm{E}y_n) S_n(x_1, y_1, \ldots, x_n, y_n)$, then

$$(\mathrm{E}!y_1) \ldots (x_n)(\mathrm{E}!y_n) S_n(x_1, y_1, \ldots, x_n, y_n).$$

Suppose $n = 1$: Let $e_{S_1} = \{2^x \cdot 3^y\colon \mathrm{seq}(y) \wedge \mathrm{lh}(y) = x + 1 \wedge (z)_{z \leqslant x} S_1(z, (y)_z - 1)\}$.

Suppose $n = j + 1$:

$$\begin{aligned} e_{S_{j+1}} = \{(2^x \cdot 3^y, z_1, \ldots, z_j)\colon &\mathrm{seq}(y) \wedge \mathrm{lh}(y) \\ &= x + 1 \wedge (z_1, \ldots, z_j) \in \circledast(e_{S_{0,j}}, \ldots, e_{S_{x,j}})\} \end{aligned}$$

where

$$S_{z,j} = (x_2)(\mathrm{E}y_2) \ldots (x_{j+1})(\mathrm{E}y_{j+1}) S_{j+1}(z, (y)_z - 1, x_2, y_2, \ldots, x_{j+1}, y_{j+1}).$$

$e_{S_{j+1}}$ is ordered by first differences and $<$, i.e., it is a subordering of the natural ordering of $\mathcal{N}^{j+1}$. It is obvious that e_{S_n} and $\circledast(e_{S_{0,n}}, \ldots, e_{S_{k,n}})$ are well-orderings and their order types are less than or equal to ω^n.

Lemma 2.4 (a) *$e_{S_n} \in W_n^R$ for all n and S_n.*

(b) *For all $x_1 \leqslant j$, $(\mathrm{E}y_1) \ldots (x_n)(\mathrm{E}y_n) S_n(x_1, y_1, \ldots, x_n, y_n)$ iff there is a y such that $|\{(2^j \cdot 3^y, z_2, \ldots, z_n) \in e_{S_n}\}| = \omega^{n-1}$.*

(c) *$|e_{S_n}| = \omega^n$ iff $(x_1)(\mathrm{E}y_1) \ldots (x_n)(\mathrm{E}y_n) S_n$.*

Proof: By induction on n, the result is immediate for $n = 1$. Consider S_n for $n > 1$, assuming the result is true for $n - 1$. By the previous lemma and definition it is clear that (i) and (iii) hold for e_{S_n}. Suppose that $|\{(2^x \cdot 3^y, z_2, \ldots, z_n) \in e_{S_n}\}| = \omega^{n-1}$, then $\mathrm{seq}(y)$, $\mathrm{lh}(y) = x + 1$, and for $i \leqslant x$, $|e_{S_{i,n}}| = \omega^{n-1}$ by Lemma 2.3, (3), where

$$(x_2)(\mathrm{E}y_2) \ldots (x_n)(\mathrm{E}y_n) S_{i,n} = (x_2)(\mathrm{E}y_2) \ldots (x_n)(\mathrm{E}y_n) S_n(i, (y)_i - 1, x_2, y_2, \ldots, x_n, y_n).$$

Hence, by our inductive hypothesis (c),

$$(x_2)(\mathrm{E}y_2) \ldots (x_n)(\mathrm{E}y_n) S_n(i, (y)_i - 1, x_2, y_2, \ldots, x_n, y_n)$$

is true, but then $(y)_i - 1$ is uniquely determined by i. Hence, y is unique and (ii) holds for e_{S_n}. Moreover, for $z < x$, take $y_z = 2^{y_0} \ldots p_z^{y_z}$ and by the definition of S_n and the previous Lemma, it is clear that $|\{(2^z \cdot 3^{y_z}, z_2, \ldots, z_n) \in e_{S_n}\}| = \omega^{n-1}$ and $y_z < y$. Thus, (iv) holds for e_{S_n} and $e_{S_n} \in W_n^R$. (b) is clear and (c) follows from (b) and properties (ii) and (iv) of elements of W_n^R.

Q.E.D.

We now define mappings $\circledast_k$: $W_k^R \times W_n^R \to W_n^R$ for $n = k, k+1$ whose definitions are motivated by the notion of relativization of quantifiers as follows.

Definition 2.3 $e_1 \circledast_1 e_1' = \{(2^x \cdot 3^{2^y \cdot 3^{y'}}): 2^x \cdot 3^y \in e_1' \text{ and } 2^{2^x \cdot 3^y} \cdot 3^{y'} \in e_1\}$.

$$e_1 \circledast_1 e_2 = \{(2^x \cdot 3^{2^y \cdot 3^{y'}}, 2^z \cdot 3^{2^w \cdot 3^{w'}}):$$
$$(2^x \cdot 3^y, 2^z \cdot 3^w) \in e_2, 2^{2^x \cdot 3^y} \cdot 3^{y'} \in e_1, \text{ and } 2^{2^z \cdot 3^w} \cdot 3^{w'} \in e_1\}.$$

Below we let $e_n(z) = \{(z_2, \ldots, z_n): (z, z_2, \ldots, z_n) \in e_n\}$. Now suppose $\circledast_j$ has been defined for all $j < k$ where $k \geq 2$. Let

$$e_k \circledast_k e_k' = \{(2^x \cdot 3^{2^y \cdot 3^{y'}}, z_2, \ldots, z_k):$$
$$(z_2, \ldots, z_k) \in (e_k(2^{2^x \cdot 3^y} \cdot 3^{y'}) \circledast_{k-1} e_k'(2^x \cdot 3^y))\}$$

and define

$$e_k \circledast_k e_{k+1} = \{(2^x \cdot 3^{2^x \cdot 3^{y'}}, z_2, \ldots, z_{k+1}):$$
$$(z_2, \ldots, z_{k+1}) \in (e_k(2^{2x \cdot 3y} \cdot 3^{y'}) \circledast_{k-1}(e_k \circledast_k e_{k+1}(2^x \cdot 3^y)))\}.$$

The following result establishes that $\circledast_k$ is well-defined and its fundamental properties:

Lemma 2.5 *Let* $e_k \in W_k^R$, $e_n \in W_n^R$ *for* $n = k$ *or* $n = k+1$, *then*

(a) $e_k \circledast_k e_n \in W_n^R$,

(b) *If* $|e_k| < \omega^k$, *then* $|e_k \circledast_k e_n| < \omega^k$,

(c) *If* $|e_k| = \omega^k$ *and* $|e_n| < \omega^n$, *then* $|e_k \circledast_k e_n| \leq |e_n| + \omega^{n-1}$

for $n = k$ *or* $n = k+1$,

and

(d) $|e_k \circledast_k e_n| = \omega^n$ *iff* $|e_k| = \omega^k$ *and* $|e_n| = \omega^n$.

Proof: By induction on k, consider $k = n = 1$. It is clear that if $2^x \cdot 3^{2^y \cdot 3^{y'}} \in e_1 \circledast_1 e_1'$, then y is unique for x since $2^x \cdot 3^y \in e_1'$ and y' is unique for $2^x \cdot 3^y$ since $2^{2^x \cdot 3^y} \cdot 3^{y'} \in e_1$; thus, $2^y \cdot 3^{y'}$ is unique for x. Thus, (i), (ii), and (iii) of the definition of W_1^R hold. Suppose $2^x \cdot 3^{2^y \cdot 3^{y'}} \in e_1 \circledast_1 e_1'$ and let $z < x$, then there is a $y_z < y$ such that $2^z \cdot 3^{y_z} \in e_1'$ and then, since $2^z \cdot 3^{y_z} < 2^x \cdot 3^y$, there is a $y_z' < y'$ such that $2^{2^z \cdot 3^{y_z}} \cdot 3^{y_z'} \in e_1$. Thus, $2^x \cdot 3^{2^y \cdot 3^{y'}} > 2^z \cdot 3^{2^{y_z} \cdot 3^{y_z'}} \in e_1 \circledast_1 e_1'$ and $e_1 \circledast_1 e_1' \in W_1^R$. Clearly, $|e_1 \circledast_1 e_1'| \leq |e_1'|$ and $|e_1| < \omega$ implies $|e_1 \circledast_1 e_1'| < |e_1| < \omega$ and from these (c), (b), and (d) follow.

