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NEGATION AS A SIGN OF NEGATIVE JUDGMENT

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1 *Introduction* We need to form negative as well as affirmative statements because we need to mark falsity as well as truth, to register rejection as false as well as acceptance as true, and to deny as well as to assert. But we do not need an embeddable negation operator any more than we need an embeddable affirmation operator, provided operators are available for forming conjunctions, disjunctions, conditionals, and universal and existential generalizations. This thesis, which is examined and defended in what follows, is of purely theoretical significance. Its significance is at most theoretical because there is nothing wrong, and much that is convenient, in having an embeddable negation operator. But it seems to me to be of some philosophical importance in relation to the question of the meaning of negation. In particular, it opens the way for an attempt to construe the meaning of negation as deriving from the mental or behavioral phenomena of negative judgment, disbelief, and denial.

These notions can be made more precise as follows. Let $\mathcal{L}$ be a first-order language with primitive operators $\wedge$, $\vee$, $\supset$, $\sim$, $\forall$, and $\exists$, employed and understood as usual. Let $\mathcal{L}_\alpha$ be just like $\mathcal{L}$, semantically as well as syntactically, except for lacking the negation operator. $\mathcal{L}_\alpha$ is thus the negationless sublanguage of $\mathcal{L}$. Now $\mathcal{L}_\alpha$ is expressively weaker than $\mathcal{L}$; that is, there are sentences of $\mathcal{L}$ to which no sentence of $\mathcal{L}_\alpha$ is logically equivalent. Moreover, it is hard to see how the logic of $\mathcal{L}_\alpha$ could be completely formalized in an "intrinsic" manner—i.e., without allowing in formal proofs or derivations excursions through sentences of $\mathcal{L}$ that involve negation, or using at least external signs representing falsity or denial. Both of these deficiencies, however, can be made up by extending $\mathcal{L}_\alpha$ just so as to permit formation of a sentence $\sim A$ for each *negation-free* sentence A, with $\sim$ understood just as in $\mathcal{L}$. Let us call the resulting language $\mathcal{L}^*$. It is easy to characterize $\mathcal{L}^*$, syntactically and semantically, as a self-contained language. At the same time, $\mathcal{L}^*$ is a sublanguage of $\mathcal{L}$.

Now $\mathcal{L}^*$ is like $\mathcal{L}$ except that negations, sentences of the form $\sim A$, never occur as proper constituents of sentences of $\mathcal{L}^*$. This is what is

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meant by calling ~ nonembeddable in $\mathcal{L}^*$. (Hence, of course, it is also noniterable.) Let us say, therefore, that in $\mathcal{L}^*$ negation is *external*, whereas $\mathcal{L}$ permits *internal* negations as well.

This feature of $\mathcal{L}^*$ is of philosophical interest because it makes it possible to construe the external negation operator of $\mathcal{L}^*$ as representing *negative judgment and disbelief*, or as a sign of *denial*, where denial is viewed as a form of speech act opposite to assertion, rather than as assertion of a negation. It is hard to construe in these ways the internal occurrences of negation permitted in $\mathcal{L}$; for judgments cannot be constituents of the objects of judgment (as pointed out by Frege in [1]), nor can speech acts be constituents of the vehicles of speech acts. In $\mathcal{L}^*$, however, where it is always external, the negation operator can be construed pragmatically as a sign of negative judgment, disbelief, and denial, contrasting with the null sign of affirmative judgment, belief, and assertion. (This contrast could be heightened by considering, in place of $\mathcal{L}^*$, the language $\mathcal{L}^{\pm}$ of positively or negatively signed negation-free sentences of $\mathcal{L}_\alpha$; but metatheoretic comparison of $\mathcal{L}^*$ with $\mathcal{L}$ is simpler because $\mathcal{L}^*$ is a sublanguage of $\mathcal{L}$.) For a speaker of $\mathcal{L}^*$, therefore, the objects of judgment, of belief and disbelief, and of assertion and denial, would all be negation-free. And thus a considerable range of perplexing puzzles about negation would seem to be significantly reduced. For example, contradiction can then be construed in terms of conflicting judgment or belief rather than in terms of inconsistent "propositional content," and such notions as "negative facts" are thereby decisively banished. For these reasons, it seems to be worth demonstrating that—and clarifying the sense in which—$\mathcal{L}^*$ would be no less adequate in principle as a language for inquiry, for developing a representation of reality, than $\mathcal{L}$. To this end the following propositions are made precise and demonstrated below:

