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A Constructivism Based on Classical Truth

PIERANGELO MIGLIOLI, UGO MOSCATO,
MARIO ORNAGHI, and GABRIELE USBERTI

1 Introduction Our aim, in this paper, is to study the relationships between constructive truth and classical truth under a unified point of view, namely, to analyze the properties of *constructive* systems whose language is enriched by a sentential operator **T** assumed to represent the concept of *classical* truth. By a "constructive system" we mean a system S in which if $\vdash_S A \vee B$ then $\vdash_S A$ or $\vdash_S B$ and if $\vdash_S \exists x A$ then $\vdash_S A[t/x]$, for some term t. We say that an operator **T** represents in S the concept of classical truth if

(1) $\vdash_{\text{CPrC}} A$ iff $\vdash_S \mathbf{T} A$

where CPrC is classical predicate calculus (we will also use "CPC", "IPC", and "IPrC" to denote, respectively, classical propositional calculus, intuitionistic propositional calculus, and intuitionistic predicate calculus).

The importance of this study should be evident to whoever is interested not in reducing classical truth to constructive notions or vice versa, but in determining the laws of interaction of these notions.

For brevity's sake we shall limit our exposition to the propositional fragments of our calculi, giving some indications about their predicative extensions in notes.

In Section 2 we will introduce the calculus E_0 and its Kripke semantics; in Section 3 we will deal with the problem of finding *maximal* constructive systems in which **T** represents classical truth and we will prove a theorem of maximality for a calculus we call E*. For this calculus we will introduce a kind of semantics based on the notion of "valuation form", and its specific features will be discussed in Section 3.

At this point, let us look at some problems connected with the introduction of an operator **T** in a constructive setting.

As IPrC is the most famous system and, let us say, the paradigmatic example of a constructive system, the most obvious way to reach our aim would seem to consist of adding an operator **T** to IPrC and then characterizing it by suit-

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able axioms or rules. The difficulty with this approach is that, owing to the well-known property of intuitionistic double negation

(2) $\vdash_{\overline{\mathrm{IPC}}} \sim\sim A$ iff $\vdash_{\overline{\mathrm{CPC}}} A$,

we should adopt for $\mathbf{T}$ axioms or rules for which

(3) $\vdash_{\overline{\mathrm{IPC}}} \sim\sim A$ iff $\vdash_{\overline{\mathrm{IPCT}}} \mathbf{T}A$

where IPCT is the hypothetical system obtained by adding $\mathbf{T}$ to IPC. On the other hand, the axioms or rules for $\mathbf{T}$ should *not* allow us to have $\vdash_{\overline{\mathrm{IPCT}}} \sim\sim A \leftrightarrow \mathbf{T}A$, for, in this case, we should obviously also have $\vdash_{\overline{\mathrm{IPrCT}}} \sim\sim A \leftrightarrow \mathbf{T}A$, from which it follows that $\vdash_{\overline{\mathrm{IPrC}}} \sim\sim A$ iff $\vdash_{\overline{\mathrm{IPrCT}}} \mathbf{T}A$, while it is not true that $\vdash_{\overline{\mathrm{IPrC}}} \sim\sim A$ $\vdash_{\overline{CPrC}}$ iff A (take, for instance, the formula $\sim\sim\forall x(P(x) \vee \sim P(x))$).

The difficulty pointed out seems to entail that it is not possible to represent truth in a natural way by taking IPrC as a basis.

The approach we have chosen involves the following two points:

(i) We axiomatize the operator $\mathbf{T}$ in such a way that it represents truth *uniformly*, namely, both at the propositional and at the predicative level

(ii) We take as a basis a constructive system whose negation connective $\neg$ does not have property (2).

Concerning step (i) our idea is to start by adopting for $\mathbf{T}$ a rule similar to the one of intuitionistic double negation, namely

$$(4) \quad \frac{\begin{array}{cc} [\cancel{\neg A}] & [\cancel{\neg A}] \\ \vdots & \vdots \\ B & \neg B \end{array}}{\mathbf{T}A}$$

and then to modify the Kripke semantics for intuitionism in such a way as to force $\mathbf{T}$ to represent classical truth at the predicative level.

Let us consider the Kuroda formula $\forall x{\sim\sim}P(x) \rightarrow {\sim\sim}\forall xP(x)$. It is a typical example of the divergence between double negation and an operator intended to represent classical truth, since the formula

(5) $\forall x\mathbf{T}P(x) \rightarrow \mathbf{T}\forall xP(x)$

should be intuitively valid if $\mathbf{T}$ represents classical truth.

Let us consider a Kripke countermodel $\langle N, \leq, \Psi, \Phi \rangle$ to the Kuroda formula, where N is the set of natural numbers ordered by $\leq$, Ψ (the "domain function") is the function that to every number $n \in N$ assigns $\{x\colon x \leq n\}$, and Φ (the valuation) is a function for which $\Phi(P, n) = \{x\colon x \in \Psi(n-1)\}$. A peculiar feature of this model may be intuitively described by saying that we will *never* attain a cognitive status n in which we eventually know if $\models_{\overline{n}} P(x)$ or $\models_{\overline{n}} \sim P(x)$, for every x in its domain.

In a way, it is the very legitimacy of such an *ideal* cognitive status that intuitionists would deny, and it is this denial which prevents intuitionistic double negation from representing truth at the predicative level. So our suggestion is to modify Kripke semantics for intuitionism by imposing the possibility of reaching an "ideal final status" from each cognitive status; more precisely:

every point of a Kripke model has, among its accessible points, some point w from which it is accessible to at most itself and such that, for every sentence A, $\models_w A$ or $\models_w \neg A$.

As for step (ii), we start by observing that, if a Kripke model for intuitionism is *finite*, there are necessarily some points w of it in which we have $\models_w A$ or $\models_w \sim A$ for every formula A and, consequently, for every *classical tautology* V, we have $\models_w V$. Thus, these points play a role similar to the one played by our "final" points. Because of the forcing clause of intuitionistic negation, it follows that "$\sim\sim V$" is true at every point of any *finite* Kripke model. From this, and from the fact that $\vdash_{\mathrm{IPC}} A$ iff, for every finite Kripke model K, $\models_K A$, property (2) of intuitionistic double negation follows. But we are seeking a negation that does *not* have property (2), even if we want to have final points. So, our idea has been to adopt a concept of negation different from intuitionistic negation, in that it is not sufficient to know that A will not be verified at any future time in order to assert A at a given time; it is necessary to have actually verified the falsity of A, where the concept of falsity of an atomic formula is as primitive as the concept of its truth.

It is not difficult to recognize that this notion of negation is similar to the notion of constructible falsity introduced by Nelson [4] and studied by Thomason [10]; but in Sections 2.2 and 2.3 it will be seen that E_0 is not a conservative extension of Thomason's calculus, even at the propositional level.

In intensional mathematics (see Shapiro [9]) the relations between classical and constructive truth are studied in the framework of classical systems to which an operator **K** (of "knowability") is added. Our systems are constructive, but the operator **T** is intended to represent classical truth. So, in a sense, the two approaches are reciprocally "dual". An exact comparison between them deserves closer attention.

2 The logic E_0

Now we present the propositional fragment of the logic E_0 (the letter E occurring in the names of our systems stands for "effective"), which formalizes the operator **T** and the constructible $\neg$ along the lines previously discussed.

The language of E_0 is the standard propositional language (with $\neg$, $\wedge$, $\vee$, $\rightarrow$ primitives), enriched with the unary operator **T**. The notation $A \leftrightarrow B$ will be taken as an abbreviation of $(A \rightarrow B) \wedge (B \rightarrow A)$.

2.1 The calculus

We present E_0 in the form of a natural-like calculus. We say "natural-like" rather than "natural" since no inverse rule is given for the **T**-rule and the $\neg$**T** rule below, even if these rules allow us to introduce nonatomic formulas in an inference. However, the "naturalness" of the calculus can perhaps be defended, if one takes as "elementary" the **T**-formulas and the $\neg$**T**-formulas.[1]

The formal setting of the rules and the notational conventions will be similar to the ones in [5] and in [6], to which the reader is referred. For instance, we will enclose in square brackets classes of occurrences of undischarged assumptions to be put into evidence; we will "slash" classes of occurrences of undischarged assumptions in order to show that they become discharged in correspondence with the application of rules such as ($\rightarrow$I), ($\vee$E), (**T**), ($\neg$**T**), etc.

With each of the connectives $\wedge$, $\vee$, $\rightarrow$ we will associate four rules: an introduction rule, an elimination rule, an introduction of the negation related to the connective, and an elimination of the negation related to the connective. We will also have an introduction and an elimination of double negation. Finally, we will have the contradiction rule and the key rules (**T**) (allowing us to introduce the operator **T**) and ($\neg$**T**) (allowing us to introduce a $\neg$**T**-formula). The rules are:

(Basic): every wff A is a proof where A is the consequence and the only undischarged assumption;

$$(\wedge \mathrm{I}):\ \frac{A \quad B}{A \wedge B}; \qquad (\wedge \mathrm{E}):\ \frac{A \wedge B}{A}, \ \frac{A \wedge B}{B};$$

$$(\neg \wedge \mathrm{I}):\ \frac{\neg A}{\neg(A \wedge B)}, \ \frac{\neg B}{\neg(A \wedge B)}; \qquad (\neg \wedge \mathrm{E}):\ \frac{\neg(A \wedge B) \quad \begin{array}{c}[\cancel{\neg A}]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[\cancel{\neg B}]\\ \vdots \\ C\end{array}}{C};$$

$$(\vee \mathrm{I}):\ \frac{A}{A \vee B}, \ \frac{B}{A \vee B}; \qquad (\vee \mathrm{E}):\ \frac{A \vee B \quad \begin{array}{c}[\cancel{A}]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[\cancel{B}]\\ \vdots \\ C\end{array}}{C};$$

$$(\neg \vee \mathrm{I}):\ \frac{\neg A \quad \neg B}{\neg(A \vee B)}; \qquad (\neg \vee \mathrm{E}):\ \frac{\neg(A \vee B)}{\neg A}, \ \frac{\neg(A \vee B)}{\neg B};$$

$$(\rightarrow \mathrm{I}):\ \frac{\begin{array}{c}[\cancel{A}]\\ \vdots \\ B\end{array}}{A \rightarrow B}; \qquad (\rightarrow \mathrm{E}):\ \frac{A \quad A \rightarrow B}{B};$$

$$(\neg \rightarrow \mathrm{I}):\ \frac{A \quad \neg B}{\neg(A \rightarrow B)}; \qquad (\neg \rightarrow \mathrm{E}):\ \frac{\neg(A \rightarrow B)}{A}, \ \frac{\neg(A \rightarrow B)}{\neg B};$$

$$(\neg\neg \mathrm{I}):\ \frac{A}{\neg\neg A}; \qquad (\neg\neg \mathrm{E}):\ \frac{\neg\neg A}{A};$$

$$(\mathrm{Contr}):\ \frac{A \quad \neg A}{p}, \ \frac{A \quad \neg A}{\neg p} \quad p \text{ any atomic formula};$$

$$(\mathbf{T}):\ \frac{\begin{array}{c}[\cancel{\neg A}]\\ \vdots \\ B\end{array} \quad \begin{array}{c}[\cancel{\neg A}]\\ \vdots \\ \neg B\end{array}}{\mathbf{T}A}; \qquad (\neg\mathbf{T}):\ \frac{\begin{array}{c}[\cancel{A}]\\ \vdots \\ B\end{array} \quad \begin{array}{c}[\cancel{A}]\\ \vdots \\ \neg B\end{array}}{\neg\mathbf{T}A}.$$

If we omit the rules (**T**) and ($\neg$**T**) then E_0 becomes the propositional fragment $\mathrm{F}^{\mathrm{prop}}$ of the logic F with "constructible" falsity, described in appendix B of [5]; apart from the different presentation, the logic F coincides with the logic studied in Thomason [10], we call it F again, F^{prop} being again its propositional fragment. As concerns (Contr), in E_0 it can be shown to be equivalent to each of the following two rules:

(Contr′): $\dfrac{A \quad \neg A}{B}$ for *any* wff B

(Contr″): $\dfrac{\mathbf{T}A \quad \mathbf{T}\neg A}{B}$ for *any* wff B.