Now for $k = 1$ consider $n = 2$. Note first that

$$e_1 \circledast_1 e_2 = \{(2^x \cdot 3^{2^y \cdot 3^{y'}}, z): z \in (e_1 \circledast_1 e_2(2^x \cdot 3^y)) \text{ and } (2^{2^x \cdot 3^y} \cdot 3^{y'} \in e_1)\}$$

and thus (i) and (iii) hold. Suppose now $|\{(2^x \cdot 3^{2^y \cdot 3^{y'}}, z) \in e_1 \circledast_1 e_2\}| = \omega$, i.e.,

$|e_1 \circledast_1 e_2(2^x \cdot 3^y)| = \omega$, and thus, by (d) for $k = n = 1$, we have $|e_1| = |e_2(2^x \cdot 3^y)| = \omega$. Since $e_2 \in W_2^R$, we have that y is unique for x, but $2^{2^x \cdot 3^y} \cdot 3^{y'} \in e_1$ and hence y' is unique for $2^x \cdot 3^y$. Thus, $2^y \cdot 3^{y'}$ is the unique number such that $|(e_1 \circledast_1 e_2)(2^x \cdot 3^{2^y \cdot 3^{y'}})| = \omega$, and (ii) holds for $e_1 \circledast_1 e_2$. By a similar type argument, as above, (iv) holds for $e_1 \circledast_1 e_2$ and $e_1 \circledast_1 e_2 \in W_2^R$. Suppose now $|e_1| < \omega$, then by the inductive hypothesis (b) for all z $|e_1 \circledast_1 e_2(z)| < \omega$, and it follows that $(e_1 \circledast_1 e_2)(z) = \emptyset$ for all but finitely many z. Thus, $|e_1 \circledast_1 e_2| < \omega$ and (b) holds. Suppose $|e_1| = \omega$ and $|e_2| < \omega^2$, there is some j such that for $i < j$ there is a unique y_i such that $|e_2(2^i \cdot 3^{y_i})| = \omega$ and for any $2^x \cdot 3^y \neq 2^i \cdot 3^{y_i}$ for all $i < j$, $|e_2(2^x \cdot 3^y)| < \omega$. It follows readily that there are exactly j numbers z of the form $z = 2^i \cdot 3^{2^{y_i} \cdot 3^{y_i'}}$ for some $i < j$ and y_i' such that $|(e_1 \circledast_1 e_2)(z)| = \omega$, using inductive hypothesis (d), and for all other z, $|(e_1 \circledast_1 e_2)(z)| < \omega$. Clearly, then,

$$|e_1 \circledast_1 e_2| \leq \omega \cdot (j) + \omega \leq |e_2| + \omega,$$

since $\omega \cdot (j) \leq |e_2|$, and (c) holds. If $|e_1| = \omega$ and $|e_2| = \omega^2$, then $|e_1 \circledast_1 e_2| = \omega^2$. Conversely, suppose $|e_1 \circledast_1 e_2| = \omega^2$, then for some y, y',

$$|(e_1 \circledast_1 e_2)(2^0 \cdot 3^{2^y \cdot 3^{y'}})| = \omega = |e_1 \circledast_1 e_2(2^0 \cdot 3^y)|$$

and by inductive hypothesis (d) $|e_1| = \omega$. If $|e_2| < \omega^2$, then $|e_1 \circledast_1 e_2| \leq |e_2| + \omega < \omega^2$ by (c), contrary to hypothesis, and thus (d) holds.

Suppose the result is true for $k - 1$ and $n = k - 1$, k where $k \geq 2$. Consider now k and $n = k$,

$$\begin{aligned} e_k \circledast_k e_k' = \{(2^x \cdot 3^{2^y \cdot 3^{y'}}, z_2, \ldots, z_k)&: \\ (z_2, \ldots, z_k) &\in (e_k(2^{2^x \cdot 3^y} \cdot 3^{y'}) \circledast_{k-1} e_k'(2^x \cdot 3^y))\}. \end{aligned}$$

Clearly, (i) and (iii) hold by the *inductive hypothesis*. Suppose $|(e_k \circledast_k e_k')(2^x \cdot 3^{2^y \cdot 3^{y'}})| = \omega^{k-1}$, then by definition and (d), $|e_k(2^{2^x \cdot 3^y} \cdot 3^{y'})| = \omega^{k-1}$ and $|e_k'(2^x \cdot 3^y)| = \omega^{k-1}$. Consequently, by definition of W_k^R, y is unique for x and y' is unique for $2^x \cdot 3^y$; thus $2^y \cdot 3^{y'}$ is uniquely determined by x. Likewise, (iv) holds and $e_k \circledast_k e_k' \in W_k^R$. Suppose $|e_k| < \omega^k$, then for at most finitely many numbers of the form $2^{2^x \cdot 3^y} \cdot 3^{y'}$ $|e_k(2^{2^x \cdot 3^y} \cdot 3^{y'})| = \omega^{k-1}$, and corresponding to each of these,

$$\begin{aligned} |(e_k \circledast_k e_k')(2^x \cdot 3^{2^y \cdot 3^{y'}})| &= |e_k(2^{2^x \cdot 3^y} \cdot 3^{y'}) \circledast_{k-1} e_k'(2^x \cdot 3^y)| \\ &\leq |e_k'(2^x \cdot 3^y)| + \omega^{k-2} \end{aligned}$$

by (c) if $|e_k'(2^x \cdot 3^y)| < \omega^{k-1}$ or, otherwise,

$$|(e_k \circledast_k e_k')(2^x \cdot 3^{2^y \cdot 3^{y'}})| = \omega^{k-1} \text{ if } |e_k'(2^x \cdot 3^y)| = \omega^{k-1}.$$

For all but finitely many of the numbers of the form $2^x \cdot 3^{2^y \cdot 3^{y'}}$, $|e_k(2^{2^x \cdot 3^y} \cdot 3^{y'})| < \omega^{k-1}$ and consequently by (b) it follows that $|(e_k \circledast_k e_k')(2^x \cdot 3^{2^y \cdot 3^{y'}})| < \omega^{k-1}$. Thus, $|e_k \circledast_k e_k'| < \omega^k$ and (b) holds. Suppose now that $|e_k| = \omega^k$ and $|e_k'| < \omega^k$, we wish to establish (c).

Case 1. Suppose for all numbers of the form $2^x \cdot 3^y$, $|e_k'(2^x \cdot 3^y)| < \omega^{k-1}$, then for any y' $|(e_k \circledast_k e_k')(2^x \cdot 3^{2^y \cdot 3^{y'}})| < \omega^{k-1}$, since if $|e_k(2^{2^x \cdot 3^y} \cdot 3^{y'})| = \omega^{k-1}$, then by (c) $|(e_k \circledast_k e_k')(2^x \cdot 3^{2^y \cdot 3^{y'}})| \leq |e_k'(2^x \cdot 3^y)| + \omega^{k-2} < \omega^{k-1}$ and if $|e_k(2^{2^x \cdot 3^y} \cdot 3^{y'})| < \omega^{k-1}$, then by (b) $|(e_k \circledast_k e_k')(2^x \cdot 3^{2^y \cdot 3^{y'}})| < \omega^{k-1}$. Thus,

$$|e_k \circledast_k e_k'| \leq \omega^{k-1} \leq |e_k'| + \omega^{k-1}.$$

Case 2. Suppose for each $i \leq j$, there is a y_i such that $|e_k'(2^i \cdot 3^{y_i})| = \omega^{k-1}$ and for $2^x \cdot 3^y \neq 2^i \cdot 3^{y_i}$ for all $i \leq j$, then $|e_k'(2^x \cdot 3^y)| < \omega^{k-1}$. Thus, since $|e_k| = \omega^k$, there is for every $i \leq j$ a unique y_i' such that $|(e_k \circledast_k e_k')(2^i \cdot 3^{2^{y_i} \cdot 3^{y_i'}})| = \omega^{k-1}$ by inductive hypothesis (d). For any number $2^x \cdot 3^{2^y \cdot 3^{y'}}$ different from $2^i \cdot 3^{2^{y_i} \cdot 3^{y_i'}}$ for all $i \leq j$, we have $|(e_k \circledast_k e_k')(2^x \cdot 3^{2^y \cdot 3^{y'}})| < \omega^{k-1}$ by the argument given in *Case* 1 above. Consequently, $|e_k \circledast_k e_k'| \leq \omega^{k-1} \cdot (j+1) + \omega^{k-1} \leq |e_k'| + \omega^{k-1}$. Thus, (c) holds. Clearly, $|e_k| = \omega^k$ and $|e_k'| = \omega^k$ implies $|e_k \circledast_k e_k'| = \omega^k$ and (d) follows from this, (b) and (c).

