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TWO COMMENTS ON LEMMON'S *BEGINNING LOGIC*

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This paper is largely of pedagogic interest, for no fundamental discoveries are reported herein. But we believe that what we have to say will be of interest to many of those who use Lemmon's admirable textbook *Beginning Logic*. Our remarks are confined to Lemmon's development of the sentential calculus, which is presented in natural deduction form. A summary of the rules of derivation is to be found on pp. 39f. of the text, and familiarity with these rules will be assumed without further ado.

1 *Augmentation of premises* Lemmon employs rules of conditional proof and reductio ad absurdum, stated in the following form:

Conditional Proof (**CP**) Given a proof of B from A as assumption, we may derive A → B as conclusion on the remaining assumptions (if any).

Reductio ad Absurdum (**RAA**) Given a proof of B & -B from A as assumption, we may derive -A as conclusion on the remaining assumptions (if any).

A significant feature of these rules is that each requires the assumption A actually to have been used in the proof of B (in the case of **CP**) or B & -B (in the case of **RAA**). Thus the following two proofs are incorrect in the system as advertised.[1]

(a)		$P \vdash Q \rightarrow P$		(b)		$P \,\&\, -P \vdash Q$	
1	(1)	P	**A**	1	(1)	$P \,\&\, -P$	**A**
2	(2)	Q	**A**	2	(2)	$-Q$	**A**
1	(3)	$Q \rightarrow P$	2, 1 **CP**	1	(3)	$--Q$	2, 1 **RAA**
				1	(4)	Q	3 **DN**

The illicit move is in each case at step (3), where there is discharged an assumption that has not in fact played any role in the proof. The question consequently arises of how these two sequents, which are perfectly valid, are to be proved.

1. We follow Lemmon in using 'P', 'Q', and suchlike as propositional variables. 'A', 'B', on the other hand, are metavariables.

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Lemmon offers a proof of (a) as example 50 on p. 59. This proof, when written out in full, occupies 24 lines. Example 51 is a proof of $-P \vdash P \rightarrow Q$, a sequent that is obviously closely related to (b); it too runs to 24 lines. Two pages later, in discussing these 'paradoxes of material implication' Lemmon says: 'Anyway, 50 and 51 can be proved using only the rules **A**, **&I**, **&E**, **RAA**, **DN**, and **CP**, in each case in nine lines; it is an interesting exercise to discover these "independent" proofs, since they reveal how difficult it is to "escape" the paradoxes.'

Now although there seems to be no proof of 51 in fewer than nine lines, there is in fact a proof of 50 in as few as five. This proof, which requires neither **RAA** nor **DN**, proceeds as follows:

1	(1)	P	**A**
2	(2)	Q	**A**
1, 2	(3)	$P \& Q$	1, 2 **&I**
1, 2	(4)	P	3 **&E**
1	(5)	$Q \rightarrow P$	2, 4 **CP**.

A similar manipulation of the rules **&I** and **&E** leads to the nine-line proof of 51 that Lemmon makes mention of in the passage quoted above.

It can be seen that this trick, which allows us to involve a premise that is in principle redundant, is quite generally performable. Its availability allows to establish the validity of the *rule of (finite) augmentation of premises*, that if $A_0, \ldots A_{m-1} \vdash C$ is valid, then so is $A_0, \ldots, A_{m-1}, B_0, \ldots, B_{n-1} \vdash C$.

2 *Derived rules of inference* On p. 57 Lemmon calls *derived rules* rules 'which expedite our proof-techniques but can be shown not to increase our derivational power, . . . in contrast to our basic ten rules, which may be called *primitive rules*'. Shortly afterwards he backtracks to note (p. 62) that '**MTT** need not have been taken as a primitive rule, but can be obtained as a derived rule from the others'. The proof is a simple application of modus ponens and **RAA**. In a similar manner we can show that **MPP** need not be taken as primitive; for with the assistance of **MTT**, **RAA**, and **DN** we can derive B from A and $A \rightarrow B$. What is much more interesting, however, is that the rule of reductio ad absurdum is itself redundant, in the presence of the remaining rules. It seems clear from his remark (p. 26) that **RAA** is 'in many ways the most powerful and the most useful' of the rules that Lemmon failed to appreciate this point, which may come as a surprise even to skilled exponents of natural deduction techniques.

To achieve full generality we must state **RAA** as a family of rules, indexed by the number of premises mentioned. Let $\mathbf{RAA}_n$ be the rule: If $A_0, \ldots, A_{n-1} \vdash D \& -D$, then $A_1, \ldots, A_{n-1} \vdash -A_0$. It is our task to show that for every n the rule $\mathbf{RAA}_n$ can be understudied by other rules. (Lemmon never considers sequents with more than finitely many premises, so no further cases arise.) We shall take advantage of Lemmon's rule **SI** of sequent introduction, a derived rule that permits us in an obvious way to utilize sequents that are already proved. We prove first that if $\mathbf{RAA}_2$ is redundant then every $\mathbf{RAA}_{n+3}$ is redundant. The proof is by induction on n, taking the redundancy of $\mathbf{RAA}_2$ as the basis.