As concerns the **T**-rules (**T**) and ($\neg$**T**), we remark the following: if from $\neg A$ both B and $\neg B$ follow, then we can assert the *classical* truth of A (and this assertion has a "constructive" character, since we don't intend to claim anything different from the classical truth of A); likewise, if from A both B and $\neg B$ follow, then we can "constructively" falsify the classical truth of A.

Let us denote by "$\vdash_{E_0} A$" and "$\Gamma \vdash_{E_0} A$" the fact that there is a proof in E_0 of A without undischarged assumptions and the fact that there is a proof in E_0 of A whose undischarged assumptions belong to the (finite or infinite) set Γ of wffs, respectively (a similar convention will hold also for the other calculi we will introduce). Now, first of all we will set forth the provability or the unprovability in E_0 of some important wffs; all unprovable formulas will also turn out to be unprovable in the stronger logic E^*.

Considering negation, we have the following obvious facts:

(p1) $\vdash_{E_0} A \leftrightarrow \neg\neg A$
(p2) $\vdash_{E_0} A \wedge B \leftrightarrow \neg(\neg A \vee \neg B)$
(p3) $\vdash_{E_0} \neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$
(p4) $\vdash_{E_0} A \vee B \leftrightarrow \neg(\neg A \wedge \neg B)$
(p5) $\vdash_{E_0} \neg(A \vee B) \leftrightarrow \neg A \wedge \neg B$
(p6) $\vdash_{E_0} \neg(A \rightarrow B) \leftrightarrow A \wedge \neg B$
(p7) $\vdash_{E_0} \neg A \vee B \rightarrow (A \rightarrow B)$.

Despite (p1)–(p7), E_0-negation is a constructible negation much weaker than classical negation; as a matter of fact, for some wffs A and B, we have:

(u1) $\nvdash_{E_0} \neg(A \wedge \neg A)$
(u2) $\nvdash_{E_0} (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$
(u3) $\nvdash_{E_0} (A \rightarrow B) \rightarrow \neg A \vee B$.

Let us briefly discuss (u1) and (u2). The failure of the contradiction law formulated in terms of the constructible $\neg$ (fact (u1)) means that, to assert the falsity of a conjunction, we have to exhibit explicitly the falsity of one of the conjuncts; the failure of the contraposition law (fact (u2)) means that, even if from the constructive truth of A we can infer the constructive truth of B, from the constructive falsity of B we cannot infer the constructive falsity of A.

On the other hand, the contradiction law and the contraposition law can be proved in E_0 if formulated in terms of the negation of classical truth, i.e.,

(p8) $\vdash_{E_0} \neg\mathbf{T}(A \wedge \neg A)$
(p9) $\vdash_{E_0} (A \rightarrow B) \rightarrow (\neg\mathbf{T}B \rightarrow \neg\mathbf{T}A)$.

Important properties of $\mathbf{T}$ and $\neg\mathbf{T}$ provable in E_0 are:

(p10) $\vdash_{\overline{\mathrm{E}_0}} A \to \mathbf{T}A$
(p11) $\vdash_{\overline{\mathrm{E}_0}} \neg A \to \neg\mathbf{T}A$.

On the other hand, for appropriate A, we have:

(u4) $\nvdash_{\overline{\mathrm{E}_0}} \mathbf{T}A \to A$
(u5) $\nvdash_{\overline{\mathrm{E}_0}} \neg\mathbf{T}A \to \neg A$.

We remark that (u4) and (u5) prevent the collapse into classical logic.

Other important properties are:

(p12) $\vdash_{\overline{\mathrm{E}_0}} \mathbf{T}A \leftrightarrow \mathbf{TT}A$
(p13) $\vdash_{\overline{\mathrm{E}_0}} \neg\mathbf{T}A \leftrightarrow \mathbf{T}\neg A$
(p14) $\vdash_{\overline{\mathrm{E}_0}} \mathbf{T}A \leftrightarrow \neg\mathbf{T}\neg\mathbf{T}A$
(p15) $\vdash_{\overline{\mathrm{E}_0}} \mathbf{T}A \wedge \mathbf{T}B \leftrightarrow \mathbf{T}(A \wedge B)$
(p16) $\vdash_{\overline{\mathrm{E}_0}} (\mathbf{T}A \to \mathbf{T}B) \leftrightarrow \mathbf{T}(A \to B)$
(p17) $\vdash_{\overline{\mathrm{E}_0}} \mathbf{T}(A \vee B) \leftrightarrow \mathbf{T}(\mathbf{T}A \vee \mathbf{T}B)$
(p18) $\vdash_{\overline{\mathrm{E}_0}} \mathbf{T}A \to \mathbf{T}(A \vee B)$
(p19) $\vdash_{\overline{\mathrm{E}_0}} \neg\mathbf{TT}A \leftrightarrow \neg\mathbf{T}A$.

We end this list of provable and unprovable facts by pointing out that *the principle of replacement of equivalents does not hold* (in E_0 as well as in the logics we will present later) as a consequence of the joint features of $\neg$ and $\mathbf{T}$. For we have

$$\vdash_{\overline{\mathrm{E}_0}} \neg(\mathbf{T}\neg A \wedge \mathbf{T}\neg B) \leftrightarrow \mathbf{T}A \vee \mathbf{T}B$$

and

$$\vdash_{\overline{\mathrm{E}_0}} \mathbf{T}\neg A \wedge \mathbf{T}\neg B \leftrightarrow \mathbf{T}\neg(A \vee B).$$

Now, if in the former equivalence we replace $\mathbf{T}\neg A \wedge \mathbf{T}\neg B$ with the equivalent $\mathbf{T}\neg(A \vee B)$, then we obtain $\vdash_{\overline{\mathrm{E}_0}} \neg\mathbf{T}\neg(A \vee B) \leftrightarrow \mathbf{T}A \vee \mathbf{T}B$, from which $\vdash_{\overline{\mathrm{E}_0}} \mathbf{T}(A \vee B) \leftrightarrow \mathbf{T}A \vee \mathbf{T}B$ follows; but the latter fact is unprovable even in the stronger logic E^*.

Now we want to justify the meaning of $\mathbf{T}$ as a classical operator. To do so, also in view of our further treatment, it is useful to consider the classical (propositional) logic CPCT *having the same set of wffs as* E_0. From the semantical point of view, of course, we will interpret the unary operator $\mathbf{T}$ of this logic as the identity unary turth-function (i.e., $\mathbf{T}(t) = t$ and $\mathbf{T}(f) = f$, where t and f denote, respectively, truth and falsehood), while the other connectives $\neg$, $\wedge$, $\vee$, $\to$ will be the usual truth-functions. A *complete* axiomatization of CPCT can be given, for instance, by *adding* to the system E_0 the following rule:

$$\text{(CL)}: \quad \frac{\begin{array}{c}[\cancel{\neg A}]\\ \vdots \\ B\end{array} \qquad \begin{array}{c}[\cancel{\neg A}]\\ \vdots \\ \neg B\end{array}}{A} \quad \text{for any formula } A.$$

Of course, with this addition, many of the rules of E_0 become redundant in CPCT: namely, in the proof of Proposition 1 below we can assume that CPCT contains exactly the rules ($\wedge$I), ($\wedge$E), ($\vee$I), ($\vee$E), ($\to$I), ($\to$E), (**T**), ($\neg$**T**), and (CL) (together with the rule allowing the introduction of assumptions).

Now we can state the following proposition, justifying the meaning of $\mathbf{T}$:

Proposition 1 $\Gamma \vdash_{CPCT} A$ *iff* $\Gamma \vdash_{E_0} \mathbf{T}A$.[2]

Proof (outline): If $\Gamma \vdash_{E_0} \mathbf{T}A$, then $\Gamma \vdash_{\mathrm{CPCT}} \mathbf{T}A$; since $\vdash_{\mathrm{CPCT}} \mathbf{T}A \to A$, $\Gamma \vdash_{E_0} \mathbf{T}A$ implies $\Gamma \vdash_{\mathrm{CPCT}} A$.

To prove the converse, let $\mathbf{T}(\Gamma) = \{\mathbf{T}B\colon B \in \Gamma\}$; then it suffices to prove that if $\Gamma \vdash_{\mathrm{CPCT}} A$ then $\mathbf{T}(\Gamma) \vdash_{E_0} \mathbf{T}A$ (since $\vdash_{E_0} B \to \mathbf{T}B$ for every B). To do so, one can show that, given a proof $\begin{array}{c}[B_1]\ldots[B_n]\\ \Pi\\ A\end{array}$ in CPCT ($[B_1]\ldots[B_n]$ the classes of undischarged assumptions of the proof Π, A its consequence), a proof $\begin{array}{c}[\mathbf{T}B_1]\ldots[\mathbf{T}B_n]\\ \Pi'\\ \mathbf{T}A\end{array}$ can be constructed in E_0. This requires a straightforward induction on the complexity of Π, where the knowledge of facts such as (p1)–(p19) is very useful.

We will consider only the cases where the last inference rule applied in Π is ($\to$I) or ($\vee$E), leaving to the reader the remaining ones.

Let Π be of the form $\dfrac{\begin{array}{c}\ldots[\cancel{A_1}]\ldots\\ \Pi_1\\ A_2\end{array}}{A_1 \to A_2}$, where Π_1 is the main subproof of Π and where $A = A_1 \to A_2$. Then, by the induction hypothesis, one can construct in E_0 the proof $\dfrac{\begin{array}{c}\ldots[\cancel{\mathbf{T}A_1}]\ldots\\ \Pi_1'\\ \mathbf{T}A_2\end{array}}{\mathbf{T}A_1 \to \mathbf{T}A_2}$; the assertion then follows from (p16).

Let Π be of the form $\dfrac{\begin{array}{c}\\ \Pi_0\\ B \vee C\end{array} \quad \begin{array}{c}..[\cancel{B}]..\\ \Pi_1\\ A\end{array} \quad \begin{array}{c}..[\cancel{C}]..\\ \Pi_2\\ A\end{array}}{A}$; then, by applying the induction hypothesis to the subproofs Π_0, Π_1, Π_2, one can construct the proof

$$\dfrac{\dfrac{\dfrac{\Pi_0'}{\mathbf{T}(B \vee C)\ (\text{ind.hyp.})}}{\mathbf{T}(\mathbf{T}B \vee \mathbf{T}C)\ ((\text{p17}))} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}..[\cancel{\mathbf{T}B}]..\ (2)\\ \Pi_1'\\ \mathbf{T}A\ (\text{ind.hyp.})\end{array} \quad \dfrac{\cancel{\neg A}\ (1)}{\neg\mathbf{T}A\ ((\text{p11}))}}{\neg\mathbf{TT}B\ ((\neg\mathbf{T})\ \text{disc.}2)}}{\neg\mathbf{T}B\ ((\text{p19}))} \quad \dfrac{\dfrac{\begin{array}{c}..[\cancel{\mathbf{T}C}]..\ (3)\\ \Pi_2'\\ \mathbf{T}A\ (\text{ind.hyp.})\end{array} \quad \dfrac{\cancel{\neg A}\ (1)}{\neg\mathbf{T}A\ ((\text{p11}))}}{\neg\mathbf{TT}C\ ((\neg\mathbf{T})\ \text{disc.}3)}}{\neg\mathbf{T}C\ ((\text{p19}))}}{\neg\mathbf{T}B \wedge \neg\mathbf{T}C}}{\neg(\mathbf{T}B \vee \mathbf{T}C)\ ((\text{p5}))}}{\neg\mathbf{T}(\mathbf{T}B \vee \mathbf{T}C)\ ((\text{p11}))}}{\mathbf{T}A\ ((\mathbf{T})\ \text{disc.}1)}$$

where "═══" stands for an inference derivable in E_0.