Suppose the result is true for $k - 1$ and $n = k - 1$, k and for k and $n = k$ and consider $n = k + 1$, then

$$\begin{aligned} e_k \circledast_k e_{k+1} = \{&(2^x \cdot 3^{2^y \cdot 3^{y'}}, z_2, \ldots, z_{k+1}): \\ &(z_2, \ldots, z_{k+1}) \in (e_k(2^{2^x \cdot 3^y} \cdot 3^{y'}) \circledast_{k-1}(e_k \circledast_k e_{k+1}(2^x \cdot 3^y)))\}. \end{aligned}$$

Clearly, (i) and (iii) hold by our inductive hypothesis. Suppose now $|(e_k \circledast_k e_{k+1})(2^x \cdot 3^{2^y \cdot 3^{y'}})| = \omega^k$, then we have $|e_k(2^{2^x \cdot 3^y} \cdot 3^{y'})| = \omega^{k-1}$, $|e_k \circledast_k e_{k+1}(2^x \cdot 3^y)| = \omega^k$ by our inductive hypotheses (d), and, hence also, $|e_k| = |e_{k+1}(2^x \cdot 3^y)| = \omega^k$. Hence, y is uniquely determined by x and y' is uniquely determined by $2^x \cdot 3^y$. Hence, $2^y \cdot 3^{y'}$ is uniquely determined by x, and (iii) holds. Suppose $z < x$, then there is a $y_z < y$ such that $|e_{k+1}(2^z \cdot 3^{y_z})| = \omega^k$, and there is a $y_z' < y'$ such that $|e_k(2^{2^z \cdot 3^{y_z}} \cdot 3^{y_z'})| = \omega^{k-1}$. Thus,

$$\begin{aligned} |(e_k \circledast_k e_{k+1})(2^z \cdot 3^{2^{y_z} \cdot 3^{y_z'}})| = {} & |e_k(2^{2^z \cdot 3^{y_z}} \cdot 3^{y_z'}) \circledast_{k-1} \\ & (e_k \circledast_k e_{k+1})2^z \cdot 3^{y_z}))| = \omega^k \end{aligned}$$

by our inductive hypothesis (d), and (iv) holds. Suppose $|e_k| < \omega^k$, then by our inductive hypothesis (b), for every $2^x \cdot 3^y$ $|e_k \circledast_k e_{k+1}(2^x \cdot 3^y)| < \omega^k$. Moreover, there are at most finitely many numbers of the form $2^x \cdot 3^{2^y \cdot 3^{y'}}$ such that $|e_k(2^{2^x \cdot 3^y} \cdot 3^{y'})| = \omega^{k-1}$ and for each of these we have

$$\begin{aligned} |(e_k \circledast_k e_{k+1})(2^x \cdot 3^{2^y \cdot 3^{y'}})| &= |e_k(2^{2^x \cdot 3^y} \cdot 3^{y'}) \circledast_{k-1}(e_k \circledast_k e_{k+1}(2^x \cdot 3^y)| \\ &\leq |e_k \circledast_k e_{k+1}(2^x \cdot 3^y)| + \omega^{k-1} < \omega^k \end{aligned}$$

by our inductive hypotheses (c). For all other numbers of the form $2^x \cdot 3^{2^y \cdot 3^{y'}}$, we have $|e_k(2^{2^x \cdot 3^y} \cdot 3^{y'})| < \omega^{k-1}$ and thus $|(e_k \circledast_k e_{k+1})(2^x \cdot 3^{2^y \cdot 3^{y'}})| < \omega^{k-1}$ by our inductive hypothesis (b). Thus, $|e_k \circledast_{k+1} e_{k+1}| < \omega^k$. Suppose now for (c) that $|e_k| = \omega^k$ and $|e_{k+1}| < \omega^{k+1}$.

Case 1. Suppose for all $2^x \cdot 3^y$ $|e_{k+1}(2^x \cdot 3^y)| < \omega^k$, then

$$|e_k(2^{2^x \cdot 3^y} \cdot 3^{y'}) \circledast_{k-1}(e_k \circledast_k e_{k+1}(2^x \cdot 3^y))| \leq |e_k \circledast_k e_{k+1}(2^x \cdot 3^y)| + \omega^{k-1}$$

if $|e_k(2^{2^x \cdot 3^y} \cdot 3^{y'})| = \omega^{k-1}$ and $|e_k \circledast_k e_{k+1}(2^x \cdot 3^y)| \leq |e_{k+1}(2^x \cdot 3^y)| + \omega^{k-1}$ by (c) since $|e_k| = \omega^k$. Thus, $|(e_k \circledast_k e_{k+1})(2^x \cdot 3^{2^y \cdot 3^{y'}})| < \omega^k$ if $|e_k(2^{2^x \cdot 3^y} \cdot 3^{y'})| = \omega^{k-1}$. If $|e_k(2^{2^x \cdot 3^y} \cdot 3^{y'})| < \omega^{k-1}$, then by (b) $|(e_k \circledast_k e_{k+1})(2^x \cdot 3^{2^y \cdot 3^{y'}})| < \omega^{k-1}$. Thus,

$$|e_k \circledast_k e_{k+1}| \leq \omega^k \leq |e_{k+1}| + \omega^k.$$

Case 2 Suppose there is a number j such that for all $i \leqslant j$, there is a y_i such that $|e_{k+1}(2^i \cdot 3^{y_i})| = \omega^k$ and for $2^x \cdot 3^y \neq 2^i \cdot 3^{y_i}$ for all $i \leqslant j$, $|e_{k+1}(2^x \cdot 3^y)| < \omega^k$. There exists exactly $(j+1)$ numbers $2^i \cdot 3^{2^{y_i} \cdot 3^{y_i'}}$ such that $|e_k(2^{2^i \cdot 3^{y_i}} \cdot 3^{y_i'})| = \omega^{k-1}$ for some $i \leqslant j$ and, consequently, $|(e_k \circledast e_{k+1})(2^i \cdot 3^{2^{y_i} \cdot 3^{y_i'}})| = \omega^k$ for all $i \leqslant j$. For any number $2^x \cdot 3^{2^y \cdot 3^{y'}} \neq 2^i \cdot 3^{2^{y_i} \cdot 3^{y_i'}}$ for all $i \leqslant j$ $e_{k+1}(2^x \cdot 3^y) < \omega^k$ and, consequently, as in *Case* 1, $|(e_k \circledast_k e_{k+1})(2^x \cdot 3^{2^y \cdot 3^{y'}})| < \omega^k$. Thus,

$$|e_k \circledast_k e_{k+1}| \leqslant \omega^k \cdot (j+1) + \omega^k \leqslant |e_{k+1}| + \omega^k.$$

Thus, (c) holds for $e_k \circledast_k e_{k+1}$. Suppose $|e_k| = \omega^k$ and $|e_{k+1}| = \omega^{k+1}$, then it readily follows that $|e_k \circledast_k e_{k+1}| = \omega^{k+1}$ and (d) results from this, (b) and (c). This completes the entire lemma. Q.E.D.

The above result can be used to establish the following main result concerning $\circledast_k$, which is fundamental to this paper.

Theorem 2.1 *If* $k \geqslant 1$ *and* $|e_k| < \omega^j$ *where* $0 \leqslant j \leqslant k$, *then* $|e_k \circledast_k e_n| < \omega^j$ *for* $n = k$ *or* $n = k + 1$.

Proof: By induction, consider $k = 1$. Suppose $|e_1| < \omega$, then by the previous Lemma $|e_1 \circledast_1 e_1'| < \omega$ and $|e_1 \circledast_1 e_2| < \omega$. If $|e_1| < \omega^0 = 1$, then $|e_1| = 0$ and thus $|e_1 \circledast_1 e_1'| = 0$ and $|e_1 \circledast_1 e_2| = 0$.