(1.1) *For any sentence of* $\mathcal{L}$ *there is a logically equivalent sentence of* $\mathcal{L}^*$.
(1.2) *For any theory* T *in* $\mathcal{L}$, *there is a theory* T* *in* $\mathcal{L}^*$ *such that, for each thesis of* T, *there is a logically equivalent thesis of* T*, *and such that* T* *is axiomatizable if* T *is*.
(1.3) *The logic of* $\mathcal{L}^*$ *can be completely formalized in an intrinsic manner; that is, a finitary concept of deducibility can be specified such that for any sentences* $A_1, \ldots, A_n (n \geqslant 0)$ *and* B *of* $\mathcal{L}^*$, *if* $A_1, \ldots, A_n$ *logically imply* B, *then (and only then)* B *can be formally deduced from* $A_1, \ldots, A_n$ *using only sentences of* $\mathcal{L}^*$.

2 *Exnegation* A useful means for verifying (1.1)-(1.3) is provided by a normal form result which seems to be of some interest in its own right, yet somehow to have escaped inclusion in the traditional canon of first-order logic. Given any formula C of $\mathcal{L}$, it is easy to see that by successive replacements in accordance with well known equivalences we can move internal negations outward, and cancel double negations, thus obtaining a unique formula D which is both logically equivalent to C and either negation-free or the single external negation of a negation-free formula. Let us call this procedure—the steps of which could evidently be uniquely ordered—*exnegation*, and the formula so obtained from a formula C, *the exnegate of* C, or briefly, ex$C$. For example, if C is $(\exists x)(\sim Fx \wedge (\exists y)(Gay \wedge \sim Rxy))$, then ex$C$ is $\sim(\forall x)(Fx \vee (\forall y)(Gay \supset Rxy))$; and if C is $((\sim(\forall x)Fx \wedge (\exists x) \sim Gx) \supset \sim(\forall x)(Fx \supset \sim \sim Rxy))$, then ex$C$ is $((\forall x)(Fx \supset Rxy) \supset ((\forall x)Fx \vee (\forall x)Gx))$. In every case, notice, ex$C$ will contain just the same predicates, parameters, and variables as C, and will be a sentence (i.e., a closed formula) iff C is a sentence. This simple "normal form" result can be recorded as follows:

(2.1) *For any formula (sentence)* C *of* $\mathcal{L}$, ex$C$ *is a formula (sentence) that is logically equivalent to* C, *and for some unique negation-free formula (sentence)* A *of* $\mathcal{L}$ *and of* $\mathcal{L}_{|\alpha}$, *either* A = ex$C$ *or* $\sim A$ = ex$C$.

But for any sentence C of $\mathcal{L}$, ex$C$ is a sentence of $\mathcal{L}^*$. Hence (2.1) verifies (1.1).

Assuming the most natural replacements to be built into the procedure of exnegation (as presupposed in the preceding illustrations), the following proposition (2.2) can easily be established. Alternatively, (2.2) could be taken as a recursive definition of exnegation.

(2.2) (*i*) *For any negation-free formulas* A, B *of* $\mathcal{L}$ *and any variable* x *of* $\mathcal{L}$,

(1) $\text{ex}A = A$
(2) $\text{ex} \sim A = \sim A$
(3) $\text{ex} \sim \sim A = A$
(4) $\text{ex}(A \wedge \sim B) = \sim(A \supset B)$
(5) $\text{ex}(\sim A \wedge B) = \sim(B \supset A)$
(6) $\text{ex}(\sim A \wedge \sim B) = \sim(A \vee B)$
(7) $\text{ex}(A \vee \sim B) = (B \supset A)$
(8) $\text{ex}(\sim A \vee B) = (A \supset B)$
(9) $\text{ex}(\sim A \vee \sim B) = \sim(A \wedge B)$
(10) $\text{ex}(A \supset \sim B) = \sim(A \wedge B)$
(11) $\text{ex}(\sim A \supset B) = (A \vee B)$
(12) $\text{ex}(\sim A \supset \sim B) = (B \supset A)$
(13) $\text{ex}(\forall x) \sim B = \sim(\exists x)B$
(14) $\text{ex}(\exists x) \sim B = \sim(\forall x)B$;