We remark that the proof of the above proposition doesn't require the use of the rule (Contr) of E_0. In other words, Proposition 1 holds as well for the logic we call $E_0^{\min}$, where $E_0^{\min}$ is the sublogic of $E^{\min}$ obtained by deleting the rule (Contr). In this sense, the comparison we shall make in Section 2.3 between E_0 and IPC cannot be generalized to a comparison between $E_0^{\min}$ and minimal logic: for, while for IPC one has $\vdash_{\overline{IPC}} {\sim}{\sim}A$ iff $\vdash_{\overline{CPC}} A$, the same doesn't hold for minimal logic.

2.2 Semantical characterization of E_0 The semantics for E_0 is given by appropriately adapting the one of [10].

An E_0-model is a triple $\langle \mathbf{K},\mathbf{R},\mathbf{I}\rangle$, where:

(a) $\mathbf{K}$ is a nonempty set of states
(b) $\mathbf{R}$ is a reflexive and transitive relation on $\mathbf{K}$ satisfying the following *additional* property:
(b_1) for every $w \in \mathbf{K}$ there is a $w' \in \mathbf{K}$ such that $w\mathbf{R}w'$ and, for every $w'' \in \mathbf{K}$, if $w'\mathbf{R}w''$ then $w' = w''$ (in other words, some final state w' is accessible from every state w of $\mathbf{K}$)
(c) $\mathbf{I}$ (the valuation function) associates with every $w \in \mathbf{K}$ a *partial* function $\mathbf{I}_w$ from the set of atomic formulas to the set $\{t,f\}$ of the truth-values; moreover, the two following properties are satisfied:
(c_1) for every w, w' and atomic formula p, if $w\mathbf{R}w'$ and p belongs to the domain of $\mathbf{I}_w$, then p belongs to the domain of $\mathbf{I}_{w'}$, and $\mathbf{I}_w(p) = \mathbf{I}_{w'}(p)$
(c_2) for every w and p, there is a w' such that $w\mathbf{R}w'$ and either $\mathbf{I}_{w'}(p) = t$ or $\mathbf{I}_{w'}(p) = f$.

We remark that in the semantics of [10] conditions (b_1) and (c_2) *are not required.*

Now, given an E_0-model $\langle \mathbf{K},\mathbf{R},\mathbf{I}\rangle$, one extends the valuation function $\mathbf{I}$ to the wffs of any kind as follows:

(1) $\mathbf{I}_w(\neg A) = t$ iff $\mathbf{I}_w(A) = f$; $\mathbf{I}_w(\neg A) = f$ iff $\mathbf{I}_w(A) = t$
(2) $\mathbf{I}_w(A \wedge B) = t$ iff $\mathbf{I}_w(A) = t$ and $\mathbf{I}_w(B) = t$;
$\mathbf{I}_w(A \wedge B) = f$ iff $\mathbf{I}_w(A) = f$ or $\mathbf{I}_w(B) = f$
(3) $\mathbf{I}_w(A \vee B) = t$ iff $\mathbf{I}_w(A) = t$ or $\mathbf{I}_w(B) = t$;
$\mathbf{I}_w(A \vee B) = f$ iff $\mathbf{I}_w(A) = f$ and $\mathbf{I}_w(B) = f$
(4) $\mathbf{I}_w(A \to B) = t$ iff, for every w' such that $w\mathbf{R}w'$, either $\mathbf{I}_{w'}(A)$ is undefined, or $\mathbf{I}_{w'}(A) = f$, or $\mathbf{I}_{w'}(B) = t$; $\mathbf{I}_w(A \to B) = f$ iff $\mathbf{I}_w(A) = t$ and $\mathbf{I}_w(B) = f$;
(5) $\mathbf{I}_w(\mathbf{T}A) = t$ iff, for every w' such that $w\mathbf{R}w'$ there is a w'' such that $w'\mathbf{R}w''$ and $\mathbf{I}_{w''}(A) = t$; $\mathbf{I}_w(\mathbf{T}A) = f$ iff, for every w' such that $w\mathbf{R}w'$, either $\mathbf{I}_{w'}(A)$ is undefined or $\mathbf{I}_{w'}(A) = f$.

One easily extends by induction the above properties (c_1) and (c_2), involving atomic formulas, to any wff A, i.e.,

(P_1) if $w\mathbf{R}w'$ and A is in the domain of $\mathbf{I}_w$, then A is in the domain of $\mathbf{I}_{w'}$ and $\mathbf{I}_w(A) = \mathbf{I}_{w'}(A)$
(P_2) for every w there is a w' such that $w\mathbf{R}w'$ and either $\mathbf{I}_{w'}(A) = t$ or $\mathbf{I}_{w'}(A) = f$.[3]

By $\Gamma \models_{E_0} A$ we mean that $\mathbf{I}_w(A) = t$ whenever w is a state of an E_0-model such that $\mathbf{I}_w(B) = t$ for every $B \in \Gamma$; then, we can state the soundness and completeness of E_0 with respect to this semantics.

Theorem 1 $\Gamma \models_{E_0} A$ *iff* $\Gamma \vdash_{E_0} A$.[4]

The proof of the soundness is given by induction on the complexity of an E_0-proof, where property (P_1) is used for the rule $(\to I)$ and both properties (P_1) and (P_2) are required for the rules $(\mathbf{T})$ and $(\neg\mathbf{T})$.

The proof of completeness is an easy generalization of the technique of [10]. Here the canonical model is given by a set of states consisting of E_0-saturated sets of formulas, where the relation $\mathbf{R}$ is defined in terms of the inclusion of these sets; for every state w, the set of the final states following w is the set of all maximal consistent extensions of w.

We remark that E_0 can be proved to satisfy the finite model property, so that it is decidable.

Now we briefly outline the semantics of the sublogic $E_0^{\min}$, defined at the end of the previous section. This semantics is obtained by modifying the one of E_0 as follows:

- for every w, $\mathbf{I}_w$ is not necessarily a function, but only a relation; i.e., we may have both $p\mathbf{I}_w t$ and $p\mathbf{I}_w f$
- if for some w and some p we have $p\mathbf{I}_w t$ and $p\mathbf{I}_w f$, then for every atomic formula q there is a w' such that $w\mathbf{R}w'$ and $q\mathbf{I}_{w'} t$ and $q\mathbf{I}_{w'} f$.

With this semantics we obtain a soundness and completeness result, i.e.:

$$\Gamma \models_{E_0^{\min}} A \text{ iff } \Gamma \vdash_{E_0^{\min}} A.$$

Finally, we state that E_0 is a constructive logic, i.e., it satisfies the disjunction property. This result can be established using purely syntactical tools, but the proof requires a formal setting which exceeds the scope of the present paper. Having the semantical characterization at our disposal, this result can be obtained with a standard technique (see, e.g., [10]).

Theorem 2 *If* $\Gamma \vdash_{E_0} A \vee B$, *then* $\Gamma \vdash_{E_0} A$ *or* $\Gamma \vdash_{E_0} B$; *if* $\Gamma \vdash_{E_0} \neg(A \wedge B)$, *then* $\Gamma \vdash_{E_0} \neg A$ *or* $\Gamma \vdash_{E_0} \neg B$.

This result holds as well for $E_0^{\min}$.

2.3 A comparison with other known logics The first comparison is with IPC. As seen in Section 2.1, intuitionistic negation and E_0-negation have different properties; for $\vdash_{E_0} A \leftrightarrow \neg\neg A$ while $\nvdash_{IPC} A \leftrightarrow {\sim}{\sim}A$, and $\nvdash_{E_0} \neg(A \wedge \neg A)$ while $\vdash_{IPC} {\sim}(A \wedge {\sim}A)$. Nevertheless, we can translate intuitionistic negation into E_0 in three different but equivalent ways. To be more precise, let $\mathfrak{I}_1$ be the following translation-map from the set of wffs of IPC to the set of wffs of E_0:

(1) $\mathfrak{I}_1(p) = p$
(2) $\mathfrak{I}_1({\sim}A) = \neg\mathbf{T}(\mathfrak{I}_1(A))$
(3) $\mathfrak{I}_1(A \wedge B) = \mathfrak{I}_1(A) \wedge \mathfrak{I}_1(B)$
(4) $\mathfrak{I}_1(A \vee B) = \mathfrak{I}_1(A) \vee \mathfrak{I}_1(B)$
(5) $\mathfrak{I}_1(A \to B) = \mathfrak{I}_1(A) \to \mathfrak{I}_1(B)$.

The translation-map $\mathfrak{I}_2$ is obtained by modifying clause (2) of $\mathfrak{I}_1$ into the following:

(2′) $\mathfrak{I}_2(\sim A) = \mathbf{T}(\neg\mathfrak{I}_2(A))$.

The translation-map $\mathfrak{I}_3$ is defined by modifying clause (2) of $\mathfrak{I}_1$ into the following:

(2″) $\mathfrak{I}_3(\sim A) = \mathfrak{I}_3(A) \rightarrow p \wedge \neg p$, for some fixed propositional variable p.

Then we can prove the following proposition:

Proposition 2 $\vdash_{IPC} A$ *iff* $\vdash_{E_0} \mathfrak{I}_1(A)$ *iff* $\vdash_{E_0} \mathfrak{I}_2(A)$ *iff* $\vdash_{E_0} \mathfrak{I}_3(A)$.[5]

Proof: That $\vdash_{\mathrm{IPC}} A$ simultaneously implies $\vdash_{\mathrm{E}_0} \mathfrak{I}_1(A)$, $\vdash_{\mathrm{E}_0} \mathfrak{I}_2(A)$, and $\vdash_{\mathrm{E}_0} \mathfrak{I}_3(A)$ can be easily proved using a standard axiomatization of IPC and remarking that (as already seen) $\vdash_{\mathrm{E}_0} \neg\mathbf{T}B \leftrightarrow \mathbf{T}\neg B$ (for every B) and that:

(p20) $\vdash_{\mathrm{E}_0} \neg\mathbf{T}B \leftrightarrow (B \rightarrow C \wedge \neg C)$ (for every B and C).

To simultaneously prove the converse implications, we assume that $\not\vdash_{\mathrm{IPC}} A$. Then, there is a finite Kripke model **K** for IPC such that, for some state w of **K**, w doesn't force A. If for every final state w' of **K** and every atomic formula q one sets $\mathbf{I}_{w'}(q) = f$ iff w' doesn't force q in **K** (leaving unchanged the positive forcing of the variables in **K**, i.e., for every w'' of **K**, $\mathbf{I}_{w''}(q) = t$ iff w'' forces q in **K**), then one gets an E_0-model $\mathbf{K}'$: by induction on A one easily sees that, for every w, w forces A in **K** iff in $\mathbf{K}'$ $\mathbf{I}_w(\mathfrak{I}_1(A)) = t$ iff $\mathbf{I}_w(\mathfrak{I}_2(A)) = t$ iff $\mathbf{I}_w(\mathfrak{I}_3(A)) = t$.