Let $k \geqslant 2$ and suppose the result is true for all $s < k$ and $n = s$ or $n = s + 1$. Consider first $n = k$, i.e., $e_k \circledast_k e_k'$. Suppose $|e_k| < \omega^j$ for $j \leqslant k$.

Case 1 Suppose $j = k$, then $|e_k| < \omega^k$ which implies by the preceding lemma (b) that $|e_k \circledast_k e_n'| < \omega^k = \omega^j$.

Case 2 Suppose $j < k$, then for at most finitely many x, y, $\omega^j > |e_k(2^x \cdot 3^y)| \geqslant \omega^{j-1}$ and for all but finitely many x, y, $|e_k(2^x \cdot 3^y)| < \omega^{j-1}$. Thus,

$$|(e_k \circledast_k e_k')(2^x \cdot 3^{2^y \cdot 3^{y'}})| = |e_k(2^{2^x \cdot 3^y} \cdot 3^{y'}) \circledast_{k-1} e_k'(2^x \cdot 3^y)| < \omega^{j-1}$$

for all but finitely many $2^x \cdot 3^y$, y', since $j - 1 < k - 1$ and our inductive hypothesis. Suppose now $\omega^{j-1} \leqslant |e_k(2^{2^x \cdot 3^y} \cdot 3^{y'})| < \omega^j$, then since $j \leqslant k - 1$ by the inductive hypothesis,

$$|e_k(2^{2^x \cdot 3^y} \cdot 3^{y'}) \circledast_{k-1} e_k'(2^x \cdot 3^y)| < \omega^j$$

where this can occur at most for finitely many $2^x \cdot 3^y$, y'. Thus, $|e_k \circledast_k e_k'| < \omega^j$.

Suppose the result is true for $e_k \circledast_k e_k'$, consider now $e_k \circledast_k e_{k+1}$ where $|e_k| < \omega^j$ for some $j \leqslant k$.

Case 1 Suppose $j = k$, then the result follows from the preceding Lemma, i.e., $|e_k \circledast_k e_{k+1}| < \omega^k$.

Case 2 Suppose $j < k$ and $|e_k| < \omega^j$. For at most finitely many x, y, $\omega^{j-1} \leqslant |e_k(2^x \cdot 3^y)| < \omega^j$ and for all other x, y, $|e_k(2^x \cdot 3^y)| < \omega^{j-1}$. Suppose $\omega^{j-1} \leqslant |e_k(2^{2^x \cdot 3^y} \cdot 3^{y'})| < \omega^j$, then

$$|(e_k \circledast_k e_{k+1})(2^x \cdot 3^{2^y \cdot 3^{y'}})| = |e_k(2^{2^x \cdot 3^y} \cdot 3^{y'}) \circledast_{k-1} (e_k \circledast_k e_{k+1}(2^x \cdot 3^y))| < \omega^j$$

by our inductive hypothesis since $j \leqslant k - 1$ and $e_k \circledast_k e_{k+1}(2^x \cdot 3^y) \in W_k^R$, and this happens for at most finitely many $2^x \cdot 3^{2^y \cdot 3^{y'}}$. If $|e_k(2^{2^x \cdot 3^y} \cdot 3^{y'})| < \omega^{j-1}$, then

$$|(e_k \circledast_k e_{k+1})(2^x \cdot 3^{2^y \cdot 3^{y'}})| = |e_k(2^{2^x \cdot 3^y} \cdot 3^{y'}) \circledast_{k-1} (e_k \circledast_k e_{k+1}(2^x \cdot 3^y))| < \omega^{j-1}$$

by our inductive hypothesis since $j - 1 < k - 1$ and $e_k \circledast_k e_{k+1}(2^x \cdot 3^y) \in W_k^R$. Thus $|e_k \circledast_k e_{k+1}| < \omega^j$. Q.E.D.

Note that we can define $e_k \circledast_k e_n$ for any $n > k + 1$ by

$$e_k \circledast_k e_n = ((e_k \circledast_k e'_{k+1}) \circledast_{k+1} e'_{k+2}) \ldots \circledast_{n-1} e_n,$$

where $|e'_{k+1}| = \omega^{k+1}, \ldots, |e'_{n-1}| = \omega^{n-1}$, and $e'_i \in W_i^R$ for $i = k + 1, \ldots, n - 1$.

3 *The construction of well-orderings of determined type* Below H_ω, $\mathsf{H}_{\omega\cdot 2}, \ldots, \mathsf{H}_{\omega\cdot n}, \ldots$ are certain fixed hyperarithmetic sets determined by $a \in O$, e.g., $\mathsf{H}_\omega = \mathsf{H}_a$ where $|a| = \omega$ and $a \in O$, chosen so that we know $|a_i|$ "effectively" (this certainly can be done up through ε_O). W refers to the set of all indices of recursive well-orderings as in [10]. W_k and W_k^A refer to sets of all indices of those well-orderings recursive and recursive in A defined in section **2**.

Lemma 3.1 *For any x, we can effectively find $e_x(k) \in W_k$ such that*

$$|e_x(k)| < \omega^k \textit{ if } x \in \mathrm{O}^{(2k)}$$
$$|e_x(k)| = \omega^k \textit{ if } x \in \overline{\mathrm{O}^{(2k)}}.$$

Proof: $x \in \mathrm{O}^{(2k)} \equiv \mathrm{E}x_1\, \forall y_1 \ldots \mathrm{E}x_k\, \forall y_k\, R(x, x_1, y_1, \ldots, x_k, y_k)$, with R recursive. Thus, $x \notin \mathrm{O}^{(2k)} \equiv \forall x_1 \mathrm{E} y_1 \ldots \forall x_k \mathrm{E} y_k\, \neg R(x, x_1, \ldots, x_k, y_k)$ and, by Lemma 2.2,

$$x \notin \mathrm{O}^{(2k)} \equiv \forall x_1 \mathrm{E}!y_1, \ldots \forall x_k \mathrm{E}!y_k\, S(x, x_1, \ldots, x_k, y_k).$$

By Lemma 2.4, take $e_x = e_{S_k}$ where $S_k = S(x, x_1, \ldots, x_k, y_k)$, then $|e_x| \leqslant \omega^k$ and $|e_x| = \omega^k$ iff $x \notin \mathrm{O}^{(2k)}$.

For the following we modify Kleene's T-predicate of [2] so that $\mathsf{T}_1^A(x, y, z, n)$ iff $\mathsf{T}_1^A(x, y, z)$ and only questions of the form $a \in A$ for $(a)_1 \leqslant n$ are asked of the oracle. Then it follows that for $a' = 3 \cdot 5^a \in O$ $\mathrm{E}z\mathsf{T}_1^{\mathsf{H}_{a'}}(x, y, z) \equiv \mathrm{E}n\mathrm{E}z\mathsf{T}_1^{\mathsf{H}_{a'}}(x, y, z, n) \equiv \mathrm{E}n\mathrm{E}z\mathsf{T}_1^{\mathsf{H}_{a'_n}}(x, y, z, n)$ where $\mathsf{H}_{a'_n} = \{w\colon (w)_1 \leqslant n \wedge (w)_O \in \mathsf{H}_{a(w)_1}\}$. Clearly $\mathsf{H}_{a_n} \leqslant_1 \mathsf{H}_{a'_n}$ and $\mathsf{H}_{a'_n} \leqslant_m \mathsf{H}_{a_n}$, using Lemma 3 [4]. Consequently, for a primitive recursive $\phi(x, n)$, $(\mathrm{E}z)\mathsf{T}_1^{\mathsf{H}_{a'_n}}(x, y, z, n) \equiv (\mathrm{E}z)\mathsf{T}_1^{\mathsf{H}_{a_n}}(\phi(x, n), y, z)$. Thus, $\mathrm{E}z\mathsf{T}_1^{\mathsf{H}_{a'}}(x, y, z) \equiv \mathrm{E}n\mathrm{E}z\mathsf{T}_1^{\mathsf{H}_{a_n}}(\phi(x, n), y, z)$. In particular,

$$(z)\, \neg \mathsf{T}_1^{\mathsf{H}_{a'}}(x, y, z) \equiv (n)(z)\, \neg \mathsf{T}_1^{\mathsf{H}_{a_n}}(\phi(x, n), y, z).$$

