Notre Dame Journal of Formal Logic
Volume 30, Number 2, Spring 1989


# Some Compactness Results for Modal Logic

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**Abstract** A modal logic $L$ is said to be compact if every $L$-consistent set of formulas has a model on a frame for $L$. Some large classes of compact (noncompact) logics are identified, and it is shown that there are uncountably many compact (noncompact) logics.

A modal logic $L$ is *compact* if every $L$-consistent set of formulas has a model on a frame for $L$, *classically compact* if a set of formulas fails to have a model on a frame for $L$ only if some finite subset fails to have a model on a frame for $L$, *canonical* if determined by its canonical frame, and *complete* if determined by some class of frames. These four properties are related in an obvious way:

$$\begin{array}{ccccc} \text{CANONICALCY} & \Rightarrow & \text{COMPACTNESS} & \Rightarrow & \text{COMPLETENESS} \\ & & \Downarrow & & \\ & & \text{CLASSICAL COMPACTNESS} & & \end{array}$$

Given the amount of attention that has been lavished upon canonicalcy and completeness, it is therefore mildly surprising that compactness has enjoyed relatively little press. The first explicit mention of it would seem to be found in Corcoran and Weaver [1]. However, they were working with a different concept of a model than the now standard one used here, and as a result the nice connection between canonicalcy and compactness is lost. (See [2], in which they show that the canonical logics $T$ and $B$ are, on their account, noncompact.) Fine raises the issue of compactness at the end of [5], but his important work on this notion did not appear in print until more than a decade later with the publication of [6]. Even then, although he was undoubtedly aware that his argument could be generalized, only one example of a familiar, noncompact logic is actually mentioned. Hughes and Cresswell [10] go a bit further, giving several examples. Unfortunately, their proofs (and suggested proofs) contain a seductive error, and

*An earlier version of this paper was presented to the Annual Meeting of the Association for Symbolic Logic held in Chicago, Illinois, April 26–27, 1985. With the exception of Theorem 4, these results were obtained before I saw [6], [10], and [11].

*Received August 31, 1987*

correct proofs appear only in [11]. Finally, Fitting [8] and van Fraassen [9] show how to obtain compactness without going through canonicalcy, though all of the logics they consider are in fact canonical.

For complete logics compactness and classical compactness come to the same thing. But in general the two concepts pull apart. In [13] it is shown that there exists an incomplete logic $L$ with the same frames as $S4.3$. Given the canonicalcy, and hence classical compactness, of the latter, it follows that $L$ is also classically compact. Classical compactness therefore implies neither completeness, compactness, nor canonicalcy. Perhaps this helps explain why it is compactness rather than classical compactness which is generally discussed in modal contexts.

As we shall see, completeness fails to imply compactness or even classical compactness—and fails with a vengeance. The tougher question, apparently still open, is whether compactness implies canonicalcy. It is becoming increasingly clear, however, that if there is a counterexample, it is not likely to be simple. Fine (in [6] and [7]) has shown that the two notions are equivalent over two broad and important classes of logics (the complete elementary and "subframe" logics), and other results of this sort are also to be found in the literature. Here I shall identify some additional classes, running across and up through the lattice of modal logics, over which compactness and canonicalcy stand or fall together.

Let us say that a logic $L$ is *finitely accessible* if no frame for $L$ contains a world having infinitely many worlds accessible from it.

**Theorem 1** *If L is finitely accessible, then L is compact if and only if L is canonical.*

*Proof:* It is enough to show that if $L$ is finitely accessible and compact, then $L$ is canonical.

By compactness, every $L$-consistent set $\Gamma$ has a model $\mathfrak{A}_\Gamma$ based upon a frame for $L$. Form the "union" of these models. To be precise, where $\{(W_i, R_i, \phi_i)\}_i$ is a class of pairwise disjoint isomorphic copies of these $\mathfrak{A}_\Gamma$'s, put $W = \bigcup_i W_i$, $R = \bigcup_i R_i$, and $\phi = \bigcup_i \phi_i$, and set $\mathfrak{A} = (W, R, \phi)$. Define a function $f$ from $W$ to the domain of $\mathfrak{A}_L = (W_L, R_L, \phi_L)$, the canonical model for $L$, by $f(u) = \{A \mid (\mathfrak{A}, u) \vDash A\}$. Since $\mathfrak{F} = (W, R)$ is a frame for $L$, to see that $\mathfrak{F}_L = (W_L, R_L)$ is also a frame for $L$ one needs only verify that $\mathfrak{F}_L$ is a $p$-morphic image of $\mathfrak{F}$ under $f$.