Suppose that for some $n \geqslant 2$ the rule $\mathbf{RAA}_n$ is redundant. Let the antecedent of $\mathbf{RAA}_{n+1}$ hold for some sentences $A_0, \ldots, A_n$. Then by use of **&E** and **SI** we can convert this proof of a contradiction from $A_0, \ldots, A_n$ into a proof of a contradiction from $A_0, \ldots, A_{n-1}$ & A_n. The induction hypothesis then tells us that $A_1, \ldots, A_{n-1}$ & $A_n \vdash -A_0$. The use of **&I** and **SI** will transform this proof to a proof of the sequent $A_1, \ldots, A_n \vdash -A_0$. Thus every proof that involves $\mathbf{RAA}_{n+1}$ can be reduced to one that involves only $\mathbf{RAA}_n$. Since the latter rule is superfluous, so too is the former.

It should be noted that we cannot adapt this proof to establish that if $\mathbf{RAA}_1$ is redundant, so is $\mathbf{RAA}_2$. We can, however, show the converse, that if $\mathbf{RAA}_2$ is redundant, so is $\mathbf{RAA}_1$. For suppose that $A_0 \vdash D$ & $-D$. By augmentation of premises (which, as we saw above, does not depend on **RAA**) we have $A_0, A_0 \rightarrow A_0 \vdash D$ & $-D$, so that by $\mathbf{RAA}_2$ we can conclude that $A_0 \rightarrow A_0 \vdash -A_0$. The following microproof establishes that $\vdash -A_0$.

1	(1)	A_0	**A**
	(2)	$A_0 \rightarrow A_0$	1, 1 **CP**
	(3)	$-A_0$	2 **SI**.

It remains only to prove that RAA_2 is redundant. We assume therefore that $A_0, A_1 \vdash D$ & $-D$, and use this sequent to show that there is a proof of the sequent $A_1 \vdash -A_0$ that nowhere uses $\mathbf{RAA}_2$.

1	(1)	A_0	**A**
2	(2)	A_1	**A**
1, 2	(3)	D & $-D$	1, 2 **SI**
1, 2	(4)	D	3 **&E**
1	(5)	$A_1 \rightarrow D$	2, 4 **CP**
1, 2	(6)	$-D$	3 **&E**
1, 2	(7)	$-A_1$	5, 6 **MTT**
2	(8)	$A_0 \rightarrow -A_1$	1, 7 **CP**
2	(9)	$--A_1$	2 **DN**
2	(10)	$-A_0$	8, 9 **MTT**.

Analogously we can give a direct proof of the redundancy of $\mathbf{RAA}_1$. We assume that $A_0 \vdash D$ & $-D$. In this proof we will make use of Lemmon's example 50, which as we saw can be proved without any use of reductio ad absurdum.

1	(1)	A_0	**A**
1	(2)	D & $-D$	1 **SI**
1	(3)	D	2 **&E**
1	(4)	$(A_0 \rightarrow A_0) \rightarrow D$	3 **SI** 50
1	(5)	$-D$	2 **&E**
1	(6)	$-(A_0 \rightarrow A_0)$	4, 5 **MTT**
	(7)	$A_0 \rightarrow -(A_0 \rightarrow A_0)$	1, 6 **CP**
	(8)	$A_0 \rightarrow A_0$	1, 1 **CP**
	(9)	$--(A_0 \rightarrow A_0)$	8 **DN**
	(10)	$-A_0$	7, 9 **MTT**

It is obviously essential to these last two proofs that we have at our disposal the intuitionistic half of the law of double negation (from A to derive --A). In the presence of reductio ad absurdum, however, this half-rule is itself clearly redundant.

Note added November 1976. It seems that it should be possible to prove the rule of augmentation of premises (and thus examples 50 and 51) by using the rules vI and vE, rather than &I and &E. The proof for a single augmented premise would run:

1	(1)	P	A
2	(2)	Q	A
1	(3)	$P \vee (Q \rightarrow P)$	1 vI
4	(4)	$Q \rightarrow P$	A
2, 4	(5)	P	2, 4 MPP
1, 2	(6)	P	3, 1, 1, 4, 5 vE.

It is, however, not entirely obvious that Lemmon's statements (pp. 23, 40) of the rule vE do indeed license this derivation, since it is not obvious that assumption (1) can fail to be discharged at line (6). But since such an interpretation would render the system unsound we do better to discount it and to read Lemmon's text in the most favorable light.

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