Lemma 3.2 *For any x, $a \in O$, $|a| = \omega$, then we can effectively find $e_x(a) \in W$ such that*

$$|e_x(a)| < \omega^\omega \textit{ if } x \in \mathsf{H}'_a$$
$$|e_x(a)| = \omega^\omega \textit{ if } x \notin \mathsf{H}'_a.$$

Proof: Choose $a \in O$ such that $\mathsf{H}_{a_n} = \mathrm{O}^{(2n-1)}$. For each n,

$$\mathrm{E}z\mathsf{T}_1^{\mathsf{H}_{a_n}}(\phi(x, n), x, z) \equiv (\mathrm{E}z)\mathsf{T}_1^{\mathrm{O}^{(2n-1)}}(\phi(x, n), x, z) \equiv g(x, n) \in \mathrm{O}^{(2n)},$$

for a primitive recursive $g(x,n)$. Let $e_n = e_{g(x,n)}(n)$ of Lemma 3.1. Consequently, $x \in \mathrm{H}'_a$ iff for some n, $|e_n| < \omega^n$. Define $s_i \in W$ inductively as follows: $s_1 = e_1, \ldots, s_{i+1} = s_i \circledast_i e_{i+1}$. Finally, let

$$e_x(a) = s_1, s_2, \ldots, s_n, \ldots = \bigcup\{\{i\} \times s_i: \text{ ordered by first coordinates first and then as in } s_i\}.$$

Suppose $x \in \mathrm{H}'_a$, then let j be the smallest i such that $|e_i| < \omega^i$. By Lemma 2.5, $|s_{j-1}| = \omega^{j-1}$,

$$|s_j| = |s_{j-1} \circledast_{j-1} e_j| \leqslant |e_j| + \omega^{j-1} < \omega^j.$$

Thus, $|s_{j+1}| = |s_j \circledast_j e_{j+1}| < \omega^j$ and by Theorem 2.1 for all $k \geqslant j$, $|s_k| < \omega^j$. Thus, $|e_x(a)| = \sum_{i=0}^{\omega} |s_i| \leqslant \omega^j$. If $x \notin \mathrm{H}'_a$, then for all n, $|e_n| = \omega^n$. Consequently, by Lemma 2.5, $|s_n| = \omega^n$ for all n. Thus, $|e_x(a)| = \omega^\omega$. Q.E.D.

Let $e_i \in W_i^{\mathrm{H}_\omega}$, then $x \in e_i \equiv (\mathrm{E}z)\,\mathrm{T}_1^{\mathrm{H}_\omega}(e_{i,1}, x, z)$ and $x \notin e_i \equiv (\mathrm{E}z)\,\mathrm{T}_1^{\mathrm{H}_\omega}(e_{i,2}, x, z)$. Thus, $x \in e_i \equiv \forall z \neg \mathrm{T}_1^{\mathrm{H}_\omega}(e_{i,2}, x, z) \equiv \phi(e_{i,2}, x) \in \overline{\mathrm{H}'_\omega}$. For each $n \geqslant 1$, let $e_n(x)$ be the elements of W_n associated with the question $\phi(e_{i,2}, x) \in \mathrm{H}'_\omega$ by Lemma 3.1 (as in the proof of Lemma 3.2), i.e., for all n, $|e_n(x)| = \omega^n$ iff $\phi(e_{i,2}, x) \in \overline{\mathrm{H}'_\omega}$ iff $x \in e_i$. Consider the question $y \in \mathrm{H}'_\omega$, as above prior to Lemma 3.2, $y \in \mathrm{H}'_\omega \equiv \mathrm{E}n\mathrm{E}z\; \mathrm{T}_1^{\mathrm{H}_{a_n}}(\phi(y,n), y, z)$. Let $|a_n| = 2n$ for $n \geqslant 1$, then

$$y \in \mathrm{H}'_\omega \equiv \mathrm{E}n\mathrm{E}z\,\mathrm{T}_1^{\mathrm{O}^{2n}}(\phi(y,n), y, z) \equiv g(y,n) \in \mathrm{O}^{2n+1} \equiv f(y,n) \in \overline{\mathrm{O}^{2(n+1)}}.$$

Moreover, it is evident that we can assume $g(y,n) \in \mathrm{O}^{2n+1} \to g(y, n+1) \in \mathrm{O}^{2(n+1)+1}$. Consequently, by Lemma 2.4 we can effectively find $a_{n+1}(y) \in W_{n+1}$, $(n \geqslant 1)$ such that $y \in \mathrm{H}'_\omega$ iff $\mathrm{E}n(|a_{n+1}(y)| = \omega^{n+1})$ iff $\mathrm{E}k(n)(n \geqslant k \to |a_{n+1}(y)| = \omega^{n+1})$, i.e., $y \notin \mathrm{H}'_\omega$ iff $\forall n(|a_{n+1}(y)| < \omega^{n+1})$.

Now we define the basic construction. Given $e_1 \in W_1^{\mathrm{H}_\omega}$, an x and a y, we find recursively the sequences $e_n(x)$ for $n \geqslant 1$ and $a_{n+1}(y)$ for $n \geqslant 1$, as above. By $s(x)/a_{k+1}(y)$, we mean the element of W constructed as in Lemma 3.2 from the sequence

$$e_1(x),\ e_2(x),\ \ldots,\ a_{k+1}(y) \circledast_{k+1} e_{k+1}(x),\ e_{k+2}(x),\ \ldots.$$

Hence, if for some $i < k+1$, $|e_i(x)| < \omega^i$, then $|s(x)/a_{k+1}(y)| \leqslant \omega^i$ by Theorem 2.1. If $|a_{k+1}(y)| < \omega^{k+1}$ or $|e_{k+1}(x)| < \omega^{k+1}$, then $|a_{k+1}(y) \circledast_{k+1} e_{k+1}(x)| < \omega^{k+1}$ by Lemma 2.5, and by Theorem 2.1, $|s(x)/a_{k+1}(y)| \leqslant \omega^{k+1}$. If the smallest i such that $|e_i(x)| < \omega^i$ is larger than $k+1$ and $|a_{k+1}(y)| = \omega^{k+1}$, then $|s(x)/a_{k+1}(x)| \leqslant \omega^i$. Define

$$\theta(x,y) = s(x)/a_2(y),\ s(x)/a_3(y),\ \ldots,\ s(x)/a_{n+1}(y),\ \ldots,$$

i.e., the effective sum of these well-orderings. It has these properties:

(1) $|\theta(x,y)| = \omega^{\omega+1}$, if $y \in \mathrm{H}'_\omega$ and $x \in \overline{\mathrm{H}'_\omega}$;
(2) $|\theta(x,y)| \leqslant \omega^\omega$, if $y \notin \mathrm{H}'_\omega$ and $x \in \overline{\mathrm{H}'_\omega}$;
(3) $|\theta(x,y)| \leqslant \omega^{i+1}$, if $x \notin \overline{\mathrm{H}'_\omega}$ and i is the smallest number such that $|e_i(x)| < \omega^i$.

For any $e'_k \in W_k$, $\theta(x,y)/e'_k$ is the above except that we replace $e_k(x)$

everywhere by $e'_k \circledast_k e_k(x)$. Thus, if $|e'_k| < \omega^k$, then $|\theta(x,y)/e'_k| \leq \omega^{k+1}$ as in (3) above. If $|e'_k| = \omega^k$, then (1), (2), and (3) indicate the size of $|\theta(x,y)/e'_k|$. Similarly, for $\theta(x,y)/e'_k/e'_j$ where $e'_k \in W_k$, $e'_j \in W_j$, $|e'_j| < \omega^j$ implies $|\theta(x,y)/e'_k/e'_j| \leq \omega^{j+1}$.

Theorem 3.1 (a) *For every* $e_i \in W_i^{\mathbf{H}_\omega}$, *we can effectively find* $e_i^* \in W$ *such that* $|e_i^*| \leq \omega^{\omega+1} \cdot |e_i|$ *and* $|e_i| = \omega^i$ *implies* $|e_i^*| = \omega^{\omega+1} \cdot \omega^i = \omega^{\omega+i+1}$.

(b) *For every* $e_i \in W_i^{\mathbf{H}_{\omega\cdot n}}$, *we can effectively find* $e_i^* \in W$ *such that* $|e_i^*| \leq \omega^{\omega\cdot n+1} \cdot |e_i|$ *and* $|e_i| = \omega^i$ *implies* $|e_i^*| = \omega^{\omega\cdot n+1} \cdot \omega^i = \omega^{\omega\cdot n+i+1}$.