and (*ii*) *for any formulas* C, D *and any variable* x *of* $\mathcal{L}$,

(1) $\text{ex ex}C = \text{ex}C$
(2) $\text{ex} \sim C = \text{ex} \sim \text{ex}C$
(3) $\text{ex} \sim \sim C = \text{ex}C$
(4) $\text{ex}(C \wedge D) = \text{ex}(\text{ex}C \wedge \text{ex}D)$
(5) $\text{ex}(C \vee D) = \text{ex}(\text{ex}C \vee \text{ex}D)$
(6) $\text{ex}(C \supset D) = \text{ex}(\text{ex}C \supset \text{ex}D)$
(7) $\text{ex}(\forall x)C = \text{ex}(\forall x)\text{ex}C$
(8) $\text{ex}(\exists x)C = \text{ex}(\exists x)\text{ex}C$

Let us call a formula *simply negative* if it is of the form $\sim A$ for negation-free A. Then in view of (2.2), the following is easy to verify:

(2.3) *For any formula* C *of* $\mathcal{L}$, ex$C$ *is negation-free (simply negative) iff* ex $\sim C$ *is simply negative (negation-free)*.

A more interesting property of exnegation follows from the fact that every negation-free sentence is true under any interpretation that has a unit domain $\mathfrak{D}$ and assigns the sole element of $\mathfrak{D}$ to each term (i.e., individual constant), $\mathfrak{D}$ to each one-place predicate, and the set of all ordered n-tuples of the single element of $\mathfrak{D}$ to each n-place predicate (for $n \geq 2$). Now suppose C is any logically true sentence of $\mathcal{L}$. Then by (2.1), ex$C$ is logically true. But in that case, if ex$C$ = $\sim A$, then A would be negation-free but not satisfiable, contradicting the fact noted above. Hence ex$C$ must be negation-free if C is logically true. Similarly if ex $\sim C$ is simply negative, so that by (2.3) ex$C$ is negation-free and thus satisfiable, then by (2.1) C must be satisfiable. Thus:

(2.4) *For any sentence C of $\mathcal{L}$, if C is logically true, then* ex$C$ *is negation-free; and if* ex $\sim C$ *is (simply) negative, then C is satisfiable.*

3 *Comparison of the logics of $\mathcal{L}$ and $\mathcal{L}^*$* Some abbreviating notation becomes convenient now. Let S, S_α, and S^* be the sets of all sentences of $\mathcal{L}$, $\mathcal{L}_\alpha$, and $\mathcal{L}^*$, respectively. Where Γ is any subset of S, let exΓ be the set of all exnegates of sentences in Γ. Thus ex$S = S^*$, and $S_\alpha \subset S^* \subset S$. Next, let C_1^n be the sequence (or list) $C_1, \ldots, C_n$ ($n \geqslant 0$), and exC_1^n, the sequence ex$C_1, \ldots,$ exC_n. Also, let 'Φ' and 'ψ' ('A' and 'B'), with or without subscripts, henceforth range just over $S^*(S_\alpha)$. And finally, let us use '$\models$' as usual to assert (classical) logical truth and implication, and '$\dashv\models$' to assert logical equivalence. Now an evident consequence of (2.1) can be stated as follows:

(3.1) *For any sentences $C_1, \ldots, C_n$ ($n > 0$) and D of $\mathcal{L}$, $C_1^n \models D$ iff* ex$C_1^n \models$ ex$D$.

Where Γ and Δ are any subsets of S, let us call Γ and Δ equivalent iff both for each $C \in \Gamma$ there is some $D \in \Delta$, and for each $D \in \Delta$ there is some $C \in \Gamma$, such that $C \dashv\models D$. Then the following is also evident in view of (2.1):

(3.2) *For any set Γ of sentences of $\mathcal{L}$, Γ and the set* exΓ *of sentences of $\mathcal{L}^*$ are equivalent.*