(1) $f$ is onto.
(2) If $uRv$, then $f(u) R_L f(v)$.
(3) Supposing that $f(u) R_L \alpha$, we need to show that $\alpha = f(v)$ for some $v$, where $uRv$. Since $(\mathfrak{A}_L, f(u)) \vDash \Diamond\top$, $(\mathfrak{A}, u) \vDash \Diamond\top$. So $u$ has at least one world accessible from it under $R$. Let $v_1, \ldots, v_n$ be all such worlds, and suppose for a *reductio* that

$$(*) \quad (\forall A)((\mathfrak{A}, v_i) \vDash A \Leftrightarrow (\mathfrak{A}_L, \alpha) \vDash A)$$

for no $v_i$. Then, for each $v_i$, there is some $A_i$ such that $(\mathfrak{A}, v_i) \vDash A_i$ and $(\mathfrak{A}_L, \alpha) \nvDash A_i$. But now

$$(\mathfrak{A}, u) \vDash \Box(A_1 \vee \ldots \vee A_n),$$

so

$$(\mathfrak{A}_L, f(u)) \models \Box(A_1 \vee \ldots \vee A_n),$$

so

$$(\mathfrak{A}_L, \alpha) \models A_1 \vee \ldots \vee A_n,$$

so $(\mathfrak{A}_L, \alpha) \models A_i$ for some $A_i$, and we have a contradiction. So (*) holds for some $v_i$, whence it follows that $\alpha = f(v_i)$.

Let $K[\bigcirc]$ be the smallest extension of $K$ to contain the formula

$[\bigcirc]$ $\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow (\bigcirc p \rightarrow p)$,

where $\bigcirc$ is a finite string of $\Box$'s and $\Diamond$'s.

**Theorem 2** *Let $\bigcirc$ contain at least one $\Diamond$. Then every logic between $K[\bigcirc]$ and Sobociński's K3.1 (Makinson's D*) is classically noncompact (noncompact, non-canonical).*

*Proof:* Suppose that $L$ is any logic in question, and let $\Gamma$ be the set containing

$$\begin{array}{ll} \bigcirc p \wedge \sim p \wedge q_0 & \\ \Box^i(q_i \rightarrow \Diamond(p \wedge q_{i+1})) & (i \text{ even}) \\ \Box^i(q_i \rightarrow \Diamond(\sim p \wedge q_{i+1})) & (i \text{ odd}) \end{array}$$

where $q_i \neq p$ for all $i \in \omega$. If $\mathfrak{A} = (W, R, \phi)$ is any model for $\Gamma$, then $\mathfrak{A}$ contains an infinite ascending sequence $w_0, w_1, \ldots$ under $R$, where $(\mathfrak{A}, w_0) \models \bigcirc p$ and $(\mathfrak{A}, w_i) \models p \Leftrightarrow i$ is odd. Let $\mathfrak{B} = (W, R, \psi)$ be the model that is just like $\mathfrak{A}$ except that $\psi(p) = W - \{w_i | i \text{ is even}\}$. Then $(\mathfrak{B}, w_0) \not\models [\bigcirc]$. So $\Gamma$ has no model on a frame for $L$. But every finite subset of $\Gamma$ has a model on a finite linear ordering, hence on a frame for $K3.1$ and thus $L$.

This theorem generalizes the results of Hughes and Cresswell [11] and covers a number of familiar logics other than $K3.1$ itself, including Sobociński's $S4.02$ ($S4[\Box\Diamond\Box]$), $S4.1.1$ ($S4[\Diamond\Box]$), $S4.2.1$ ($S4.2[\Diamond\Box]$), $S4.3.1$ (Prior's $D$, Segerberg's $S4.3Grz$, $S4.3[\Diamond\Box]$), $K1.1$ (Segerberg's $S4Grz$), $K2.1$, $Z3$, $Z5$, $Z7$ and Segerberg's $KW$ (Boolos' $G$) and $K4.3W$. By contrast, we have the following positive results. The first one, when combined with Theorem 2, shows that $K3.1$ "bounds" canonicalcy (compactness, classical compactness), thus answering a question raised in [13].

**Theorem 3** *Every extension of S4.3.1 not contained in K3.1 is canonical (compact, classically compact).*

*Proof:* Using the fact that all extensions of $S4.3.1$ are complete, it is possible to catalog those logics not contained in $K3.1$ and then check to see that they are all elementary, i.e., that their accessibility relations satisfy an elementary condition. (The details can be culled from [12]). But every complete elementary modal logic is canonical (compact, classically compact) by Fine [7].

**Theorem 4** *There exist $2^{\aleph_0}$ canonical (compact, classically compact) normal modal logics.*

*Proof:* Let $A_n$ be the formula

$$\Diamond^{n+1}\Box\bot \rightarrow \Box(\Box^n\Diamond\top \rightarrow \Box\bot).$$

Where $X \subseteq \omega$, let $K_X$ be the smallest normal extension of $K$ to contain $A_n$ for all $n \in X$. $K_X$ is canonical (compact, classically compact)—as, indeed, is every normal extension of $K$ whose proper axioms are variable-free. For the truth value of a variable-free formula at a world in the canonical frame will be invariant across valuations.

