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Intuitionistic Open Induction and Least Number Principle and the Buss Operator

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Abstract In “Intuitionistic validity in T-normal Kripke structures,” Buss asked whether every intuitionistic theory is, for some classical theory T, that of all T-normal Kripke structures $\mathcal{H}(T)$ for which he gave an r.e. axiomatization. In the language of arithmetic Iop and Lop denote PA^{-} plus Open Induction or Open LNP, iop and lop are their intuitionistic deductive closures. We show $\mathcal{H}(Iop) = lop$ is recursively axiomatizable and $lop \vdash_{i\ c}\dashv iop$, while $i\forall_1 \nvdash lop$. If iT proves $\mathrm{PEM}_{\mathrm{atomic}}$ but not totality of a classically provably total Diophantine function of T, then $\mathcal{H}(T) \nsubseteq iT$ and so $iT \notin \mathrm{range}(\mathcal{H})$. A result due to Wehmeier then implies $i\Pi_1 \notin \mathrm{range}(\mathcal{H})$. We prove Iop is not $\forall_2$-conservative over $i\forall_1$. If $Iop \subseteq T \subseteq I\forall_1$, then iT is not closed under $\mathrm{MR}_{\mathrm{open}}$ or Friedman’s translation, so $iT \notin \mathrm{range}\ (\mathcal{H})$. Both Iop and $I\forall_1$ are closed under the negative translation.

1 ***Iop-normal Kripke structures vs. models of iop or lop*** We begin with a version of the Kripke semantics for (arithmetic in) intuitionistic predicate logic. The language $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ is fixed throughout the paper unless otherwise is mentioned. Let us consider Kripke structures for $\mathcal{L}$ to be of the form $\mathcal{K} = \langle K, \leq; (\mathcal{M}_\alpha)_{\alpha\in K}\rangle$. The frame $\langle K, \leq\rangle$ of $\mathcal{K}$ is a rooted poset whose elements are called nodes of $\mathcal{K}$. The attached world $\mathcal{M}_\alpha$ at a node α is a classical structure (interpreting $=$) for $\mathcal{L}$ whose universe is denoted M_α. In each such world the interpretation of the equality symbol contains, perhaps strictly, the true equality but is still an $\mathcal{L}$-congruence relation. For all $\alpha, \beta \in K$ with $\alpha \leq \beta$ (in which case β is said to be accessible from α), $\mathcal{M}_\alpha$ is a weak substructure of (or homomorphically embedded in) $\mathcal{M}_\beta$. This means that $M_\alpha \subseteq M_\beta$ and the truth in $\mathcal{M}_\alpha$ of atomic sentences with parameters from M_α is preserved in $\mathcal{M}_\beta$, although tuples of elements of M_α may acquire new atomic properties (e.g., equality) in $\mathcal{M}_\beta$. The forcing relation $\Vdash$ between nodes and atomic sentences coincides with classical truth $\models$ in the corresponding attached worlds. In particular no node forces $\perp$. The inductive definitions for $\vee, \wedge, \exists$ are as their classical counterparts while the ones for $\rightarrow$ and $\forall$ (through the latter forcing of a formula defined as