Suppose we regard a subset of $S(S^*)$ as *a belief system in $\mathcal{L}(\mathcal{L}^*)$*—i.e., as containing all and only the statements believed, or accepted as true, by some speaker of $\mathcal{L}(\mathcal{L}^*)$. Then (3.2) shows a sense in which whatever can be believed about the world by a speaker of $\mathcal{L}$ can also be believed by a speaker of $\mathcal{L}^*$. This conclusion depends on the fact that ex$C$ and C are not merely logically equivalent, but also contain just the same predicates and terms and hence, on any fixed interpretation, would describe the same state of affairs or "express the same thing about the world." But then, furthermore, by possessing the set of *beliefs and disbeliefs* which exΓ can also be thought of as representing, one whose judgments all pertain to statements in $\mathcal{L}_\alpha$ can in an obvious sense "hold the same view of the world" as a speaker of $\mathcal{L}$ whose belief system is represented by Γ. That is, suppose X is a speaker of $\mathcal{L}$ whose view of the world consists in his acceptance as true of just the statements in Γ; and suppose Y is a speaker of $\mathcal{L}^*$, whose view of the world can be described as such that for each $\Phi \in$ exΓ, (1) if Φ is negation-free then Y accepts Φ as true, and (2) if Φ is simply negative, so that $\Phi = \sim A$ for negation-free A, then Y rejects A as false. Then if we suppose, as seems quite plausible, that there is no cognitively significant difference between X's accepting C and Y's accepting A where $A =$ ex$C$, or between X's accepting C and Y's rejecting A where $\sim A =$ ex$C$, provided in both cases that X and Y interpret the common vocabulary of $\mathcal{L}$ and $\mathcal{L}^*$ in the same way, it follows that X and Y hold the same view of the world. But each statement regarding which Y holds a belief or disbelief is negation-free.

Furthermore, it is now easy to establish thesis (1.2), interpreted as follows. Let us understand a *subvocabulary* (of $\mathcal{L}$ or of $\mathcal{L}^*$) to be a set of predicates and terms of $\mathcal{L}$ containing at least one predicate. And let us say that a sentence C of $\mathcal{L}$ is *couched* in a subvocabulary V iff only predicates and terms in V occur in C. Next, let us say a subset Γ of $S(S^*)$ is *$\mathcal{L}$-closed* (*$\mathcal{L}^*$-closed*) *in* V iff each sentence in Γ is couched in V, and each sentence of $\mathcal{L}(\mathcal{L}^*)$ that is couched in V and is logically implied by a finite (or null) sequence of sentences in Γ is also in Γ. Then we can define a *theory in* $\mathcal{L}(\mathcal{L}^*)$ to be a pair $\langle \mathsf{V},\Gamma\rangle$ such that V is a subvocabulary of $\mathcal{L}(\mathcal{L}^*)$ and Γ is $\mathcal{L}$-closed in V. Now suppose $\langle \mathsf{V},\Gamma\rangle$ is an arbitrary theory in $\mathcal{L}$. Then obviously each sentence in $\mathrm{ex}\Gamma$ is couched in V. Next, let ψ be an arbitrary sentence of $\mathcal{L}^*$ couched in V, and suppose $\Phi_1^n \models \psi$ for some $\Phi_1, \ldots, \Phi_n \in \mathrm{ex}\Gamma$ $(n \geqslant 0)$. Then for each $i = 1, \ldots, n$, there is a $C_i \in \Gamma$ such that $\mathrm{ex}C_i = \Phi_i$. But by (2.2), $\mathrm{ex}\psi = \psi$. Thus for some $C_1, \ldots, C_n \in \Gamma$, $\mathrm{ex}C_1^n \models \mathrm{ex}\psi$. Hence by (3.1), $C_1^n \models \psi$. But then $\psi \in \Gamma$, and so $\psi = \mathrm{ex}\psi \in \mathrm{ex}\Gamma$. Hence $\langle \mathsf{V},\mathrm{ex}\Gamma\rangle$ is a theory in $\mathcal{L}^*$. This shows that:

(3.3) *For any theory* $\langle \mathsf{V},\Gamma\rangle$ *in* $\mathcal{L}$, $\langle \mathsf{V},\mathrm{ex}\Gamma\rangle$ *is a theory in* $\mathcal{L}^*$.

(The converse of (3.3) does not hold. For note that no theory in $\mathcal{L}^*$ is a theory in $\mathcal{L}$—since, for example, for each $\Phi \in S^*$ it is clear that $\Phi \models \sim\sim\Phi$, $\sim\sim\Phi \in S$, but $\sim\sim\Phi \notin S^*$. And note also that $\mathrm{ex}\Gamma = \mathrm{ex}\ \mathrm{ex}\Gamma$. Hence where $\langle \mathsf{V},\Gamma\rangle$ is a theory in $\mathcal{L}$, $\langle \mathsf{V},\mathrm{ex}\Gamma\rangle$ is a theory in $\mathcal{L}^*$ by (3.3), and therefore by the same token so is $\langle \mathsf{V},\mathrm{ex}\ \mathrm{ex}\Gamma\rangle$. But $\langle \mathsf{V},\mathrm{ex}\Gamma\rangle$ is not a theory in $\mathcal{L}$. Thus there are theories $\langle \mathsf{V},\mathrm{ex}\Delta\rangle$ in $\mathcal{L}^*$ such that $\langle \mathsf{V},\Delta\rangle$ is not a theory in $\mathcal{L}$.)