The above result can be stated in a different form: i.e., we can define in E_0 (in three different but equivalent ways) intuitionistic negation, thus obtaining a logic which is a conservative extension of IPC.

To be more precise, first of all we enrich the language of E_0 with the unary connective $\sim$; then we define $\mathrm{E}_0(\sim,\neg\sim)$ as the logic obtained by adding to E_0 *one* of the three *equivalent* definitions of $\sim$:

(I) $\sim A \leftrightarrow \neg\mathbf{T}A$
(II) $\sim A \leftrightarrow \mathbf{T}\neg A$
(III) $\sim A \leftrightarrow (A \rightarrow p \wedge \neg p)$,

and the following axiom characterizing the meaning of $\neg\sim$:

(IV) $\mathbf{T}A \rightarrow \neg\sim A$.

We remark that the converse of (IV), i.e., $\neg\sim A \rightarrow \mathbf{T}A$, can be proved using the rules of E_0 and the definition of $\sim$.

The semantics for $\mathrm{E}_0(\sim,\neg\sim)$ is obtained from the one of E_0 with the addition of the following clauses:

(1′) $\mathbf{I}_w(\sim A) = t$ iff, for every w' such that $w\mathbf{R}w'$, either $\mathbf{I}_{w'}(A)$ is undefined or $\mathbf{I}_{w'}(A) = f$; $\mathbf{I}_w(\sim A) = f$ iff, for every w' such that $w\mathbf{R}w'$, there is a w'' such that $w'\mathbf{R}w''$ and $\mathbf{I}_{w''}(A) = t$; $\mathbf{I}_w(\neg\sim A) = t$ iff $\mathbf{I}_w(\sim A) = f$; $\mathbf{I}_w(\neg\sim A) = f$ iff $\mathbf{I}_w(\sim A) = t$.

Thus, one obtains a soundness and completeness result for $E_0(\sim,\neg\sim)$ quite similar to the one for E_0, to be used to prove the following proposition in the nontrivial direction.

Proposition 3 *For every wff A belonging to the language of IPC,* $\vdash_{IPC} A$ *iff* $\vdash_{E_0(\sim,\neg\sim)} A$.[6]

Now we compare E_0 with the above quoted propositional fragment F^{prop} of the logic F studied in [5] and in [10]. One immediately sees that F^{prop} is a sublogic of E_0, so the question arises whether or not E_0 is a conservative extension of F^{prop}. The answer is negative. For, we have that

$$\vdash_{E_0} ((A \to p \wedge \neg p) \to p \wedge \neg p) \leftrightarrow (\neg A \to p \wedge \neg p),$$

while, in general,

$$\nvdash_{F^{prop}} (\neg A \to p \wedge \neg p) \to ((A \to p \wedge \neg p) \to p \wedge \neg p),$$

as one easily verifies using the semantics of [10].

We also remark that in F^{prop} we can introduce intuitionistic negation with the definition: $\sim A \leftrightarrow (A \to p \wedge \neg p)$. Thus, we set $F^{prop}(\sim,\neg\sim)$ to be F^{prop} with the addition of such a definition of $\sim$ and of the axiom for $\neg\sim$: $\sim\sim A \to \neg\sim A$. The semantics of $F^{prop}(\sim,\neg\sim)$ is given by simply adding *to Thomason's semantics* the above-considered (1′) for $\sim$.

In this way, one obtains a soundness and completeness result for $F^{prop}(\sim,\neg\sim)$ and can show that $F^{prop}(\sim,\neg\sim)$ is a conservative extension of IPC; i.e., for every A of the language of IPC, $\vdash_{IPC}$ A iff $\vdash_{F^{prop}(\sim,\neg\sim)} A$.[7]

Now, the previously seen formula:

$$(\neg A \to p \wedge \neg p) \to ((A \to p \wedge \neg p) \to p \wedge \neg p)$$

becomes equivalent, in $F^{prop}(\sim,\neg\sim)$, to $\sim\neg A \to \sim\sim A$. Again, in $F^{prop}(\sim,\neg\sim)$ the latter formula is unprovable, while it is provable in $E_0(\sim,\neg\sim)$; hence, $E_0(\sim,\neg\sim)$ is not conservative over $F^{prop}(\sim,\neg\sim)$.

To put the question in another way, in $F^{prop}(\sim,\neg\sim)$ we can define **T** by: $\mathbf{T}A \leftrightarrow \sim\sim A$, in such a way that $\vdash_{CPCT} A$ iff $\vdash_{F^{prop}(\sim,\neg\sim,\mathbf{T})} \mathbf{T}A$; but the properties of the **T** of $F^{prop}(\sim,\neg\sim,\mathbf{T})$ are weaker than the ones of the **T** of $E_0(\sim,\neg\sim)$, since in the latter logic **T** is equivalent to $\sim\neg$, while in the former this doesn't hold.[8]

To complete this discussion, we remark that $E_0(\sim,\neg\sim)$ is a conservative extension of $F'^{prop}(\sim,\neg\sim)$, where $F'^{prop}(\sim,\neg\sim)$ is $F^{prop}(\sim,\neg\sim)$ with the addition of the axiom-schema:

(V) $\sim\neg A \to \sim\sim A$.

This depends on the fact that the addition of (V) to $F^{prop}(\sim,\neg\sim,\mathbf{T})$ provides an equivalent presentation of $E_0(\sim,\neg\sim)$. Thus, it is just (V) that requires an essential modification in Thomason's semantics.

The differences between E_0 and F^{prop} are further stressed by taking into account Thomason's translation of F into the predicative modal logic S4 supplemented by the Barcan formula: here we will be interested only in the subtranslation of F^{prop} into propositional S4. Such a translation is given by:

(1) $\mathfrak{J}_4(p) = \Box p$
(2) $\mathfrak{J}_4(\neg p) = \Box \neg p$
(3) $\mathfrak{J}_4(A \vee B) = \mathfrak{J}_4(A) \vee \mathfrak{J}_4(B)$
(4) $\mathfrak{J}_4(\neg(A \vee B)) = \mathfrak{J}_4(\neg A) \wedge \mathfrak{J}_4(\neg B)$
(5) $\mathfrak{J}_4(A \rightarrow B) = \Box(\mathfrak{J}_4(A) \rightarrow \mathfrak{J}_4(B))$
(6) $\mathfrak{J}_4(\neg(A \rightarrow B)) = \mathfrak{J}_4(A) \wedge \mathfrak{J}_4(\neg B)$
(7) $\mathfrak{J}_4(\neg\neg A) = \mathfrak{J}_4(A)$.

Now, as proved in [10], one has: $\vdash_{F^{prop}} A$ iff $\vdash_{S4} \mathfrak{J}_4(A)$ (we remark that in Thomason's presentation F^{prop} does not have the logical constant $\wedge$, which can be defined in terms of $\vee$ and $\neg$).

Let us extend $\mathfrak{J}_4$ to cover all wffs as follows:

(8) $\mathfrak{J}_4(A \wedge B) = \mathfrak{J}_4(A) \wedge \mathfrak{J}_4(B)$
(9) $\mathfrak{J}_4(\neg(A \wedge B)) = \mathfrak{J}_4(\neg A) \vee \mathfrak{J}_4(\neg B)$
(10) $\mathfrak{J}_4(\mathbf{T}A) = \Box\Diamond\mathfrak{J}_4(A)$
(11) $\mathfrak{J}_4(\neg\mathbf{T}A) = \Box\Diamond\mathfrak{J}_4(\neg A)$.

Then one can find a formula A such that $\vdash_{E_0} A$ and $\nvdash_{S4} \mathfrak{J}_4(A)$. However, it is possible to translate E_0, according to $\mathfrak{J}_4$, into a suitable extension of S4. For consider the propositional modal logic S4.1, which can be given, for instance, by adding to S4 the axiom-schema: $\Diamond(\Diamond A \rightarrow \Box A)$ (see [3]). The semantics of S4.1 is just the semantics of S4, with the only *additional* requirement that some *final* (in the same sense as for the E_0-models) state be accessible from each state; then, using the above semantical characterization of E_0, one can prove:

Proposition 4 $\vdash_{E_0} A$ *iff* $\vdash_{S4.1} \mathfrak{J}_4(A)$.

Using the Bulldozer Theorem, the latter result can be almost immediately extended to S4 + Grz (see [7], [8]).

To conclude this section, we take into account the minimal propositional logic MPC and E_0^{min}. A possible interpretation of the $\sim$ of MPC in E_0^{min} seems to be given, e.g., by $\neg\mathbf{T}$. As a matter of fact, considering the above translation $\mathfrak{J}_1$, we have: if $\vdash_{MPC} A$ then $\vdash_{E_0^{min}} \mathfrak{J}_1(A)$. But the converse doesn't hold; e.g., we have $\vdash_{E_0^{min}} \neg\mathbf{T}\neg\mathbf{T}(A \wedge \neg\mathbf{T}A \rightarrow B)$, while $\nvdash_{MPC} \sim\sim(A \wedge \sim A \rightarrow B)$. This should better illuminate the remark at the end of Section 2.1.

*3 The logic E**

3.1 The calculus A general problem which can be raised at this point is the following: are there systems which are *maximal* among the constructive systems such that **T** represents in them classical truth? In this section we will answer this problem by introducing the system E*, which extends E_0 and has the required property.

E^* is obtained by adding to E_0 the following rules:

$$(^*1): \frac{\begin{matrix}[\cancel{\neg p}] & [\cancel{\neg p}] \\ \vdots & \vdots \\ A & \neg A\end{matrix}}{p}, \qquad (^*2): \frac{\begin{matrix}[\cancel{p}] & [\cancel{p}] \\ \vdots & \vdots \\ A & \neg A\end{matrix}}{\neg p}$$

where p is any atomic formula, and

$$(\mathrm{E}): \frac{\mathbf{T}A \to B \vee C}{(\mathbf{T}A \to B) \vee (\mathbf{T}A \to C)}.$$

Let us call "E_0^*" the calculus obtained by adding to E_0 *only* the rules (*1) and (*2). The reader will recognize (*1) and (*2) as rules quite similar to Prawitz's Λ_C-rule (see [5], [6]), which provides a natural calculus for *classical* logic, presented in a language *not containing* $\vee$ (where $A \vee B$ is intended as an abbreviation in the metalanguage of $\neg(\neg A \wedge \neg B)$, and where $\neg$ does not have the constructive properties we have seen in the logic E_0).

Indeed, (*1) and (*2) give rise to the following fact, which can be proved with a straightforward induction on the complexity of A:

Proposition 5 *For every A not containing subformulas of the form $B \vee C$ or of the form $\neg(B \wedge C)$ out of the scopes of $\mathbf{T}$, $\vdash_{E_0^*} \mathbf{T}A \leftrightarrow A$.*

However, the presence of $\vee$ and $\neg\wedge$ in our language prevents the collapse of E_0^* into classical logic. Thus, for instance, we have: $\nvdash_{E_0^*} \mathbf{T}(p \vee \neg p) \to p \vee \neg p$.