Proof: For part (a), we define e_i^* by induction on i. First let $\sigma(j) = j$th element of the form $2^x \cdot 3^y$ such that $0 \leq x \leq y$ and so that for all j, $\sigma(j) < \sigma(j+1)$, (the range of σ is by definition the candidates for elements of any $e \in W_1^A$). Define e_1^* to be the effective sum in W of $\theta(\sigma(0), y(0)), \ldots, \theta(\sigma(n), y(n)), \ldots$ where

$$y(n) \in \mathbf{H}'_\omega \equiv \sigma(n) \in e_1 \wedge (\mathrm{E}z)(z > \sigma(n) \wedge z \in e_1).$$

If $|e_1| = 0$, then $|\theta(\sigma(x), y(x))| < \omega^\omega$ by (3) above and $|e_1^*| \leq \omega^\omega$. If $|e_1| = \omega$, then for each $x \in e_1$, $y(n) \in \mathbf{H}'_\omega$, and $|\theta(\sigma(n), y(n))| = \omega^{\omega+1}$ by (1) where $\sigma(n) = x$. If $|e_1| \neq 0$, finite, then $|\theta(\sigma(x), y(x))| \leq \omega^\omega$ for the largest x such that $\sigma(x) \in e_1$ since $y(n) \notin \mathbf{H}'_\omega$ and by (2), and for all $k > x$, $|\theta(\sigma(k), y(k))| < \omega^\omega$ by (3). Thus, $|e_1^*| \leq \omega^{\omega+1} \cdot |e_1|$. Suppose $a(i) \in W_{i+1}$ for each i, by $(e_1/a(i))^*$, we mean

$$\theta(f(0), y(0))/a(0), \ldots, \theta(f(n), y(n))/a(n), \ldots$$

so that if for all i, $|a(i)| < \omega^{i+1}$, then for each n, $|\theta(f(n), y(n))/a(n)| < \omega^\omega$, and, hence, $|(e_1/a(i))^*| \leq \omega^\omega$. Similarly, for a fixed $e'_k \in W_k$,

$$(e_1/a(i)/e'_k)^* = ((\theta(f(0), y(0))/e'_k)/a(0))^*, \ldots, (\theta(f(n), y(n))/e'_k/a(n))^*, \ldots,$$

so that if $|e'_k| < \omega^k$, then $|(e_1/a(i)/e'_k)^*| \leq \omega^{k+2}$.

Suppose now we have defined for all $e_i \in W_i^{\mathbf{H}_\omega}$, e_i^*, $(e_i/a(j))^*$, $(e_i/a(j)/e'_k)^*$ as above, such that the following properties are obtained:

(i) If $|e_i| = 0$, then $|e_i^*| \leq \omega^\omega$, $|(e_i/a(j))^*| \leq \omega^\omega$, and $|(e_i/a(j)/e'_k)^*| \leq \omega^\omega$.

(ii) If $|e_i| = \omega^i$, then $|e_i^*| = \omega^{\omega+1} \cdot \omega^i$.

(iii) If, for all j, $|a(j)| < \omega^{j+1}$, then $|(e_i/a(j))^*| \leq \omega^\omega$ and $|(e_i/a(j)/e'_k)^*| \leq \omega^\omega$.

(iv) If, for some k, $|a(k)| = \omega^{k+1}$ (and for all $j \geq k$, $|a(j)| = \omega^{j+1}$), and $|e_i| = \omega^i$, then $|(e_i/a(j))^*| = \omega^{\omega+1} \cdot \omega^i$.

(v) If $|e'_k| < \omega^k$, then $|(e_i/a(j)/e'_k)^*| \leq \omega^{k+i+1}$.

(vi) If $|e'_k| = \omega^k$, $|e_i| = \omega^i$, for all $j \geq k$, $|a(j)| = \omega^{j+1}$ for some k, then $|(e_i/a(j)/e'_k)^*| = \omega^{\omega+1} \cdot \omega^i$.

(vii) If $0 \neq |e_i| < \omega^i$, then $|e_i^*| \leq \omega^{\omega+1} \cdot |e_i|$, $|(e_i/a(j))^*| \leq \omega^{\omega+1} \cdot |e_i|$, and $|(e_i/a(j)/e'_k)^*| \leq \omega^{\omega+1} \cdot |e_i|$.

Now define $(e_{i+1})^*$ for $e_{i+1} \in W_{i+1}^{\mathbf{H}_\omega}$ as follows: For each n, let

$$y(n) \in \mathbf{H}'_\omega \equiv (\mathrm{E}z)(z \in e_{i+1}(\sigma(n))) \wedge (\mathrm{E}z)(\mathrm{E}x)(x > n \wedge z \in e_{i+1}(\sigma(x))).$$

Let $y(n)(j) \in W_{j+1}$ such that

$$y(n) \in \mathsf{H}'_\omega \equiv (\mathrm{E}k)(j)(j \geqslant k \rightarrow |y(n)(j)| = \omega^{j+1}).$$

Let $a(j)$ be similar except

$$(\mathrm{E}k)(j)(j \geqslant k \rightarrow |a(j)| = \omega^{j+1}) \equiv (\mathrm{E}z)(z \in e_{i+1}) \equiv \mathrm{E}z\mathrm{E}x(z \in e_{i+1}(\sigma(x)).$$

$$e^*_{i+1} = (e_{i+1}(\sigma(0))/y(0)(j)/a(0))^*, \ldots, (e_{i+1}(\sigma(n))/y(n)(j)/a(n))^*, \ldots$$
(the effective sum in W).

Similarly,

$$(e_{i+1}/b(j))^* = (e_{i+1}(\sigma(0))/(y(0)(j)/a(0) \circledast_1 b(0))^*, \ldots, \\ (e_{i+1}(\sigma(n))/y(n)(j)/a(n) \circledast_{n+1} b(n))^*, \ldots,$$

and

$$(e_{i+1}/b(j)/e'_k)^* = (e_{i+1}(\sigma(0))/y(0)(j)/a(n) \circledast_{n+1} b(n)/e'_k)^*, \ldots, \\ (e_{i+1}(\sigma(n))/y(n)(j)/a(n) \circledast_{n+1} (b(n))/e'_k)^*, \ldots.$$

Suppose $|e_{i+1}| = 0$, then for each n, $|a(n)| < \omega^{n+1}$, $a(n) \in W_{n+1}$, consequently, *by (v)* $|e_{i+1}(\sigma(x))/y(n)(j)/a(x)| < \omega^\omega$ and, thus, $|e^*_{i+1}| \leqslant \omega^\omega$. Suppose now $|e_{i+1}| \neq 0$, and suppose $|e_{i+1}| = \omega^{i+1}$, then for infinitely many n, $|e_{i+1}(\sigma(n))| = \omega^i$, and given such n, $y(n) \in \mathsf{H}'_\omega$ and consequently, $|y(n)(j)| = \omega^{j+1}$ for all $j \geqslant k$ for some k. Hence, by (vi), if also n is sufficiently large so that $|a(n)| = \omega^{n+1}$,

$$|(e_{i+1}(\sigma(n))/y(n)(j)/a(n))^*| = \omega^{\omega+1} \cdot |e_{i+1}(\sigma(n))| = \omega^{\omega+1} \cdot \omega^i.$$

Thus,

$$|e^*_{i+1}| = (\omega^{\omega+1} \cdot \omega^i) \cdot \omega = \omega^{\omega+1} \cdot \omega^{i+1}.$$