Next, where $\langle \mathsf{V},\Gamma\rangle$ is a theory in $\mathcal{L}(\mathcal{L}^*)$, let us call a subset Σ of $S(S^*)$ an *$\mathcal{L}$-axiom* (*$\mathcal{L}^*$-axiom*) *set for* $\langle \mathsf{V},\Gamma\rangle$ iff Σ is a decidable subset of Γ and each sentence of Γ is logically implied by none or more sentences in Σ. Now suppose Σ is an $\mathcal{L}$-axiom set for a theory $\langle \mathsf{V},\Gamma\rangle$ in $\mathcal{L}$. By (3.3), $\langle \mathsf{V},\mathrm{ex}\Gamma\rangle$ is a theory in $\mathcal{L}^*$. It is natural then to suspect that $\mathrm{ex}\Sigma$ is an $\mathcal{L}^*$-axiom set for $\langle \mathsf{V},\mathrm{ex}\Sigma\rangle$. Clearly $\mathrm{ex}\Sigma \subset \mathrm{ex}\Gamma$, since $\Sigma \subset \Gamma$. Now suppose $\psi \in \mathrm{ex}\Gamma$. Then for some $D \in \Gamma$, $\psi = \mathrm{ex}D$. But then by (2.1), $D \models \psi$; and since ψ is couched in V and Γ is $\mathcal{L}$-closed in V, therefore $\psi \in \Gamma$. Thus for some $C_1, \ldots, C_n \in \Sigma$ $(n \geqslant 0)$, $C_1^n \models \psi$. Hence, by (3.1), $\mathrm{ex}C_1^n \models \mathrm{ex}\psi$; and since, by (2.2), $\mathrm{ex}\psi = \psi$, then $\mathrm{ex}C_1^n \models \psi$. But $\mathrm{ex}C_1, \ldots, \mathrm{ex}C_n \in \mathrm{ex}\Sigma$. Hence, each sentence in $\mathrm{ex}\Gamma$ is logically implied by none or more sentences in $\mathrm{ex}\Sigma$. Therefore, in order to show $\mathrm{ex}\Sigma$ to be an $\mathcal{L}^*$-axiom set for $\langle \mathsf{V},\mathrm{ex}\Gamma\rangle$, it suffices to show that $\mathrm{ex}\Sigma$ is decidable. For the sake of simplicity, now, let us suppose that Σ is either finite or specified by a finite list of axiom schemata. In the first case, $\mathrm{ex}\Sigma$ is obviously decidable; and in the second case, we can obtain an effective specification of $\mathrm{ex}\Sigma$ by forming the exnegates of the axiom schemata used to specify Σ. Hence, we can conclude:

(3.4) *If* $\langle V, \Gamma\rangle$ *is a theory in* $\mathcal{L}$ *for which* Σ *is an* $\mathcal{L}$*-axiom set which is either finite or specified by a finite list of schemata, then* $\text{ex}\Sigma$ *is an* $\mathcal{L}^*$*-axiom set for the theory* $\langle V, \text{ex}\Gamma\rangle$ *in* $\mathcal{L}^*$.

In view of (3.2)-(3.4), thesis (1.2) is evident. Moreover, if we again think of a subset Γ of S as a belief system in $\mathcal{L}$, but now as one that can be axiomatized relative to the logic of $\mathcal{L}$ by specification of finitely many axioms or axiom schemata, then in view of (3.2)-(3.4) $\text{ex}\Gamma$ can be regarded as a belief system in $\mathcal{L}^*$ which is not only equivalent to Γ—in a way that we have seen to be epistemologically strong—but which also can be axiomatized, relative to the logic of $\mathcal{L}^*$, likewise by specification of finitely many axioms or axiom schemata. (Slight modifications of the arguments given here would establish verbally identical results even if $\mathcal{L}$ lacked a disjunction operator. This is due to the fact that $(C \vee D) \dashv \models ((C \supset D) \supset D)$; for given any formula of such a language, exnegation could proceed as above except that parts of the forms $(\sim C \wedge \sim D)$ or $(\sim C \supset D)$ would be replaced by $\sim((C \supset D) \supset D)$ or $((C \supset D) \supset D)$, respectively; and of course there would be no disjunctions to replace. (This possibility was pointed out to me by Peter Harvey.) The important point is that the exnegate would be a formula of the same language.)