By rules (*1) and (*2), we have that the uniform substitution property is not satisfied in E_0^*; i.e., there are wffs $A(p)$ such that $\vdash_{E_0^*} A(p)$ (p an atomic subformula of A) but there is a wff B such that $\nvdash_{E_0^*} A(B)$, $A(B)$ being the formula obtained by substituting every occurrence of p in A with an occurrence of B.

One may wonder at the lack of uniform substitution in a system which is called "logical". On the one hand, we observe that even if this feature of E_0^* (and of E^*) makes it rather nonstandard, one could argue for the logical legitimacy of it along the following line: if one equates atomic formulas of a language (not with any sentences whatever) with *atomic* sentences of some theory, and if, moreover, one wants to express the fact that classical and constructive truth of such *atomic* sentences coincide (as is the case for the most usual theories), then the adoption of rules (*1) and (*2) seems to be sufficiently justified; in this line, one easily sees that E_0^* (as well as E^*) is closed under substitution of atomic formulas with *arbitrary T-formulas* and, more generally, with arbitrary formulas satisfying Proposition 5 (to better understand the meaning of this remark, see Section 3.4). On the other hand, starting from the maximality of E^*, we can also obtain a maximality result for a subsystem of it satisfying the uniform substitution property; in this case, however, a restricted notion of maximality is involved (see Section 3.4).

The Kripke style semantics for E_0^* is a natural extension of the one for E_0. We define the E_0^*-models to be E_0-models with the following *additional* property: for every $w \in \mathbf{K}$ and for every atomic p, if neither $\mathbf{I}_w(p) = t$ nor $\mathbf{I}_w(p) = f$, then there are $w', w'' \in \mathbf{K}$ such that $w\mathbf{R}w'$ and $w\mathbf{R}w''$ and $\mathbf{I}_{w'}(p) = t$ and

$\mathbf{I}_{w''}(p) = f$. With this notion of an E_0^*-model, we obtain a soundness and completeness result for E_0^*, i.e.,

$$\Gamma \models_{E_0^*} A \text{ iff } \Gamma \vdash_{E_0^*} A.$$[9]

Now we state a simple result which will be very useful later:

Proposition 6 *Let $A(p_1, \ldots, p_n)$ be any wff containing exactly the atomic formulas $p_1, \ldots, p_n$; then $\vdash_{E_0^*} A(p_1, \ldots, p_n) \leftrightarrow A(\mathbf{T}p_1, \ldots, \mathbf{T}p_n)$, where $A(\mathbf{T}p_1, \ldots, \mathbf{T}p_n)$ is obtained by simultaneously substituting in $A(p_1, \ldots, p_n)$ the formula $\mathbf{T}p_1$ for $p_1, \ldots,$ the formula $\mathbf{T}p_n$ for p_n.*

Proof: Rules (*1) and (*2) provide the basis, i.e., $\vdash_{E_0^*} p \leftrightarrow \mathbf{T}p$ and $\vdash_{E_0^*} \neg p \leftrightarrow \neg\mathbf{T}p$. The induction step is carried out straightforwardly, using the logic E_0.

If we add rule (E) to E_0 instead of to E_0^*, then we obtain the *sublogic* E of E^*. We remark that both E_0 and E, which do not contain (*1) and (*2), satisfy the uniform substitution property.

We point out that the rule (E) by no means can be taken as a rule of a natural calculus in the sense of [5], since on the one hand it introduces a 'nonelementary' formula (a disjunction of implications), and on the other hand there is no corresponding inverse rule. It is possible to present E^* in the form of a special sequent calculus (allowing proofs with more than one consequence) for which something similar to the inversion principle can be stated; the explanation of such a calculus, however, exceeds the purpose of the present paper.

One can show: if $\vdash_{E_0} \mathbf{T}A \to B \vee C$ then $\vdash_{E_0} \mathbf{T}A \to B$ or $\vdash_{E_0} \mathbf{T}A \to B$; the same for E_0^*. However, rule (E) cannot be eliminated in the systems E and E^*, if the formula $\mathbf{T}A \to B \vee C$ depends on undischarged assumptions: for instance, using a suitable E_0^*-model, one can falsify the formula $(\mathbf{T}A \to B \vee C) \to (\mathbf{T}A \to B) \vee (\mathbf{T}A \to C)$.

In the frame of IPC, where negation is not 'constructible', the meaning of our rule (E) can be captured by the principle $(\sim A \to B \vee C) \to (\sim A \to B) \vee (\sim A \to C)$, introduced in [2].

We end this section by stating the normal-form theorem for E^*, which requires the previously seen Proposition 6 concerning E_0^*.

Theorem 3 *For every formula A there are* $\mathbf{T}$*-formulas* $\mathbf{T}A_1, \ldots, \mathbf{T}A_m$ $(m > 0)$ *such that* $\vdash_{E^*} A \leftrightarrow \mathbf{T}A_1 \vee \ldots \vee \mathbf{T}A_m$.[10]

Proof: By Proposition 6, we can assume that A does not contain occurrences of atomic formulas which are not in the scope of $\mathbf{T}$. With this assumption, we can prove our theorem by induction on the complexity of A, starting from the $\mathbf{T}$-subformulas of A.

Basis: $A = \mathbf{T}B$: immediate.

Step: (i) $A = \neg\mathbf{T}B$: immediate, since $\vdash_{E^*} \neg\mathbf{T}B \leftrightarrow \mathbf{T}\neg B$.

(ii) $A = B \wedge C$: by induction hypothesis, $\vdash_{E^*} B \leftrightarrow \mathbf{T}B_1 \vee \ldots \vee \mathbf{T}B_m$ and $\vdash_{E^*} C \leftrightarrow \mathbf{T}C_1 \vee \ldots \vee \mathbf{T}C_n$; then $\vdash_{E^*} B \wedge C \leftrightarrow (\mathbf{T}B_1 \vee \ldots \vee \mathbf{T}B_m) \wedge (\mathbf{T}C_1 \vee \ldots \vee \mathbf{T}C_n)$, from which $\vdash_{E^*} B \wedge C \leftrightarrow (\mathbf{T}B_1 \wedge \mathbf{T}C_1) \vee \ldots \vee (\mathbf{T}B_1 \wedge \mathbf{T}C_n) \vee \ldots \vee (\mathbf{T}B_m \wedge \mathbf{T}C_1) \vee \ldots \vee (\mathbf{T}B_m \wedge \mathbf{T}C_n)$, from which, by (p15) of Section 2.1, $\vdash_{E^*} B \wedge C \leftrightarrow \mathbf{T}(B_1 \wedge C_1) \vee \ldots \vee \mathbf{T}(B_1 \wedge C_n) \vee \ldots \vee \mathbf{T}(B_m \wedge C_1) \vee \ldots \vee \mathbf{T}(B_m \wedge C_n)$, which proves our assertion.

(iii) $A = \neg(B \wedge C)$: here $\vdash_{\overline{E^*}} \neg(B \wedge C) \leftrightarrow \neg B \vee \neg C$, from which, by the induction hypotheses on $\neg B$ and $\neg C$, $\vdash_{\overline{E^*}} \neg(B \wedge C) \leftrightarrow \mathbf{T}B'_1 \vee \ldots \vee \mathbf{T}B'_m \vee \mathbf{T}C'_1 \vee \ldots \vee \mathbf{T}C'_n$, which is our assertion.

(iv) The cases $A = \neg(B \vee C)$ and $A = \neg(B \rightarrow C)$ are similar to the case $A = B \wedge C$; the case $A = B \vee C$ is similar to the case $A = \neg(B \wedge C)$; the case $A = \neg\neg B$ is obvious.

(v) $A = B \rightarrow C$: this is the *only* case requiring the rule (E); by induction hypothesis, $\vdash_{\overline{E^*}} (B \rightarrow C) \leftrightarrow (\mathbf{T}B_1 \vee \ldots \vee \mathbf{T}B_m \rightarrow \mathbf{T}C_1 \vee \ldots \vee \mathbf{T}C_n)$, from which, since $\vdash_{\overline{E_0}} (H \vee K \rightarrow Z) \rightarrow (H \rightarrow Z) \wedge (K \rightarrow Z)$ for any H, K, Z, we obtain $\vdash_{\overline{E^*}} (B \rightarrow C) \leftrightarrow (\mathbf{T}B_1 \rightarrow \mathbf{T}C_1 \vee \ldots \vee \mathbf{T}C_n) \wedge \ldots \wedge (\mathbf{T}B_m \rightarrow \mathbf{T}C_1 \vee \ldots \vee \mathbf{T}C_n)$; now, *by repeatedly applying rule (E)*, we deduce $\vdash_{\overline{E^*}} (B \rightarrow C) \leftrightarrow [(\mathbf{T}B_1 \rightarrow \mathbf{T}C_1) \vee \ldots \vee (\mathbf{T}B_1 \rightarrow \mathbf{T}C_n)] \wedge \ldots \wedge [(\mathbf{T}B_m \rightarrow \mathbf{T}C_1) \vee \ldots \vee (\mathbf{T}B_m \rightarrow \mathbf{T}C_n)]$, from which, repeatedly applying (p16) of Section 2.1, $\vdash_{\overline{E^*}} (B \rightarrow C) \leftrightarrow (\mathbf{T}(B_1 \rightarrow C_1) \vee \ldots \vee \mathbf{T}(B_1 \rightarrow C_n)) \wedge \ldots \wedge (\mathbf{T}(B_m \rightarrow C_1) \vee \ldots \vee \mathbf{T}(B_m \rightarrow C_n))$; starting from the latter fact, our proof can be concluded as in the case $A = B \wedge C$.

Let us remark that, in the proof of Theorem 3, rules (*1) and (*2) are used only to put every atomic formula in the scope of **T** (Proposition 6). Thus, we obtain:

Corollary *Let A be a formula where every occurrence of an atomic formula is in the scope of some occurrence of* **T**: *then there are* **T**-*formulas* $\mathbf{T}A_1, \ldots, \mathbf{T}A_m (m > 0)$ *such that* $\vdash_{\overline{E}} A \leftrightarrow \mathbf{T}A_1 \vee \ldots \vee \mathbf{T}A_m$.

3.2 The semantics of the valuation forms Even if one can provide a Kripke style semantics for E*, we present here a new kind of semantics, which seems to us to provide a reasonably flexible tool for the semantical analysis of constructive logics, and which, in the case of E*, is particularly simple.

The basic notion of our semantics is that of *valuation form* (vf) of a sentence. Intuitively, the vf's of A represent the possible ways of analyzing A in order to ascertain its truth value. So, the adequacy condition we impose upon the definition of this notion is that a sentence is true in a (classical) interpretation iff there is a vf of it which is true in that interpretation. Proposition 7 below shows that the adequacy condition is satisfied by the definition we are going to give. Let us consider first some cases. A sentence of the form $A \vee B$ can be analyzed in two ways: according to the former, its truth value depends on the truth value of A, according to the latter, its truth value depends on the truth value of B. So, the vf's of $A \vee B$ will be of two kinds, which we shall denote by $\dfrac{\overset{\triangledown}{A}}{A \vee B}$ and by $\dfrac{\overset{\triangledown}{B}}{A \vee B}$ respectively, where $\overset{\triangledown}{A}$ stands for any vf of A and $\overset{\triangledown}{B}$ stands for any vf of B. A sentence of the form $A \rightarrow B$ can be analyzed in as many ways as there are combinatorial possibilities of associating a way of analyzing A to a way of analyzing B. So, the vf's of $A \rightarrow B$ will be functions from the vf's of A to the vf's of B. In the case of an atomic, structureless, sentence, there is of course only one trivial way to 'analyze' it, namely, to take itself as the result of having analyzed it. So, the vf of an atomic formula will be the formula itself. In our semantics **T**-sentences are also treated as atomic, in the sense that the vf of a **T**-sentence is the formula itself; this technical peculiarity reflects the intuition that what is necessary to know in order to ascertain the truth value of $\mathbf{T}A$

is merely the truth value of A; the effect of the operator is, then, so to speak, 'to block' the constructive analysis of the sentences inside its scope.