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forcing of its universal closure) are stronger and require the corresponding classical defining clause to hold at all accessible nodes, (see Troelstra and van Dalen [7]). For example, $\alpha \Vdash \neg\varphi(\overline{x})$ (where $\neg\varphi$ is $\varphi \to \bot$) means $\forall\beta \geq \alpha,\ \forall\overline{b} \in M_\beta,\ \beta \Vdash \neg\varphi(\overline{b})$, that is, $\forall\beta \geq \alpha, \forall\overline{b} \in M_\beta, \forall\gamma \geq \beta, \gamma \nVdash \varphi(\overline{b})$. Therefore $\alpha \nVdash \neg\varphi$ if and only if $\exists\beta \geq \alpha, \exists\overline{b} \in M_\beta,\ \exists\gamma \geq \beta,\ \gamma \Vdash \varphi(\overline{b})$. It is quite possible that for some such β and $\overline{b}$, $\varphi(\overline{b})$ is forced at some $\gamma > \beta$ but not at β; in that case $\alpha \nVdash \varphi \vee \neg\varphi$. At a node α of a Kripke structure a formula $\varphi(\overline{x})$ is said to be decidable if α forces $\forall\overline{x}(\varphi(\overline{x}) \vee \neg\varphi(\overline{x}))$ (the instance on φ of the Principle of the Excluded Middle, PEM). For any formula $\psi, \alpha \Vdash \psi$ if and only if $\forall\beta \geq \alpha, \beta \Vdash \psi$. In particular if φ is decidable at a node α, then it is decidable at any node accessible from α also. A formula is decidable in a Kripke structure if it is decidable at its root (equivalently at all its nodes). An intuitionistic theory which proves $\forall\overline{x}(\varphi(\overline{x}) \vee \neg\varphi(\overline{x}))$ is said to decide φ. By soundness and completeness of the Kripke semantics (see [7]), this is equivalent to decidability of φ in any Kripke model of (i.e., one forcing all formulas in) the intuitionistic theory at hand. Some consequences of decidability of atomic formulas $\mathrm{PEM}_{\mathrm{atomic}}$ (which can be considered at a node or in a theory) are presented in Lemma 1.1 below which is essentially due to Markovic [4]. We state it in a somewhat more general form on a node-by-node basis rather than for (Kripke models of) an intuitionistic theory deciding all atomic formulas. One refers to quantifier-free (respectively prenex existential or universal) formulas as open (respectively $\exists_1$ or $\forall_1$). A $\forall_2$-formula is one of the form $\forall\overline{y}\varphi(\overline{x},\overline{y})$ where $\varphi \in \exists_1$. Decidability of open, respectively $\exists$-free, formulas is denoted by $\mathrm{PEM}_{\mathrm{open}}$, respectively $\mathrm{PEM}_{\exists-\mathrm{free}}$.

Lemma 1.1

(i) For a node α of a Kripke structure, $\alpha \Vdash \mathrm{PEM}_{\mathrm{atomic}}$ implies $\alpha \Vdash \mathrm{PEM}_{\mathrm{open}}$. If (the frame of) the structure is linear and $\alpha \Vdash \mathrm{PEM}_{\mathrm{atomic}}$, then indeed $\alpha \Vdash \mathrm{PEM}_{\exists-\mathrm{free}}$.

(ii) If $\alpha \Vdash \mathrm{PEM}_{\mathrm{atomic}}$, $\varphi(\overline{x}) \in \exists_1$, and $\overline{a} \in M_\alpha$, then $\alpha \Vdash \varphi(\overline{a})$ iff $\mathcal{M}_\alpha \models \varphi(\overline{a})$.

(iii) If α is as in (ii) and $\varphi \in \forall_2$, then $\alpha \Vdash \varphi$ iff $\forall\beta \geq \alpha$, $\mathcal{M}_\beta \models \varphi$.

Proof: (i) If for some $\beta \geq \alpha$ and $\overline{b} \in M_\beta,\ \beta \nVdash \varphi(\overline{b}) \to \psi(\overline{b})$ (the cases of $\wedge$ and $\vee$ being more trivial), then $\exists\gamma \geq \beta$ with $\gamma \Vdash \varphi(\overline{b})$ but $\gamma \nVdash \psi(\overline{b})$. Assuming decidability of φ and ψ at α, this implies $\beta \Vdash \varphi(\overline{b}) \wedge \neg\psi(\overline{b})$ and so $\forall\gamma' \geq \beta, \gamma' \nVdash \varphi(\overline{b}) \to \psi(\overline{b})$.To show decidability of $\forall y\varphi(y,\overline{x})$ at α assuming that of φ and linearity of the frame, suppose for some $\beta \geq \alpha,\ \overline{b} \in M_\beta$ and $\gamma \geq \beta,\ \gamma \Vdash \forall y\varphi(y,\overline{b})$. For any $\gamma' \geq \beta$ and $c \in M_{\gamma'}, \gamma' \leq \gamma\ \vee\ \gamma' \geq \gamma$ implies $\gamma' \nVdash \neg\varphi(c,\overline{b})$ and so $\gamma' \Vdash \varphi(c,\overline{b})$.