4 *Formal proof in* $\mathcal{L}$ *and* $\mathcal{L}^*$ In a semantic approach to logic it is easy to beg philosophical questions. This observation is pertinent to the assessment of the arguments in section **3**, inasmuch as these arguments *employ* many sorts of embedded negations. It is reasonable therefore to ask whether *deductive reasoning* from premisses stated in $\mathcal{L}^*$ to consequences stated in $\mathcal{L}^*$ can always be carried through *within* $\mathcal{L}^*$, i.e., without employing sentences with negative proper constituents. The model-theoretic arguments of section **3** do not on their face ensure an affirmative answer to this question; and a negative answer would indicate that $\mathcal{L}^*$ is *not*, even in principle, an adequate language for inquiry. It is possible, however, to establish the affirmative answer, and thus thesis (1.3), by specifying a logical calculus for $\mathcal{L}^*$ in which, for any sentences $\Phi_1, \ldots, \Phi_n, \psi$ of $\mathcal{L}^*$, ψ can be formally derived from $\Phi_1, \ldots, \Phi_n$, without use of sentences containing embedded negations, just in case $\Phi_1^n \models \psi$. Consider for example how $\sim(A \vee B)$ can be derived from $\sim A$ and $\sim B$. According to the systems of many introductory textbooks, the most natural derivation would be to infer $(\sim A \wedge \sim B)$ from $\sim A$ and $\sim B$ and then $\sim(A \vee B)$ by DeMorgan's law; but $(\sim A \wedge \sim B)$ is not a sentence of $\mathcal{L}^*$. Similarly, consider the problem of constructing a formal proof of Peirce's law, $(((A \supset B) \supset A) \supset A)$. In order to do this in any of various familiar systems which permit indirect derivation only by inferring a double negative by a *reductio* step and then eliminating the outer pair of negation operators, an excursion through sentences of $\mathcal{L}$ which are not sentences of $\mathcal{L}^*$ would be necessary. Thus everything depends on details, and the only elementary way of establishing an affirmative answer to the indicated question is to specify a definite system of formal proof and demonstrate that it has the desired property. In fact, this can be done easily and in various ways.

The most elegant method for this purpose is that of analytic tableaux, as presented e.g. by Smullyan [2]. Since in constructing tableaux negations are *added* only for proper constituents of a compound, and these are always negation-free in $\mathcal{L}^*$, we only need to stipulate that $A(\sim A)$ is shown to be deducible from $\Phi_1, \ldots, \Phi_n$ by the closure of the tableau generated by $\Phi_1, \ldots, \Phi_n, \sim A(\Phi_1, \ldots, \Phi_n, A)$—or that such a closed tableau *constitutes* the formal deduction in question. Natural deduction methods can also be adapted for this purpose. This can be shown most concisely by regarding such a system as a calculus of *sequents* of the form $C_1^n \rightarrow D$ $(n \geq 0)$. If each of C_1^n and D is a sentence of $\mathcal{L}(\mathcal{L}^*)$, then $C_1^n \rightarrow D$ will be called an *$\mathcal{L}$-sequent* (*$\mathcal{L}^*$-sequent*). A calculus of sequents is determined by an effective specification, e.g. by schemata, of a set of primitive rules of the form $\sigma_1; \ldots; \sigma_k \Rightarrow \sigma$ $(k \geq 0)$, where $\sigma_1, \ldots, \sigma_k$, and σ are sequents. A *derivation* (*in the calculus*) of a sequent ξ from sequents $\xi_1, \ldots, \xi_m$ $(m \geq 0)$ is defined to be a finite sequence θ of sequents such that the last member of θ is ξ, and for each member σ of θ, either σ is one of $\xi_1, \ldots, \xi_m$ or there is a primitive rule $\sigma_1, \ldots, \sigma_k \Rightarrow \sigma$ such that $\sigma_1, \ldots, \sigma_k$ are members of θ preceding σ in θ. A *proof* (*in the calculus*) *of* ξ is a derivation of ξ from the empty list of sequents. We say ξ is *derivable* from $\xi_1, \ldots, \xi_m$ iff there can be constructed a derivation of ξ from $\xi_1, \ldots, \xi_m$; or *provable* iff there can be constructed a proof of ξ. The proof of a sequent $C_1^n \rightarrow D$ is taken as constituting a formal deduction of D from $C_1, \ldots, C_n$.