Let us define, for each formula A, the class $\mathbf{F}(A)$ of its vf's; we recall that "$\overset{\triangledown}{A}$" will denote any element of $\mathbf{F}(A)$. The definition of $\mathbf{F}(A)$ is given inductively as follows:

(1) $\mathbf{F}(p) = \{p\}$ for every atomic p

(2) $\mathbf{F}(\neg p) = \{\neg p\}$ for every atomic p

(3) $\mathbf{F}(\mathbf{T}A) = \{\mathbf{T}A\}$

(4) $\mathbf{F}(\neg\mathbf{T}A) = \{\neg\mathbf{T}A\}$

(5) $\mathbf{F}(\neg\neg A) = \left\{\dfrac{\overset{\triangledown}{A}}{\neg\neg A} : \overset{\triangledown}{A} \in \mathbf{F}(A)\right\}$

(6) $\mathbf{F}(A \wedge B) = \left\{\dfrac{\overset{\triangledown}{A} \quad \overset{\triangledown}{B}}{A \wedge B} : \overset{\triangledown}{A} \in \mathbf{F}(A) \text{ and } \overset{\triangledown}{B} \in \mathbf{F}(B)\right\}$

(7) $\mathbf{F}(\neg(A \wedge B)) = \left\{\dfrac{\overset{\triangledown}{\neg A}}{\neg(A \wedge B)} : \overset{\triangledown}{\neg A} \in \mathbf{F}(\neg A)\right\} \cup \left\{\dfrac{\overset{\triangledown}{\neg B}}{\neg(A \wedge B)} : \overset{\triangledown}{\neg B} \in \mathbf{F}(\neg B)\right\}$

(8) $\mathbf{F}(A \vee B) = \left\{\dfrac{\overset{\triangledown}{A}}{A \vee B} : \overset{\triangledown}{A} \in \mathbf{F}(A)\right\} \cup \left\{\dfrac{\overset{\triangledown}{B}}{A \vee B} : \overset{\triangledown}{B} \in \mathbf{F}(B)\right\}$

(9) $\mathbf{F}(\neg(A \vee B)) = \left\{\dfrac{\overset{\triangledown}{\neg A} \quad \overset{\triangledown}{\neg B}}{\neg(A \vee B)} : \overset{\triangledown}{\neg A} \in \mathbf{F}(\neg A) \text{ and } \overset{\triangledown}{\neg B} \in \mathbf{F}(\neg B)\right\}$

(10) $\mathbf{F}(A \to B) = \mathbf{F}(B)^{\mathbf{F}(A)}$

(11) $\mathbf{F}(\neg(A \to B)) = \left\{\dfrac{\overset{\triangledown}{A} \quad \overset{\triangledown}{\neg B}}{\neg(A \to B)} : \overset{\triangledown}{A} \in \mathbf{F}(A) \text{ and } \overset{\triangledown}{\neg B} \in \mathbf{F}(\neg B)\right\}.$

In other words, a valuation form for $A \to B$ is any function having $\mathbf{F}(A)$ as the domain and taking values in $\mathbf{F}(B)$. It may be useful to represent such a form (which in propositional logic is always finite) by a tree such as

$$\frac{\langle \overset{\triangledown 1}{A}, \overset{\triangledown 1}{B}\rangle \dots \langle \overset{\triangledown n}{A}, \overset{\triangledown n}{B}\rangle}{A \to B},$$

where $\overset{\triangledown 1}{A}, \dots, \overset{\triangledown n}{A}$ are (enumerated in some way) *all* the valuation forms of A and $\overset{\triangledown 1}{B}, \dots, \overset{\triangledown n}{B}$ are the corresponding forms of B.

The forms (1)–(4) are called "elementary forms"; by the presence of cases (7) and (8), in general there are several nonelementary forms of a formula A.

Now we define the *truth value* $\mathbf{I}(\overset{\triangledown}{A})$ *of a vf* $\overset{\triangledown}{A}$ in a (classical) interpretation $\mathbf{I}$, where $\mathbf{I}$ is a function assigning one of the two truth values t and f to *every* propositional variable (in other words, $\mathbf{I}$ is a total interpretation). In order that every vf assumes one (and only one) truth value, we stipulate that $\mathbf{I}(\overset{\triangledown}{A}) = f$ iff $\mathbf{I}(\overset{\triangledown}{A}) \neq t$; the definition of $\mathbf{I}(\overset{\triangledown}{A})$ is completed by the following inductive clauses, which start from the elementary vf's:

(1) $\mathbf{I}(\overset{\triangledown}{p}) = \mathbf{I}(p)$

(2) $\mathbf{I}(\overset{\triangledown}{\neg p}) = t$ iff $\mathbf{I}(p) = f$

(3) $\mathbf{I}(\overset{\triangledown}{\mathbf{T}A}) = \mathbf{I}(\mathbf{T}A)$, where $\mathbf{I}(\mathbf{T}A)$ is evaluated in the usual *classical* way (we recall that $\mathbf{T}(t) = t$ and $\mathbf{T}(f) = f$)

(4) $\mathbf{I}(\overset{\triangledown}{\neg}\mathbf{T}A) = t$ iff $\mathbf{I}(\mathbf{T}A) = f$

(5) $\mathbf{I}\left(\dfrac{\overset{\triangledown}{A}}{\neg\neg A}\right) = \mathbf{I}(A)$

(6) $\mathbf{I}\left(\dfrac{\overset{\triangledown}{A} \quad \overset{\triangledown}{B}}{A \wedge B}\right) = t$ iff $\mathbf{I}(\overset{\triangledown}{A}) = t$ and $\mathbf{I}(\overset{\triangledown}{B}) = t$

(7) $\mathbf{I}\left(\dfrac{\overset{\triangledown}{\neg A}}{\neg(A \wedge B)}\right) = \mathbf{I}(\overset{\triangledown}{\neg A})$

(7′) $\mathbf{I}\left(\dfrac{\overset{\triangledown}{\neg B}}{\neg(A \wedge B)}\right) = \mathbf{I}(\overset{\triangledown}{\neg B})$

(8) $\mathbf{I}\left(\dfrac{\overset{\triangledown}{A}}{A \vee B}\right) = \mathbf{I}(\overset{\triangledown}{A})$

(8′) $\mathbf{I}\left(\dfrac{\overset{\triangledown}{B}}{A \vee B}\right) = \mathbf{I}(\overset{\triangledown}{B})$

(9) $\mathbf{I}\left(\dfrac{\overset{\triangledown}{\neg A} \quad \overset{\triangledown}{\neg B}}{\neg(A \vee B)}\right) = t$ iff $\mathbf{I}(\overset{\triangledown}{\neg A}) = t$ and $\mathbf{I}(\overset{\triangledown}{\neg B}) = t$

(10) $\mathbf{I}(A \to B) = t$ iff, for every $\overset{\triangledown}{A} \in \mathbf{F}(A)$ such that $\mathbf{I}(\overset{\triangledown}{A}) = t$, the corresponding $\overset{\triangledown}{B} \in \mathbf{F}(B)$ is such that $\mathbf{I}(\overset{\triangledown}{B}) = t$

(11) $\mathbf{I}\left(\dfrac{\overset{\triangledown}{A} \quad \overset{\triangledown}{\neg B}}{\neg(A \to B)}\right) = t$ iff $\mathbf{I}(\overset{\triangledown}{A}) = t$ and $\mathbf{I}(\overset{\triangledown}{\neg B}) = t$.

The relation between the *classical* interpretation of *formulas* and the interpretation of *vf's* (i.e., vf's can be seen as 'grounds' of the classical truth of formulas) is stated by the following proposition, which requires a straightforward induction on the complexity of A.

Proposition 7 *For every* $\mathbf{I}$, $\mathbf{I}(A) = t$ *iff there exists an* $\overset{\triangledown}{A} \in \mathbf{F}(A)$ *such that* $\mathbf{I}(\overset{\triangledown}{A}) = t$.

By "$\models A$" we will mean that A is classically valid and by "$\Gamma \models A$" we will mean that A is a classical consequence of Γ, i.e., every classical interpretation satisfying all formulas of Γ satisfies A too; by $\overset{\triangledown}{\Gamma}$ we will denote any function associating a vf $\overset{\triangledown}{B}$ to every $B \in \Gamma$; also, we will say that $\mathbf{I}$ satisfies (or verifies) $\overset{\triangledown}{\Gamma}$ iff $\mathbf{I}(\overset{\triangledown}{B}) = t$ for every $\overset{\triangledown}{B}$ associated by $\overset{\triangledown}{\Gamma}$ with $B \in \Gamma$. Since $\Gamma \models A$ implies $\Gamma' \models A$ for some finite subset Γ' of Γ, from Proposition 7 we easily deduce the following corollary:

Corollary 1 $\models A$ *iff for every* $\mathbf{I}$ *there is an* $\overset{\triangledown}{A} \in \mathbf{F}(A)$ *such that* $\mathbf{I}(\overset{\triangledown}{A}) = t$; $\Gamma \models A$ *iff for every* $\overset{\triangledown}{\Gamma}$ *and* $\mathbf{I}$, *there is an* $\overset{\triangledown}{A} \in \mathbf{F}(A)$ *such that if* $\mathbf{I}$ *satisfies* $\overset{\triangledown}{\Gamma}$ *then* $\mathbf{I}(\overset{\triangledown}{A}) = t$.

In contrast with classical validity and consequence, we now define the notions of "constructive validity" and of "constructive consequence".

- A formula A is said to be *constructively valid* (and we denote this by "$\models_{\overline{\mathrm{cs}}} A$") iff there is an $\overset{\triangledown}{A} \in \mathbf{F}(A)$ such that, for every $\mathbf{I}$, $\mathbf{I}(\overset{\triangledown}{A}) = t$.

- A formula A is said to be a *constructive consequence* of the (finite or infinite) set of formulas Γ (and we denote this by "$\Gamma \models_{\overline{\mathrm{cs}}} A$") if for every $\overset{\nabla}{\Gamma}$ there is an $\overset{\nabla}{A} \in \mathbf{F}(A)$ such that, for every $\mathbf{I}$, if $\mathbf{I}$ satisfies $\overset{\nabla}{\Gamma}$ then $\mathbf{I}(\overset{\nabla}{A}) = t$.

As compared with the nonconstructive notions analyzed in Corollary 1, *the constructive notions involve an exchange of quantifiers.*

Examples:

1. $p \vee \neg p$ *is not* constructively valid, though it is classically valid. There are exactly two vf's belonging to $\mathbf{F}(p \vee \neg p)$: $\dfrac{p}{p \vee \neg p}$ and $\dfrac{\neg p}{p \vee \neg p}$. The first vf is falsified by every interpretation with $\mathbf{I}(p) = f$ and the second one is falsified by every interpretation with $\mathbf{I}(p) = t$.