(ii) Induction on formulas is straightforward again. In fact the *if* part for $\to$ in the induction step is the only place where $\mathrm{PEM}_{\mathrm{atomic}}$ and the formula being prenex $\exists_1$ (not just $\forall$-free) are used. Note that if an atomic formula $R(a)$ with $a \in M_\alpha$ is not decidable at α, then $\mathcal{M}_\alpha \models \neg R(a)$ but $\alpha \nVdash \neg R(a)$. Also $\mathrm{PEM}_{\mathrm{atomic}}$ together with $\mathcal{M}_\alpha \models \neg\exists x\, R(x)$ (R being $\forall$-free or even atomic) does not imply $\alpha \Vdash \neg\exists x R(x)$. Observe that the *only if* part for the case of $\forall$ in the induction step works too. So, as remarked by Markovic, a prenex formula which is forced at α is classically true in $\mathcal{M}_\alpha$.

(iii) This is an immediate consequence of part (ii). □

Here are some sets of axioms which will be used in this paper. We conceive of PA^- as the usual set of axioms for nonnegative parts of discrete strictly ordered commutative rings with 1, (see Kaye [3]). This contains the set

$$\mathrm{SLO} = \{\forall x \neg x < x, \forall xyz((x < y \wedge y < z) \rightarrow x < z), \forall xy(x < y \vee x = y \vee y < x)\}$$

of axioms for strict linear orders. Given a formula $\varphi(x, \overline{y})$, let $I_x\varphi$ denote the instance of induction scheme with respect to x on the formula $\varphi(x, \overline{y})$, that is, the sentence $\forall \overline{y}[\varphi(0, \overline{y}) \wedge \forall x(\varphi(x, \overline{y}) \rightarrow \varphi(x+1, \overline{y})) \rightarrow \forall x\varphi(x, \overline{y})]$. Let $I\exists_1$, respectively $I\forall_1$, respectively Iop, denote the union of PA^- with the set of all instances of induction with respect to any free variable on prenex existential, respectively prenex universal, respectively open formulas. For a set T of sentences in the language $\mathcal{L}$, let iT denote the intuitionistic theory axiomatized by T, that is $iT = \{\varphi : T \vdash_i \varphi\}$. Note that iT contains T (but not its classical deductive closure unless it includes all formulas of the form $\neg\neg\varphi \rightarrow \varphi$). We abbreviate $iIop$ as iop. Similarly $i\exists_1 = iI\exists_1$ and $i\forall_1 = iI\forall_1$. Recall from [1] that for a classical theory T, $\mathcal{H}(T)$ denotes the intuitionistic theory of the class of (i.e. formulas forced in all) T-normal Kripke structures for $\mathcal{L}$ (those whose worlds are classical models of T). The third part of Proposition 1.2 below is Buss's Theorem 7 in [1], where PA (Peano Arithmetic, that is PA^- plus all instances of induction) is weakened in its statement to just SLO plus the appropriate instance of $\exists_1$-induction. It has a similar spirit as the *if* part of Lemma 1.1(iii).

Proposition 1.2

(i) $\mathcal{H}(\mathrm{SLO}) \vdash \mathrm{PEM}_{\mathrm{atomic}} \dashv i\mathrm{SLO}$.

(ii) For $\mathrm{SLO} \subseteq T \subseteq \mathrm{PA}^-$*, a Kripke structure for $\mathcal{L}$ forces iT iff it is T-normal. Therefore* $\mathrm{SLO} \subseteq T \subseteq \mathrm{PA}^-$ *implies $\mathcal{H}(T) = iT$.*

(iii) For any SLO*-normal Kripke model and $\exists_1$-formula $\varphi(x, \overline{y})$, if $\mathcal{M}_\alpha \models I_x\varphi$ for all α, then $\alpha \Vdash I_x\varphi$ for all α. Therefore* $\mathrm{PA}^- \subseteq T \subseteq I\exists_1$ *implies $iT \subseteq \mathcal{H}(T)$. In particular $\mathcal{H}(Iop) \vdash_i iop$.*

Proof: (i) These are immediate from the axioms in SLO (indeed by (ii) which uses both provabilities here, $\mathcal{H}(\mathrm{SLO}) = i\mathrm{SLO}$).

(ii) The axioms in SLO are $\forall_1$ so $\forall_2$. Also note that replacing the axiom $\forall xy(x < y \rightarrow \exists z(x + z = y))$ by its intuitionistically equivalent (prenex) $\forall_2$-formula $\forall xy\exists z(x < y \rightarrow x + z = y)$, PA^- is $\forall_2$-axiomatized too. Now use Lemma 1.1(iii) and (i) above to get the equivalence. The latter statement is then a consequence of soundness and completeness of Kripke semantics.

