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A PATCHING LEMMA

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Let S be a set. A local system L on S is a collection of subsets of S such that for each finite subset $\{x_1, \ldots, x_n\} \subseteq S$ there is an $H \in \mathsf{L}$ with $\{x_1, \ldots, x_n\} \subseteq H$. In [2] we stated the following patching lemma which we called Theorem H and which group theorists have found useful in proving local theorems (see [8], pp. 96-100):

Patching lemma *Let L be a local system on S, F a set, n a positive integer. Suppose that for each $H \in \mathsf{L}$ there is a function f_H: $H^n \to F$ and $\{f_H(x) \mid H \in \mathsf{L}\}$ is finite for each $x \in S^n$. Then there is a function f: $S^n \to F$ such that for any finite subset $K \subseteq S^n$ there is an $H \in \mathsf{L}$ with $K \subseteq H^n$ and $f|K = f_H|K$.*

We now give a proof of this lemma based on the Boolean prime ideal theorem (**BPI**) and some simple properties of ultrafilters. By [1] this really avoids the axiom of choice.

Proof: For each $x \in S^n$ let $I_x = \{H \in \mathsf{L} \mid x \in H^n\}$. $I_x \subseteq \mathsf{L}$ and by the properties of local systems $\{I_x \mid x \in S^n\}$ has the finite intersection property. By **BPI** there is a nontrivial ultrafilter $\mathcal{M}$ on L such that $I_x \in \mathcal{M}$ for each $x \in S^n$.

For each $x \in S^n$ let $A_x = \{f_H(x) \mid H \in \mathsf{L}\}$. By assumption each A_x is finite. For $x \in S^n$, $a \in A_x$ let $V(x, a) = \{H \in \mathsf{L} \mid x \in H^n, f_H(x) = a\}$. It is easy to see that $I_x = \bigcup\{V(x, a) \mid a \in A_x\}$. Hence $\bigcup\{V(x, a) \mid a \in A_x\} \in \mathcal{M}$. But $\{V(x, a) \mid a \in A_x\}$ is a finite collection of disjoint sets whose union belongs to the ultrafilter $\mathcal{M}$. Thus there is a unique $a^* \in A_x$ such that $V(x, a^*) \in \mathcal{M}$. We now define f: $S^n \to F$ as follows: $f(x) = a^*$ where $V(x, a^*) \in \mathcal{M}$. Let K be a finite subset of S^n. $\{V(x, f(x)) \mid x \in K\}$ is a finite collection of elements of $\mathcal{M}$. Hence $\bigcap\{V(x, f(x)) \mid x \in K\} \in \mathcal{M}$ and there is an $H \in \mathsf{L}$ which is in this intersection. If $x \in K$ then $x \in H^n$ and $f_H(x) = f(x)$. And f has the desired property.

Remarks With $n = 1$ and $\mathsf{L} = \{H \mid H$ finite subset of $S\}$ the patching lemma is the well-known Rado selection lemma [6]. In [3] W. A. J. Luxemburg gave a proof of Rado's lemma using ultraproducts. Our proof avoids the mention of ultraproducts. With $n = 1$ and L a net (in the sense of A. Robinson) we obtain Robinson's valuation lemma [7].

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The patching lemma can also be of use when S and F support additional structure. For example, if S and F are universes of relational systems of the same similarity type and F is finite and $\mathbf{L} = \{H | H \text{ a finite subsystem of } S\}$ and each f_H is a homomorphism, then f is a homomorphism. The local nature of the definition of homomorphism makes this property of f easily verifiable. This example is called Grätzer's theorem by Y. Nakano in [5]. Employing this example when S is a Boolean algebra, F is the two-element Boolean algebra and f_H is a homomorphism such that $f_H[H \cap I] = \{0\}$ where I is a given ideal of S leads to a proof of **BPI**. In this application one employs the axiom of choice for families of finite sets (**ACF**) in picking an f_H for each finite subalgebra H. Thus in Zermelo-Fraenkel set theory (**ZF**) together with **ACF** we have the equivalence of the patching lemma and **BPI**. And further we can say that the patching lemma is independent of **ZF** + **ACF**. In the model of **ZF** + **ACF** which appears in [1] the patching lemma holds and in the model of **ZF** + **ACF** which appears in [4] the patching lemma fails. In fact, these two models show the independence of the patching lemma with respect to the stronger axiom system **ZF** + "The universe is linearly ordered."

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