2. $(\mathbf{T}(\neg p \vee \neg q) \to \mathbf{T}p \vee \mathbf{T}q) \to \mathbf{T}p \vee \mathbf{T}q$ *is* constructively valid. For, there are four vf's for this formula, among which is the vf

$$\frac{\left\langle \dfrac{\overline{\langle \mathbf{T}(\neg p \vee \neg q), \overset{\mathbf{T}p}{\overline{\mathbf{T}p \vee \mathbf{T}q}} \rangle}}{\mathbf{T}(\neg p \vee \neg q) \to \mathbf{T}p \vee \mathbf{T}q}, \dfrac{\mathbf{T}p}{\mathbf{T}p \vee \mathbf{T}q} \right\rangle \quad \left\langle \dfrac{\langle \mathbf{T}(\neg p \vee \neg q), \overset{\mathbf{T}q}{\overline{\mathbf{T}p \vee \mathbf{T}q}} \rangle}{\mathbf{T}(\neg p \vee \neg q) \to \mathbf{T}p \vee \mathbf{T}q}, \dfrac{\mathbf{T}q}{\mathbf{T}p \vee \mathbf{T}q} \right\rangle}{(\mathbf{T}(\neg p \vee \neg q) \to \mathbf{T}p \vee \mathbf{T}q) \to \mathbf{T}p \vee \mathbf{T}q}$$

which is easily seen to be satisfied by every interpretation.

3.3 The soundness theorem, the completeness theorem, and the maximality theorem for E^* Now we state the soundness theorem for E^* in the following *general* form:

Theorem 4 *If* $\Gamma \vdash_{\overline{E^*}} A$ *then* $\Gamma \models_{\overline{cs}} A$.

Proof: It suffices to prove the following: given an E^*-proof $\begin{matrix} [A_1] \ldots [A_n] \\ \Pi \\ A \end{matrix}$ ($A_1, \ldots, A_n$ the undischarged assumptions, A the consequence) and given any $\overset{\nabla}{A}_1 \in \mathbf{F}(A_1), \ldots, \overset{\nabla}{A}_n \in \mathbf{F}(A_n)$, there exists (and one can build it up starting from $\overset{\nabla}{A}_1, \ldots, \overset{\nabla}{A}_n$ and the proof Π) a vf $\overset{\nabla}{A} \in \mathbf{F}(A)$ such that every $\mathbf{I}$ simultaneously satisfying $\overset{\nabla}{A}_1, \ldots, \overset{\nabla}{A}_n$ satisfies $\overset{\nabla}{A}$ too.

The proof of the latter fact is given by induction on the complexity of Π, where the basis, corresponding to the introduction of an assumption, is immediate. We will treat only the cases corresponding to the rules ($\neg \wedge$ E), ($\to$I), ($\to$E), (**T**), and (E), leaving to the reader the remaining ones.

If $\begin{matrix} \ldots \\ \Pi \\ A \end{matrix}$ is $\dfrac{\begin{matrix} \ldots \\ \Pi_1 \\ \neg(B \wedge C) \end{matrix} \qquad \begin{matrix} \ldots [\cancel{\neg B}] \ldots \\ \Pi_2 \\ A \end{matrix} \qquad \begin{matrix} \ldots [\cancel{\neg C}] \ldots \\ \Pi_3 \\ A \end{matrix}}{A}$, then given $\overset{\nabla}{A}_1, \ldots, \overset{\nabla}{A}_n$ one has, in particular, a vf for every undischarged assumption of Π_1. Thus, by applying the induction hypothesis to Π_1, one has a vf $\overset{\nabla}{\neg}(B \wedge C)$ satisfied by every interpretation satisfying $\overset{\nabla}{A}_1, \ldots, \overset{\nabla}{A}_n$; let, for definiteness, this vf be $\dfrac{\neg \overset{\nabla}{B}}{\neg(B \wedge C)}$. Then one takes the subform $\neg \overset{\nabla}{B}$ and has a vf for all undischarged assumptions of Π_2 (including $\neg B$), where these vf's are satisfied by every interpretation satisfying $\overset{\triangledown}{A}_1, \dots, \overset{\triangledown}{A}_n$: then our assertion follows by applying the induction hypothesis to Π_2.

If $\begin{matrix}\dots\\ \Pi\\ A\end{matrix}$ is $\dfrac{\begin{matrix}\dots[\cancel{B}]\dots\\ \Pi_1\\ C\end{matrix}}{B\to C}$, let $\overset{\triangledown}{A}_1, \dots, \overset{\triangledown}{A}_n$ be given and let $\overset{\triangledown}{B} \in \mathbf{F}(B)$ be any vf of B: by applying the induction hypothesis to Π_1, we can construct a $\overset{\triangledown}{C}$ such that, for every $\mathbf{I}$ satisfying $\overset{\triangledown}{A}_1, \dots, \overset{\triangledown}{A}_n$, if $\mathbf{I}$ satisfies $\overset{\triangledown}{B}$ then $\mathbf{I}$ satisfies $\overset{\triangledown}{C}$. By repeating this reasoning for every $\overset{\triangledown}{B} \in \mathbf{F}(B)$, we can correctly associate a $\overset{\triangledown}{C} \in \mathbf{F}(C)$ with every $\overset{\triangledown}{B} \in \mathbf{F}(B)$, which gives rise to be required vf $B \overset{\triangledown}{\to} C$, with $A = B \to C$.

If $\begin{matrix}\dots\\ \Pi\\ A\end{matrix}$ is $\dfrac{\begin{matrix}\dots\\ \Pi_1\\ B\end{matrix} \quad \begin{matrix}\dots\\ \Pi_2\\ B\to A\end{matrix}}{A}$, then given $\overset{\triangledown}{A}_1, \dots, \overset{\triangledown}{A}_n$, one applies the induction hypothesis to Π_1 and obtains a vf $\overset{\triangledown}{B}$ satisfying the theorem; likewise, one applies the induction hypothesis to Π_2 and obtains a vf $B \overset{\triangledown}{\to} A$ satisfying the theorem. Since the domain of the vf $B \overset{\triangledown}{\to} A$ coincides with $\mathbf{F}(B)$, the vf $\overset{\triangledown}{B}$ *is in the domain of* $B \overset{\triangledown}{\to} A$; the vf $\overset{\triangledown}{A}$ associated by $B \overset{\triangledown}{\to} A$ with $\overset{\triangledown}{B}$ is easily seen to be the required vf for A.

If $\begin{matrix}\dots\\ \Pi\\ A\end{matrix}$ is $\dfrac{\begin{matrix}\dots[\cancel{\neg B}]\dots\\ \Pi_1\\ C\end{matrix} \qquad \begin{matrix}\dots[\cancel{\neg B}]\dots\\ \Pi_2\\ \neg C\end{matrix}}{\mathbf{T}B}$, then, by applying the induction hypothesis to both Π_1 and Π_2, one sees that there is no interpretation $\mathbf{I}$ which satisfies $\overset{\triangledown}{A}_1, \dots, \overset{\triangledown}{A}_n$ and some $\neg\overset{\triangledown}{B}$: for, by Proposition 7, we would have both $\mathbf{I}(C) = t$ and $\mathbf{I}(C) = f$. Hence, every $\mathbf{I}$ satisfying $\overset{\triangledown}{A}_1, \dots, \overset{\triangledown}{A}_n$ satisfies $\mathbf{T}B$.

If $\begin{matrix}\dots\\ \Pi\\ A\end{matrix}$ is $\dfrac{\begin{matrix}\dots\\ \Pi_1\\ \mathbf{T}B\to C\vee D\end{matrix}}{(\mathbf{T}B\to C)\vee(\mathbf{T}C\to D)}$, then we apply the induction hypothesis to Π_1 and obtain a vf $\mathbf{T}B \overset{\triangledown}{\to} C \vee D$ satisfying our theorem. Now, since the latter vf is an implication and $\mathbf{F}(\mathbf{T}B)$ is the *one-element* set $\{\mathbf{T}B\}$, $\mathbf{T}B \overset{\triangledown}{\to} C \vee D$ will be of *one of the two* following kinds:

$$\dfrac{\langle \mathbf{T}B, \overline{C \vee D}^{\,\overset{\triangledown}{C}} \rangle}{\mathbf{T}B \to C \vee D}; \quad \dfrac{\langle \mathbf{T}B, \overline{C \vee D}^{\,\overset{\triangledown}{D}} \rangle}{\mathbf{T}B \to C \vee D}.$$

In the first case we take the subforms $\mathbf{T}B$ and $\overset{\triangledown}{C}$ and construct the required vf:

$$\dfrac{\dfrac{\langle \mathbf{T}B, \overset{\triangledown}{C}\rangle}{\mathbf{T}B \to C}}{(\mathbf{T}B \to C) \vee (\mathbf{T}B \to D)}.$$

In the second case we take the subforms $\mathbf{T}B$ and $\overset{\triangledown}{D}$ and proceed similarly.

Using the soundness theorem and Example 1 of Section 3.2, we immediately deduce $\not\vdash_{\mathrm{E}^*} p \vee \neg p$; likewise, we can show $\not\vdash_{\mathrm{E}^*} (p \to q) \vee (q \to p)$ and the unprovability of all formulas quoted as unprovable in the previous sections, there considered as unprovable in subsystems of E*.

As a corollary of Theorem 4, we obtain:

Corollary 2 *If* $\vdash_{E^*} A \vee B$ *then* $\models_{cs} A$ *or* $\models_{cs} B$; *if* $\vdash_{E^*} \neg(A \wedge B)$ *then* $\models_{cs} \neg A$ *or* $\models_{cs} \neg B$.

Proof: Let us consider only the first case. If $\vdash_{\mathrm{E}^*} A \vee B$, then, by Theorem 4, there is a vf $A \overset{\triangledown}{\vee} B$ satisfied by every interpretation; but $A \overset{\triangledown}{\vee} B = \dfrac{\overset{\triangledown}{A}}{A \vee B}$ or $A \overset{\triangledown}{\vee} B = \dfrac{\overset{\triangledown}{B}}{A \vee B}$, from which our assertion follows.

Corollary 2 doesn't state the disjunction property, since $\models_{\mathrm{cs}} A$ might not imply $\vdash_{\mathrm{E}^*}$A. The disjunction property can be obtained using a direct proof; we will state it as a consequence of Corollary 2 and of the completeness theorem in the following *special* form:

Theorem 5 *If* $\models_{cs} A$ *then* $\vdash_{E^*} A$.

Proof: By Theorem 3, there are $\mathbf{T}A_1, \ldots, \mathbf{T}A_m$ such that:

(1) $\vdash_{\mathrm{E}^*} A \to \mathbf{T}A_1 \vee \ldots \vee \mathbf{T}A_m$;
(2) $\vdash_{\mathrm{E}^*} \mathbf{T}A_1 \vee \ldots \vee \mathbf{T}A_m \to A$.

From implication (1) we immediately deduce that there is an E*-proof

$$\begin{array}{c}[A]\\ \Pi\\ \mathbf{T}A_1 \ldots \mathbf{T}A_m\end{array}$$

whose only undischarged assumption is A. On the other hand, $\models_{\mathrm{cs}} A$, i.e., there is a vf $\overset{\triangledown}{A} \in \mathbf{F}(A)$ true in every interpretation $\mathbf{I}$. Then, by Theorem 4, there is a vf $\mathbf{T}A_1 \vee \overset{\triangledown}{\ldots} \vee \mathbf{T}A_m$ true in every interpretation $\mathbf{I}$. By the definition of a vf of a disjunction of $\mathbf{T}$-formulas, the latter fact implies that, for some i, $1 \leq i \leq m$, $\mathbf{T}A_i$ is satisfied by every interpretation $\mathbf{I}$, i.e., $\models_{\mathrm{cs}} \mathbf{T}A_i$. By the *completeness theorem of classical logic*, we therefore deduce $\vdash_{\mathrm{CPCT}} \mathbf{T}A_i$, from which, by Proposition 1 (see Section 2.1) we deduce $\vdash_{\mathrm{E}_0} \mathbf{TT}A_i$, which gives rise, by (p12) of Section 2.1, to $\vdash_{\mathrm{E}_0} \mathbf{T}A_i$. The latter fact implies, *a fortiori*, $\vdash_{\mathrm{E}^*} \mathbf{T}A_1 \vee \ldots \vee \mathbf{T}A_m$. By implication (2) we obtain $\vdash_{\mathrm{E}^*} A$.