(iii) Let $\varphi(x, \overline{y})$ be the formula $\exists \overline{z}\psi(x, \overline{y}, \overline{z})$, where ψ is open. Let $\beta \geq \alpha$ be an arbitrary node and $\overline{b} \in M_\beta$. We need to show $\beta \Vdash \exists \overline{z}\,\psi(0, \overline{b}, \overline{z}) \wedge \forall x(\exists \overline{z}\,\psi(x, \overline{b}, \overline{z}) \rightarrow \exists \overline{z}\,\psi(x+1, \overline{b}, \overline{z})) \rightarrow \forall x\exists \overline{z}\,\psi(x, \overline{b}, \overline{z})$. Let $\gamma \geq \beta, \gamma \Vdash \exists \overline{z}\,\psi(0, \overline{b}, \overline{z})$, and $\gamma \Vdash \forall x(\exists \overline{z}\psi(x, \overline{b}, \overline{z}) \rightarrow \exists \overline{z}\psi(x+1, \overline{b}, \overline{z}))$. By Lemma 1.1(iii) and (i) above it is enough to show for any $\eta \geq \gamma$, we have $\mathcal{M}_\eta \models \forall x\exists \overline{z}\psi(x, \overline{b}, \overline{z})$. Since $\eta \geq \gamma$, we have $\eta \Vdash \exists \overline{z}\psi(0, \overline{b}, \overline{z})$ and $\eta \Vdash \forall x(\exists \overline{z}\psi(x, \overline{b}, \overline{z}) \rightarrow \exists \overline{z}\psi(x+1, \overline{b}, \overline{z}))$. So by Lemma 1.1(ii) and (i) again, $\mathcal{M}_\eta \models \exists \overline{z}\psi(0, \overline{b}, \overline{z})$ and $\mathcal{M}_\eta \models \forall x(\exists \overline{z}\psi(x, \overline{b}, \overline{z}) \rightarrow \exists \overline{z}\psi(x+1, \overline{b}, \overline{z}))$. Then by $\mathcal{M}_\eta \models I_x\varphi$, we will have $\mathcal{M}_\eta \models \forall x\exists \overline{z}\psi(x, \overline{b}, \overline{z})$. The relation $iT \subseteq \mathcal{H}(T)$ for $\mathrm{PA}^- \subseteq T \subseteq I\exists_1$ is now a consequence of soundness of the Kripke semantics. $\square$

Remark 1.3 For formulas φ and ψ, Friedman's translation of φ by ψ denoted φ^ψ is obtained by simultaneously replacing each atomic subformula P of φ by $P \vee \psi$, renaming any bound variables of φ which are free in ψ. As Friedman observed in Friedman [2], $\psi \vdash_i \varphi^\psi$ and if $T \vdash_i \varphi$, then $T^\psi \vdash_i \varphi^\psi$. Buss axiomatized the intuitionistic theory $\mathcal{H}(T)$ by formulas of the form $(\neg\varphi)^\psi$, where φ is a semipositive formula (i.e., each subformula of φ of the form $\varphi_1 \to \varphi_2$ has φ_1 atomic) such that $T \vdash_c \neg\varphi$ and ψ is arbitrary. It is immediate from the Buss soundness and completeness theorems in [1] that for any set of axioms T, $iT \subseteq \mathcal{H}(T)$ if and only if (*if* by completeness, *only if* by soundness) every T-normal Kripke structure forces iT. Furthermore using the Buss soundness theorem, it is clear that if every Kripke model of iT is T-normal, then $\mathcal{H}(T) \subseteq iT$. For a recursively enumerable set T of axioms, the Buss axiomatization for $\mathcal{H}(T)$ is recursively enumerable. Given a formula θ, the problem of whether it has the form $(\neg\varphi)^\psi$ for a semipositive formula φ is decidable, whereas the problem of whether T classically proves $\neg\varphi$ has only a partial decision procedure which may well not halt if $T \nvdash_c \neg\varphi$. In Theorem 1.4 below we give a recursive axiomatization of $\mathcal{H}(Iop)$. For a formula $\varphi(x, \overline{y})$, the instance of the Least Number Principle, LNP, on φ with respect to x is the sentence $L_x\varphi : \forall\overline{y}(\exists x\varphi(x, \overline{y}) \to \exists x(\varphi(x, \overline{y}) \wedge \forall z < x\neg\varphi(z, \overline{y})))$. Let Lop denote the union of PA$^-$ with the set of sentences $L_x\varphi(x, \overline{y})$ for open formulas φ and lop abbreviate $iLop$.