Now to show $|e^*_{i+1}| \leqslant \omega^{\omega+1} \cdot |e_{i+1}|$, it is clear that $|e_{i+1}| = \sum_{n \in \omega} |e_{i+1}(\sigma(n))|$. Suppose for some k $|e_{i+1}(\sigma(j))| = 0$ for all $j > k$. Let k_1 be the largest number such that $|e_{i+1}(\sigma(k_1))| \neq 0$, then $y(k_1) \notin \mathsf{H}'_\omega$, and therefore $|y(k_1)(j)| < \omega^{j+1}$ for all j. Consequently,

$$|(e(\sigma(k_1))/y(k_1)(j)/a(k_1))^*| \leqslant \omega^\omega$$

by (iii) and also

$$|(e(\sigma(n))/y(n)(j)/a(n))^*| \leqslant \omega^\omega, \text{ for } n \geqslant k_1,$$

$$\begin{aligned} |e^*_{i+1}| &= \sum_{j \in \omega} |(e_{i+1}(\sigma(j))/y(n)(j)/a(j))^*| \\ &\leqslant \Big(\sum_{j \leqslant k_1 - 1} \omega^{\omega+1} \cdot |e_{i+1}(\sigma(j))|\Big) + \omega^\omega \cdot \omega \leqslant \omega^{\omega+1} \cdot \Big(\sum_{j \leqslant k_1 - 1} |e_{i+1}(\sigma(j)|) + 1\Big) \\ &\leqslant \omega^{\omega+1} \cdot |e_{i+1}|, \text{ (since } |e_{i+1}(k_1)| \neq 0). \end{aligned}$$

If there is no k such that for all $j \geqslant k$, $|e_{i+1}(\sigma(j))| = 0$, then

$$\begin{aligned} |e^*_{i+1}| &= \sum_{n \in \omega} |(e_{i+1}(\sigma(n))/y(n)(j)/a(n))^*| \\ &\leqslant \sum_{n \in \omega} \omega^{\omega+1} \cdot (1 + |e_{i+1}(\sigma(n)|) \leqslant \omega^{\omega+1} \cdot \sum (1 + e_{i+1}(\sigma(n))) \\ &= \omega^{\omega+1} \cdot |e_{i+1}|. \end{aligned}$$

By similar arguments the other conditions (i)-(vii) can be checked at $i + 1$. This verifies part (a).

The procedure is completely analogous for $\mathbf{H}_{\omega\cdot 2}, \mathbf{H}_{\omega\cdot 3}, \ldots$. For example, given a question $x \in \overline{\mathbf{H}'_{\omega\cdot 2}}$, we find effectively $e_n(x) \in W_n^{\mathbf{H}_{\omega\cdot 1}}$ such that for all n, $|e_n(x)| = \omega^n$ iff $x \in \overline{\mathbf{H}'_{\omega}}$. Define $s_1(x) = e_1(x), \ldots, s_{i+1}(x) = s_i(x) \circledast_{i+1} e_{i+1}(x), \ldots$, and define $e_x = (s_1(x))^*, \ldots, (s_n(x))^*, \ldots$ effective sum in W of those well-orderings constructed in part (a). If $|e_i(x)| < \omega^i$ for some i, then $|s_n(x)| < \omega^i$ for all n, and, hence,

$$|e_x| = \sum_{n\in\omega} |(s_n(x))^*| \leqslant \sum_{n\in\omega} \omega^{\omega+1}\cdot(1 + |s_n(x)|) \leqslant \omega^{\omega+1}\cdot \sum_{n\in\omega} |1 + s_n(x)|$$
$$\leqslant \omega^{\omega+1}\cdot\omega^i = \omega^{\omega+i+1}.$$

This gives the basic construction for the $W_i^{\mathbf{H}_{\omega\cdot 2}}$ argument. Thus, (b) holds by induction. Q.E.D.

This technique yields the following result:

Theorem 3.2 *Let $n \geqslant 1$, then there is an effective procedure to find $e_x \in W$ such that*

$$|e_x| = \omega^{\omega\cdot n} \textit{ if } x \in \overline{\mathbf{H}'_{\omega\cdot n}}$$
$$|e_x| < \omega^{\omega\cdot n} \textit{ if } x \in \mathbf{H}'_{\omega\cdot n}.$$

Corollary 3.1 *Let $k \geqslant 1$, then there is an effective procedure to jinu $e_x \in W$ such that*

$$|e_x| = \omega^{\omega\cdot n+k} \textit{ if } x \in \overline{\mathbf{H}^{2k+1}_{\omega\cdot n}}$$
$$|e_x| < \omega^{\omega\cdot n+k} \textit{ if } x \in \mathbf{H}^{2k+1}_{\omega\cdot n}.$$

Proof: $x \in \overline{\mathbf{H}^{2k+1}_{\omega\cdot n}}$ iff $(x_1)\mathrm{E}y_1 \ldots (x_k)\mathrm{E}y_k(z)\, R(x, x_1, y_1, \ldots, x_k, y_k, z)$, with R recursive in $\mathbf{H}_{\omega\cdot n}$. By the techniques of section 2 we can find S_k a set of k-tuples of $\mathcal{N}$ such that for recursive f, $(x_1, \ldots, x_k) \in S_k$ iff $f(x_1, \ldots, x_k) \in \overline{\mathbf{H}'_{\omega\cdot n}}$, since the set $\overline{\mathbf{H}'_{\omega\cdot n}}$ is a complete set for predicates $\forall zR$ with R recursive in $\mathbf{H}_{\omega\cdot n}$, and $|S_k| = \omega^k$ iff $x \in \overline{\mathbf{H}^{2k+1}_{\omega\cdot n}}$. Let $e_x =$ the effective sum of $e_{f(x_1,\ldots,x_k)}$ such that $(x_1, \ldots, x_k) \in \mathcal{N}^k$ ordered first by $(x_1, \ldots, x_k)$ and next as in $e_{f(x_1,\ldots,x_n)}$, there $e_{f(x_1,\ldots,x_n)}$ is obtained as in Theorem 3.1. Clearly, if $|S_k| = \omega^k$, then $|e_x| = \omega^{\omega\cdot n}\cdot\omega^k = \omega^{\omega\cdot n+k}$. If $|S_k| < \omega^k$, then since $\omega^{\omega\cdot n}$ is a principle number for addition it follows easily that $|e_x| < \omega^{\omega\cdot n}\cdot\omega^k = \omega^{\omega\cdot n+k}$. Q.E.D.

The following result generalizes a result of [8] originally noticed by S. Tennenbaum.

Lemma 3.3 *For any set A, $\{x\}^A(x)$ grows faster than any function recursive in A.*

Proof: By grows faster we mean there is no function f recursive in A such that whenever $\{x\}^A(x)$ is defined $\{x\}^A(x) \leqslant f(x)$. If $F(x)$ were such a function, take e to be a Gödel number of $f(x) + 1$ in A, then $\{e\}^A(e) = f(e) + 1 \leqslant f(e)$, a contradiction.

Now we show Theorem 1.2 (b) is best possible. Namely, we construct

$a \in O$ such that $|a| = \omega^{k+2}$ such that there is no function f recursive in $O^{(2k)}$ such that $a_i \leqslant \omega^{k+1} \cdot f(i)$ for all i. Consider $T_1^{O^{(2k)}}(x, x, w)$ by Post's Theorem and Lemma 2.2 this is equivalent to $(E!z)(x_1)(E!y_1) \ldots (x_k)(E!y_k)\, R(x, w, z, x_1, \ldots, y_k)$ with R recursive. Noting that $T_1^{O^{(2k)}}(x, x, w)$ true implies w is unique, consider for each number $2^w \cdot 3^z$

$$(x_1)(E!y_1) \ldots (x_k)(E!y_k)\, R(x, w, z, x_1, \ldots, x_k).$$

By Lemma 2.4, find $e_k(2^w \cdot 3^z, x) \in W_k$ such that $|e_k(2^w \cdot 3^z, x)| = \omega^k$ iff $(x_1)(E!y_1) \ldots (x_k)(E!y_k)\, R(x, w, z, x_1, \ldots, x_k)$. Let $e(2^w \cdot 3^z, x) \in W$ be w-copies of $e_k(2^w \cdot 3^z, x)$, and let $e(x) \in W$ be the effective sum over $e(2^w \cdot 3^z, x)$ for all possible w, z ordered first by $2^w \cdot 3^z$ next as in $e(2^w \cdot 3^z, x)$. Clearly, $|e(x)| \leqslant \omega^k$, if $\neg(Ew)\, T_1^{O^{(2k)}}(x, x, w)$. However, if $Ew\, T_1^{O^{(2k)}}(x, x, w)$, then $\omega^k \cdot \{x\}^{O^{(2k)}}(x) < |e(x)| < \omega^{k+1}$. Let $a(i) \in O$ be found as in [5] Theorem 1 such that $|a_i| = \omega \cdot |1 + e(i)|$. Let $a_O = a(O), \ldots, a_{i+1} = a_i +_O a(i)$, clearly, $|3 \cdot 5^a| = \omega^{k+2}$, and for all i, $\omega^{k+1} \cdot \{i\}^{O^{(2k)}}(i) \leqslant |a_i|$. Consequently, by Lemma 3.3 $a' = 3 \cdot 5^a \not\prec^{O^{(2k)}} b'$ where $|b_i| = \omega^{k+1} \cdot i$. Thus, by Theorem 1.1, $H_{a'} \not\leqslant_1^{O^{(2k)}} H_{b'}$, but by Theorem 1.2 (b) $H_{a'}$ isomorphic to $H_{b'}$ by function recursive in $O^{(2k+1)}$. Thus, Theorem 1.2 (b) is the best possible result.