Now consider the calculus **N** of $\mathcal{L}$-sequents determined by the following primitive rule schemata:

(N1) $\Rightarrow C_1^n \rightarrow C_i$ provided $n \geq 1$ and $i = 1, \ldots, n$
(N2) $C_1^n \rightarrow C \Rightarrow D_1^m \rightarrow C$ provided each of C_1^n is one of D_1^m
(N3) $C_1^n \rightarrow D;\ C_1^n \rightarrow E \Rightarrow C_1^n \rightarrow (D \wedge E)$
(N4a) $C_1^n \rightarrow (D \wedge E) \Rightarrow C_1^n \rightarrow D$
(N4b) $C_1^n \rightarrow (D \wedge E) \Rightarrow C_1^n \rightarrow E$
(N5a) $C_1^n \rightarrow D \Rightarrow C_1^n \rightarrow (D \vee E)$
(N5b) $C_1^n \rightarrow E \Rightarrow C_1^n \rightarrow (D \vee E)$
(N6) $C_1^n \rightarrow (D \vee E);\ C_1^n, D \rightarrow C;\ C_1^n, E \rightarrow C \Rightarrow C_1^n \rightarrow C$
(N7) $C_1^n, D \rightarrow E \Rightarrow C_1^n \rightarrow (D \supset E)$
(N8) $C_1^n \rightarrow (D \supset E);\ C_1^n \rightarrow D \Rightarrow C_1^n \rightarrow E$
(N9) $C_1^n, D \rightarrow E;\ C_1^n, D \rightarrow \sim E \Rightarrow C_1^n \rightarrow \sim D$
(N10) $C_1^n, \sim D \rightarrow E;\ C_1^n, \sim D \rightarrow \sim E \Rightarrow C_1^n \rightarrow D$
(N11) $C_1^n \rightarrow Ht/x \Rightarrow C_1^n \rightarrow (\forall x)H$ provided t is foreign to each of C_1^n and to H
(N12) $C_1^n \rightarrow (\forall x)H \Rightarrow C_1^n \rightarrow Ht/x$
(N13) $C_1^n \rightarrow Ht/x \Rightarrow C_1^n \rightarrow (\exists x)H$
(N14) $C_1^n \rightarrow (\exists x)H;\ C_1^n, Ht/x \rightarrow C \Rightarrow C_1^n \rightarrow C$ provided t is foreign to each of C_1^n and to H and C

Here C_1^n, D_1^n, C, D, and E are arbitrary sentences of $\mathcal{L}$, x is any variable, H is any formula in which no variable other than x has a free occurrence, t is any term, and Ht/x is the result of replacing each free occurrence of x in H by t. I shall use '$\vdash$' in the usual manner to assert derivability and provability in **N**. The calculus **N** is classically sound and complete: that is, for any sentences C_1^n, D of $\mathcal{L}$, $\vdash C_1^n \rightarrow D$ iff $C_1^n \models D$. In checking this claim by reference to more familiar systems, it is useful to note that $C_1^n \rightarrow \sim\sim D \vdash C_1^n \rightarrow D$, as the following derivation (schema) shows:

1.	$C_1^n \rightarrow \sim\sim D$	
2.	$C_1^n, \sim D \rightarrow \sim D$	(N1)
3.	$C_1^n, \sim D \rightarrow \sim\sim D$	1 (N2)
4.	$C_1^n \rightarrow D$	2,3 (N10)

Now consider the system **N*** of $\mathcal{L}^*$-sequents determined by the same schemata, (N1)-(N14). In derivations in **N***, of course, only $\mathcal{L}^*$-sequents may occur. Hence in (N1)-(N14) regarded as postulates of **N***, D and E must be *negation-free* sentences of $\mathcal{L}^*$, and H must be a *negation-free* formula of $\mathcal{L}^*$. In connection with **N***, I shall use '$\overset{*}{\vdash}$' to assert derivability and provability.