As an immediate consequence of Theorem 5 and Corollary 2, we can state the disjunction property for E*.

Corollary 3 *If* $\vdash_{E^*} A \vee B$ *then* $\vdash_{E^*} A$ *or* $\vdash_{E^*} B$; *if* $\vdash_{E^*} \neg(A \wedge B)$ *then* $\vdash_{E^*} \neg A$ *or* $\vdash_{E^*} \neg B$.

Let us remark that Theorem 5, together with the normal form theorem, provides a decision procedure for E*; another decision procedure directly follows, of course, from the definition of an interpretation of a vf and from the circumstance that the number of vf's of any wff is finite.

The proof of the above Theorem 5 is very simple, but requires the completeness theorem of CPCT and the normal form theorem for E*. As such, it uses a rather particular technique which cannot be generalized to wider contexts. Also, we have obtained a *special* completeness theorem dealing only with constructive validity.

With a more general technique, which can be extended to the predicative frame, we can prove the following *general* completeness theorem:

$$\text{if } \Gamma \models_{\overline{cs}} A \text{ then } \Gamma \vdash_{\overline{E^*}} A.$$

For the sake of brevity we omit the proof.

Even if particular, the technique used in the proof of Theorem 5 allows a very simple proof of the next theorem also. To state this theorem, the following definitions are needed.

- By an E_0-logic L we mean any set of wffs containing all the E_0-provable formulas and closed under modus ponens (i.e., $A \in L$ and $A \to B \in L \Rightarrow B \in L$).
- We say that an E_0-logic is *constructive* iff: $A \vee B \in L \Rightarrow A \in L$ or $B \in L$.
- We say that an E_0-logic is *classically valid* iff: $A \in L \Rightarrow \vdash_{\overline{CPCT}} A$.

Now we can state the following maximality theorem for E^*:

Theorem 6 *Let L be any constructive and classically valid E_0-logic such that: if $\vdash_{\overline{E^*}} A$ then $A \in L$. Then $L = \{A\colon \vdash_{\overline{E^*}} A\}$.*

Proof: That $\{A\colon \vdash_{\overline{E^*}} A\} \subseteq L$ is a hypothesis of the theorem. To prove the converse, we remark that, since L is an extension of E^*, the normal form theorem holds in L too; i.e., for every wff A, there are $\mathbf{T}$-formulas $\mathbf{T}A_1, \ldots, \mathbf{T}A_m$ such that $A \leftrightarrow \mathbf{T}A_1 \vee \ldots \vee \mathbf{T}A_m \in L$. Now, let $A \in L$: since L is an E_0-logic, $\mathbf{T}A_1 \vee \ldots \vee \mathbf{T}A_m \in L$. Since L is constructive, for some i, $1 \leq i \leq m$, $\mathbf{T}A_i \in L$. Since L is classically valid, $\vdash_{\overline{CPCT}} \mathbf{T}A_i$: it follows, by Proposition 1 of Section 2.1, $\vdash_{\overline{E_0}} \mathbf{TT}A_i$, from which, by (p12) of Section 2.1, $\vdash_{\overline{E_0}} \mathbf{T}A_i$. *A fortiori*, $\vdash_{\overline{E^*}} \mathbf{T}A_1 \vee \ldots \vee \mathbf{T}A_m$, from which, by the normal form theorem for E^*, $\vdash_{\overline{E^*}} A$.

We remark that a normal form theorem (Theorem 3 bis) holds for the sublogic E with respect to the wffs containing only atomic formulas in the scope of $\mathbf{T}$. Then, without any change in the proofs of Theorem 5 and Theorem 6 respectively, we obtain:

Theorem 5 bis *If A doesn't contain atomic formulas out of the scope of the occurrences of $\mathbf{T}$, then $\models_{\overline{cs}} A$ implies $\vdash_{\overline{E}} A$.*

Theorem 6 bis *Let L be any constructive and classically valid E_0-logic such that for every A not containing atomic formulas out of the scopes of the occurrences of $\mathbf{T}$, $\vdash_{\overline{E}} A$ implies $A \in L$. Then, for every A not containing atomic formulas out of the scopes of the occurrences of $\mathbf{T}$, $A \in L$ implies $\vdash_{\overline{E}} A$.*

3.4 Concluding remarks The maximality of E^* doesn't imply that, for every constructive and classically valid E_0-logic L, L is a sublogic of E^*. For, by adapting a proof of [1], one can show that the greatest E_0-logic cannot exist.

E^* doesn't satisfy the uniform substitution property, but satisfies the following *restricted substitution property*:

(rsp) if $\vdash_{E^*} A(p_1,\ldots,p_n)$ then, for *any* **T**-formulas $\mathbf{T}A_1,\ldots,\mathbf{T}A_n$, $\vdash_{E^*} A(\mathbf{T}A_1,\ldots,\mathbf{T}A_n)$.

One easily proves the following proposition:

if L is any consistent E_0-logic satsifying (rsp), then L is classically valid.

Thus, the above Theorem 6 can be stated as a maximality result involving constructive and consistent E_0-logics satisfying (rsp). But we can get more.

Let SE^* (stable part of E^*) be the set of formulas so defined:

$$SE^* = \{A(p_1,\ldots,p_n)\colon n > 0 \text{ and, for every } B_1,\ldots,B_n,\ \vdash_{E^*} A(B_1,\ldots,B_n)\}.$$

Then, SE^* turns out to be consistent, closed under modus ponens and satisfying the uniform substitution property; moreover, one can prove the following facts:

- SE^* satisfies the disjunction property
- SE^* is maximal in the family of consistent sets of formulas closed under modus ponens and satisfying the uniform substitution property and the disjunction property.

We don't know of an axiomatization of E^* (we don't even know whether SE^* is decidable).

NOTES

1. If the **T**-formulas and the $\neg$**T**-formulas are assumed to be atomic, one obtains a normalization result for E_0. For a discussion of the plausibility of this assumption see Section 3.

2. We can extend in a quite natural way propositional E_0 into predicative E_0, we will call E_0^{pred}, by introducing the usual rules ($\forall$I), ($\forall$E), ($\exists$I), ($\exists$E), and the following dual rules ($\neg\exists$I), ($\neg\exists$E), ($\neg\forall$I), and ($\neg\forall$E):

$$(\neg\exists\mathrm{I})\colon \frac{\neg A(a)}{\neg\exists x A(x)}; \qquad (\neg\exists\mathrm{E})\colon \frac{\neg\exists x A(x)}{\neg A(t)};$$

$$(\neg\forall\mathrm{I})\colon \frac{\neg A(t)}{\neg\forall x A(x)}; \qquad (\neg\forall\mathrm{E})\colon \frac{\neg\forall x A(x) \qquad \begin{matrix}[\cancel{\neg A(a)}]\\ \vdots \\ B\end{matrix}}{B}.$$

The cautions to be taken in order to correctly apply these rules are the usual (and obvious) ones. In particular, in ($\neg\exists$I) and ($\neg\forall$E), a is a parameter in the sense of [5] and [6]; in ($\neg\exists$E) and ($\neg\forall$I), t is a term free for x in $A(x)$. Now, Proposition 1 can be straightforwardly extended to E_0^{pred}, taking CPrCT instead of CPCT. Thus, our **T**-rules allow a grasp of the essential meaning of the operator **T** also in a predicative frame, without requiring *special* rules (different from the introduction and elimination ones) for the quantifiers. We recall that the classical provability of a formula A does not imply, in a predicative frame, the intuitionistic provability of $\sim\sim A$; to get the latter fact one has to add to IPrC a *special* principle such as Kuroda's.

3. In the predicative frame, (P_2) cannot be stated without the explicit requirement of condition (b_1) involving the existence of the final states. On the other hand, at the propositional level (P_2) is a consequence of conditions (c_1) and (c_2) alone; this means that in this case the latter conditions automatically imply that for every wff A there is a set of states (depending on A) which behave as final with respect to A, i.e., we can develop the semantics of E_0 without requiring (b_1).

4. The above semantics is extended without any difficulty to the predicative case, the only novelty being the existence of (possibly) growing domains associated with the states (where $\neg\forall$ and $\exists$ are interpreted in a similar way, the like for $\forall$ and $\neg\exists$). With this semantics we can prove a soundness and completeness result for E_0^{pred}.

5. This proposition and Proposition 3 no longer hold in the predicative frame. Take, for instance, the Kuroda formula, which is unprovable in IPrC while its translations according to $\mathfrak{J}_1$, $\mathfrak{J}_2$, and $\mathfrak{J}_3$ are provable in E_0^{pred}.

6. With a slightly more complicated proof, involving the transformation of a possibly infinite Kripke countermodel for IPC into a possibly infinite $E_0(\sim,\neg\sim)$ model, one can generalize the above proposition in the following form: for every set Γ of wffs and every wff A such that $\Gamma \cup \{A\}$ is contained in the language of IPC, $\Gamma \vdash_{\text{IPC}} A$ iff $\Gamma \vdash_{E_0(\sim,\neg\sim)} A$. In the predicative frame, on the other hand, we can prove the following: for every set Γ of wffs and every wff A such that $\Gamma \cup \{A\}$ is contained in the language of IPrC, $\Gamma \vdash_{K} A$ iff $\Gamma \vdash_{E_0^{\text{pred}}} A$, where K is the logic obtained by adding the Kuroda principle to IPrC.

7. The soundness and the completeness, as well as this result, hold in the predicative frame.

8. In the predicative frame the $\mathbf{T}$ of $\mathbf{F}(\sim,\neg\sim,\mathbf{T})$ no longer has the meaning of the classical operator, since the Kuroda formula doesn't hold; here it is equivalent to $\sim\sim$, but $\sim$ has the properties of intuitionistic negation. On the other hand, in $E_0^{\text{pred}}(\sim,\neg\sim)$ the operator $\mathbf{T}$ is *still equivalent* to $\sim\sim$, but $\sim$ no longer has the properties of intuitionistic negation; in other words, $\mathbf{T}$-rules modify in an essential way the meaning of $\sim$.

9. The predicative logic $E_0^{*\text{pred}}$ is obtained by adding (*1) and (*2) to E_0^{pred} (here the atomic formulas are the ones of predicate calculus). Then, a soundness and completeness result can be proved for $E_0^{*\text{pred}}$ with respect to the obvious predicative extension of the notion of an E_0^*-model.

10. In the predicative frame one cannot state a normal form theorem even with the addition of the following rule:

$$(\mathrm{E}^{\text{pred}})\colon\ \frac{\mathbf{T}A \to \exists x B(x)}{\exists x(\mathbf{T}A \to B(x))}.$$

Also, one can devise more than one extension of the propositional notion of vf, so that a number of different predicative extensions of E* might be proposed. To deal adequately with these aspects, the authors are collecting the material for a subsequent paper.

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Pierangelo Miglioli
Ugo Moscato
Mario Ornaghi
Dipartimento di Scienze dell' Informazione
Universita' di Milano

Gabriele Usberti
Dipartimento di Filosofia
Universita' di Milano