To see that these $K_X$'s are all distinct, consider the frames $\mathfrak{F}_n = (W_n, R_n)$, where

$$W_n = \{0, \ldots, n+2\},$$
$$R_n = \{(i, i+1) | 0 \leq i \leq n\} \cup \{(0, n+2), (n+2, n+2)\}.$$

Suppose that $n \notin X$ and $\mathfrak{A}$ is any model based upon $\mathfrak{F}_n$. If $j > n$, then $(\mathfrak{A}, i) \nvDash \Diamond^{j+1}\Box\bot$ and hence $(\mathfrak{A}, i) \vDash A_j$ for all $i \in W_n$. On the other hand, if $j < n$, then $(\mathfrak{A}, i) \vDash \Diamond^{j+1}\Box\bot$ only if $i = n - j$. Suppose that $iR_nk$. Then $k = (n - j) + 1$, so $(\mathfrak{A}, k) \vDash \Box^j\Diamond\top \rightarrow \Box\bot$ since $(\mathfrak{A}, (n - j) + 1) \nvDash \Box^j\Diamond\top$. So $(\mathfrak{A}, i) \vDash \Box(\Box^j\Diamond\top \rightarrow \Box\bot)$, and again we have that $(\mathfrak{A}, i) \vDash A_j$ for all $i \in W_n$. However, $(\mathfrak{A}, 0) \nvDash A_n$. $\mathfrak{F}_n$ is therefore a frame for $K_X$ which fails to validate $A_n$. It follows that

$$A_n \in K_X \Leftrightarrow n \in X$$

and thus that $K_X \neq K_Y$ when $X \neq Y$.

That there is an equal number of noncompact (noncanonical) modal logics follows at once from Fine's observation (in [4]) that there are uncountably many incomplete extensions of $S4$. The more interesting question in the present context is the number of *complete* noncompact logics. Here, too, there are uncountably many. The following quick proof of this fact shows that there are this many even among the logics of width 2 and arbitrary finite depth.

**Theorem 5** *There exist $2^{\aleph_0}$ classically noncompact (noncompact, noncanonical) extensions of S4 with the finite model property.*

*Proof:* Fine [3] constructs a sequence of frames (Figure 1), distinct sets of which determine distinct (normal) extensions of $S4$. Let $L$ be any logic determined by an infinite class of these frames. (There are $2^{\aleph_0}$ such logics, of course.) Every $\mathfrak{F}_i$ validates the $S4.02$ axiom $[\Box\Diamond\Box]$, so $L$ contains $S4.02$. On the other hand, $K3.1$ is determined by the class of all finite linear orderings. Therefore, to show that every nontheorem of $K3.1$ is also a nontheorem of $L$, it is enough to observe that every finite linear ordering $(n, \leq)$ is a $p$-morphic image of $\mathfrak{F}_m$ $(m > n)$ under $f$, where

$$f(i) = \begin{cases} j, \text{ if } j \leq n-1 \text{ and } i = 2j \text{ or } 2j-1 \\ n-1, \text{ otherwise.} \end{cases}$$

This is shown diagrammatically in Figure 2.

It follows that $L$ is intermediate between $S4.02$ and $K3.1$, and hence classically noncompact (noncompact, noncanonical) by Theorem 2.

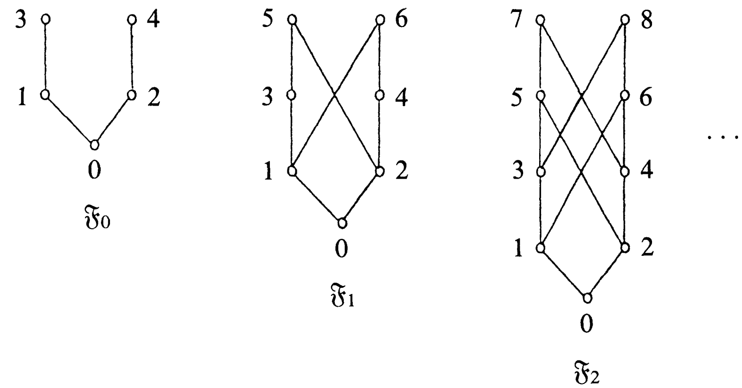


Figure 1.

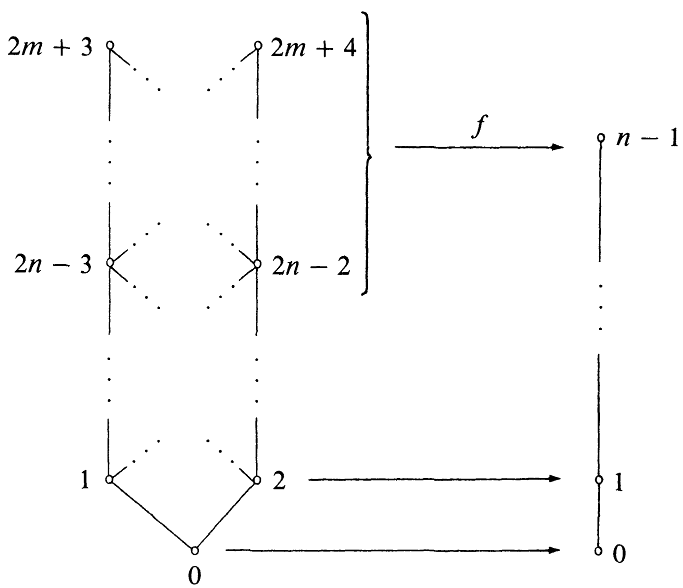


Figure 2.

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