From this characterization of **N*** as a subcalculus of **N**, it is evident that any proof in **N*** of an $\mathcal{L}^*$-sequent $\phi_1^n \rightarrow \psi$ is likewise a proof of $\phi_1^n \rightarrow \psi$ in **N**. Hence by the soundness of **N**, if $\overset{*}{\vdash}\phi_1^n \rightarrow \psi$, then $\phi_1^n \models \psi$. Conversely, suppose $\phi_1^n \models \psi$. Now supposing $\psi = \sim A(A)$ for negation-free A, then the tableau generated by the method of Smullyan [2] for ϕ_1^n, A $(\phi_1^n, \sim A)$ will be closed. But it is easy to specify an effective procedure by which from such a tableau a proof in **N*** of $\phi_1^n \rightarrow \psi$ can be retrieved. (Verification of this claim is left to the reader.) Hence **N*** is both complete and sound in relation to $\mathcal{L}^*$ in the following sense:

(4.1) *For any sentences* $\phi_1, \ldots, \phi_n$ $(n \geq 0)$ *and* ψ *of* $\mathcal{L}^*$, $\overset{*}{\vdash}\phi_1^n \rightarrow \psi$ *iff* $\phi_1^n \models \psi$.

The significance of (4.1) depends on the fact that *only* $\mathcal{L}^*$*-sequents can occur in derivations and proofs in* **N***. Hence by (4.1), if $\phi_1^n \models \psi$, then there is a proof in **N*** which represents a formal deduction of ψ from $\phi_1, \ldots, \phi_n$ in which only sentences of $\mathcal{L}^*$ occur. And this established thesis (1.3). In view of (4.1), (3.1), and the completeness and soundness of **N**, the following proof-theoretic analogue of (3.1) is also evident:

(4.2) *For any sentences* $C, \ldots, C$ $(n \geq 0)$ *and* D *of* $\mathcal{L}$, $\vdash C_1^n \rightarrow D$ *iff* $\overset{*}{\vdash} \mathrm{ex}C_1^n \rightarrow \mathrm{ex}D$.

(It is also possible, though a bit tedious, to prove (4.2) by a purely proof-theoretic argument, without assuming the completeness or soundness of **N**.)

It is clear from (4.1) that the internal logic of $\mathcal{L}^*$ can be completely formalized in a quite natural way. And (4.2) makes it clear that proof-theoretic analogues of (3.3) and (3.4) can be proved in relation to **N** and **N***. It thus becomes evident that for any belief system Γ in $\mathcal{L}$ that can be deductively organized on the basis of the system **N**, there is an equivalent belief system in $\mathcal{L}^*$, namely $\mathrm{ex}\Gamma$, which can be deductively organized on the basis of the system **N***. This conclusion explicates and sums up (1.2) and (1.3). Hence $\mathcal{L}^*$ is as adequate a language for inquiry as $\mathcal{L}$, at least as far as deductive, and thus purely hypothetico-deductive methods are concerned.

Recall now the relationship of $\mathcal{L}^*$ to $\mathcal{L}_\alpha$, the negationless language corresponding to $\mathcal{L}$ and to $\mathcal{L}^*$. The sole feature distinguishing $\mathcal{L}^*$ from $\mathcal{L}$ is that the negation operator, common to both, is held external and non-embeddable in $\mathcal{L}^*$. And the philosophical motive for considering $\mathcal{L}^*$ and its logic is that each sentence of $\mathcal{L}^*$ can be regarded as representing either the acceptance as true or the rejection as false of some negation-free statement in $\mathcal{L}$—that is to say, some statement in $\mathcal{L}_\alpha$. We have thus shown that for any system of affirmative judgments, or beliefs, regarding statements in $\mathcal{L}$, there is an equivalent system of affirmative and negative judgments, or beliefs and disbeliefs, regarding statements in $\mathcal{L}_\alpha$. However, if sentences of $\mathcal{L}^*$ are thus construed as representing judgments regarding negation-free statements, then deductive reasoning on the part of a speaker of $\mathcal{L}^*$ must be construed as involving, as its immediate constituents, so to speak, not the statements he judges but rather judgments regarding them. Construing negation as a sign of negative judgment, disbelief, or denial thus calls for a revision of the traditional concepts of deductive reasoning and deducibility.

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