In order to show that Theorem 1.2 (a) at $|a'| = \omega^\omega$ is necessary, we first build $|a(k)'| = \omega^\omega$ such that $a(k)_i$ grows more rapidly to ω^ω than $\omega^{f(i)}$ for any function f recursive in $O^{(2k)}$. As in the previous sections, obtain for each x, $e_k(2^w \cdot 3^z, x) \in W_k$, if $w > k$, let $e_{k,w}(2^w \cdot 3^z, x) = e_k(2^w \cdot 3^z, x) \circledast_k e_w$ where $e_w \in W_w$ and $|e_w| = \omega^w$, and if $w \not> k$, $e_{k,w}(2^w \cdot 3^z, x) = e_k(2^w \cdot 3^z, x)$. Let $e(x) \in W$ be the effective sum of $e_{k,w}(2^w \cdot 3^z, x)$ for all w, z. If $Ew\, T_1^{O^{(2k)}}(x, x, w)$, then $\omega^w \leqslant |e(x)| < \omega^\omega$, since for all $2^w \cdot 3^z$ except one $|e_k(2^w \cdot 3^z, x)| < \omega^k$ while $|e_{k,w}(2^w \cdot 3^z, x)| = \omega^w$ for some z if $T_1^{O^{(2k)}}(x, x, w)$, by Theorem 2.1. If $\neg Ew\, T_1^{O^{(2k)}}(x, x, w)$, then $|e(x)| \leqslant \omega^k$ by Theorem 2.1. By Theorem 1 [5], let $c(k)_i \in O$ such that $|c(k)_i| = \omega \cdot |1 + e(i)|$. Define $a(k)' = 3 \cdot 5^{a(k)}$ so that $a(k)_O = c(k)_O$ and $a(k)_{i+1} = a(k)_i +_O c(k)_{i+1}$. Clearly, $a(k)'$ grows faster to ω^ω than $\omega^{f(i)}$ for any f recursive on $O^{(2k)}$. Finally, define

$$a_O = a(1)_O, \ldots, a_{i+1} = a_i +_O (a(1)_{i+1} +_O a(2)_{i+1} + \ldots +_O a(i+1)_{i+1}).$$

Clearly, $|a'| = |3 \cdot 5^a| = \omega^\omega$ and $a' \not\prec^{O^{(2k)}} b'$, if $b' \in O$ and $|b_i| = \omega^i$ (suppose $a' \prec b'$ via f recursive in $O^{(2k)}$ just choose an index e for f larger than k, then $\{e\}^{O^{(2k)}}(e) < |a(k)_e| \leqslant \omega^{f(e)}$, a contradiction). Thus, by Theorem 1.1, $H_{a'} \not\leqslant_1^{O^{(2k)}} H_{b'}$ for any k but by Theorem 1.2 (a) $H_{a'}$ and $H_{b'}$ are isomorphic by a permutation of $\mathcal{N}$ recursive in H_ω.

A similar argument works for building $a' \in O$ such that $|a'| = \omega^{\omega \cdot n + k}$ $(k \neq O)$ such that a_i grows faster to $|a'|$ than does $\omega^{\omega \cdot n + (k-1)} \cdot f(i)$ for any function f recursive in $H_{\omega \cdot n + 2k - 1}$. Since for $k = 2$,

$$T_1^{H_{\omega \cdot n + 2k - 1}}(x, x, w) \equiv E!z(x_1)(E!y_1)(x_2)\, R(x, w, z, x_1, y_1, x_2)$$

with R recursive in $H_{\omega \cdot n}$. By Corollary 3.1 we find $e_x \in W$ such that $\omega^{\omega \cdot n + 1} \cdot w \leqslant |e_x| < \omega^{\omega \cdot n + 2}$ if $T_1^{H^{(3)}_{\omega \cdot n}}(x, x, w)$, and $|e_x| \leqslant \omega^{\omega \cdot n + 1}$, otherwise. Let $c_i \in O$ such that $|c_i| = \omega \cdot |1 + e_i|$ by Theorem 1 in [5], then $\omega^{\omega \cdot n + 1} \cdot w \leqslant |c_i| < \omega^{\omega \cdot n + 2}$, if $T_1^{H^{(3)}_{\omega \cdot n}}(i, i, w)$ and $|c_i| \leqslant \omega^{\omega \cdot n + 1}$, otherwise. Let $a_O = c_O, \ldots, a_{i+1} =$

$a_i +_O c_{i+1}$ and $a' = 3 \cdot 5^a$. Clearly, $|a'| = \omega^{\omega \cdot n+2}$ and $a' \not\leq_{\mathsf{H}^{(3)}_{\omega\cdot n}} b'$, if $|b_i| = \omega^{\omega \cdot n+1} \cdot i$. By Theorem 1.1 and 1.2, $\mathsf{H}_{a'} \not\equiv_1^{\mathsf{H}^{(3)}_{\omega\cdot n}} \mathsf{H}_{b'}$ but $\mathsf{H}_{a'}$ and $\mathsf{H}_{b'}$ are isomorphic by a permutation of $\mathcal{N}$ recursive in $\mathsf{H}^{(4)}_{\omega\cdot n}$. Similarly, for $\mathsf{H}_{\omega\cdot n}$ being necessary for showing $\mathsf{H}_{a'}$ and $\mathsf{H}_{b'}$ isomorphic if $|a'| = |b'| = \omega^{\omega\cdot n}$.

These results are summarized by the following theorem.

Theorem 3.3 *For all* $\beta < \omega^2$, $|a'| = |b'| = \omega^\beta$, *the* H_c *determined as in* Theorem 1.2 *is the smallest level possible in the hyperarithmetic hierarchy in order to have all* $\mathsf{H}_{a'}$ *and* $\mathsf{H}_{b'}$ *isomorphic, using functions recursive in* H_c.

REFERENCES

[1] Davis, M., *On the Theory of Recursive Unsolvability*, Ph.D. Thesis, Princeton University (1950).

[2] Kleene, S. C., *Introduction to Metamathematics*, New York, Toronto, Amsterdam, and Gronigen (1952).

[3] Kleene, S. C., "On the forms of the predicate in the theory of constructive ordinals," *American Journal of Mathematics*, vol. 66 (1944), pp. 41-58.

[4] Kleene, S. C., "Arithmetical predicates and function quantifiers," *Transactions of the American Mathematical Society*, vol. 79 (1955), pp. 312-340.

[5] Kreisel, G., J. Shoenfield, and Hao Wang, "Number theoretic concepts and recursive well-orderings," *Archiv für mathematische Logik und Grundlagen forschung*, vol. 5 (1960), pp. 42-64.

[6] Moschovakis, Y. N., "The many-one degrees of the predicates $\mathsf{H}_a(x)$," *Pacific Journal of Mathematics*, vol. 18 (1966), pp. 329-342.

[7] Nelson, G. C., "Isomorphism types of the hyperarithmetic sets," Abstract in *The Journal of Symbolic Logic*, vol. 36 (1971), p. 374.

[8] Nelson, G. C., "Many-one reducibility within the Turing degrees of the hyperarithmetic sets $\mathsf{H}_a(x)$," *Transactions of the American Mathematical Society*, vol. 191 (1974), pp. 1-44.

[9] Rogers, H., Jr., *Theory of Recursive Functions and Effective Computability*, New York (1967).

[10] Spector, C., "Recursive well-orderings," *The Journal of Symbolic Logic*, vol. 20 (1955), pp. 151-163.

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