Theorem 1.4 $\mathcal{H}(Iop) = lop$.

Proof: It suffices to show that a Kripke structure for $\mathcal{L}$ is *Iop*-normal if and only if it is Lop-normal if and only if it forces lop. As for the former equivalence here, first note that clearly $L_x\neg\varphi \vdash_c I_x\varphi$ for any φ and so $Lop \vdash_c Iop$. This is indeed true intuitionistically as one can see easily by a direct method or by combining this theorem with Proposition 1.2(iii).

The argument for $Iop \vdash_c Lop$ (which will be shown in the next section to fail intuitionistically) is deeper and is based on an important theorem due to Shepherdson [5]. He characterized the rings generated by models of *Iop* as integer parts of real closed fields, that is, discrete subrings which have elements within 1 (equivalently within a finite distance) of every element in the field. Take any $M \models Iop$ and open formula φ. Then φ is a Boolean combination of polynomial inequalities (with coefficients in $\mathbb{N}$). So it defines, after fixing the parameters in M, a finite union of (closed, some of the bounded ones may be single points) intervals in the real closure $RC(M)$ of (the fraction field, ordered in the obvious way, of the ring generated by) M. By Shepherdson's theorem, the initial point of the left-most interval intersecting M has an integer part in M. Either this integer part or its successor in M (depending on whether it belongs to M or not) is the least element of the set defined by φ in M.

Turning to the second equivalence, we know from Proposition 1.2(ii) that a Kripke structure for $\mathcal{L}$ forces iPA$^-$ if and only if it is PA$^-$-normal. So it suffices to show that for any Kripke model of iPA$^-$ all instances of open LNP are classically true in each world if and only if they are forced at every node of the structure.

if: Using Lemma 1.1(ii), this is easily verified on an instance-by-instance and node-by-node basis.

only if: Let $\mathcal{K}$ be an Lop-normal Kripke structure, α a node of $\mathcal{K}$, and $\varphi(x, \overline{y})$ an open formula. To prove $\alpha \Vdash \forall\overline{y}(\exists x\varphi(x, \overline{y}) \to \exists x(\varphi(x, \overline{y}) \wedge \forall z < x\neg\varphi(z, \overline{y})))$, let $\beta \geq \alpha$, $\overline{b} \in M_\beta$, $\gamma \geq \beta$ such that $\gamma \Vdash \exists x\varphi(x, \overline{b})$. Consider the set $\{z \in M_\gamma : \mathcal{M}_\gamma \models \varphi(z, \overline{b})\}$

which by Lemma 1.1(ii) and $\gamma \Vdash \exists x \varphi(x, \overline{b})$ is nonempty and so by $\mathcal{M}_\gamma \models Lop$ has a least element m. By Lemma 1.1(ii) again it is enough to show $\gamma \Vdash \forall z < m \neg\varphi(z, \overline{b})$. If that were not the case, then for some $\delta \geq \gamma$ and $d \in M_\delta$, we would have $\delta \Vdash d < m \wedge \varphi(d, \overline{b})$. We claim $d \in M_\gamma$, contradicting the definition of m.

To prove this claim note that $\varphi(x, \overline{b})$ is a Boolean combination of polynomial inequalities with respect to x with coefficients in M_γ. So $d \in RC(M_\gamma)$. By $Lop \vdash Iop$ and Shepherdson's theorem there exists $d' \in M_\gamma$ which is strictly within 1 of d. But then $d, d' \in M_\delta$ are strictly within 1 of each other. So $d = d' \in M_\gamma$. □

2 Examples for some obstacles to $iT \in \mathrm{range}(\mathcal{H})$ For a formula φ, (a slight variant of) the (Gödel-Gentzen) negative translation of φ denoted $\overline{\varphi}$ is the formula obtained from φ by replacing any subformula of φ of the form $\psi \vee \eta$, respectively $\exists x \psi$, by $\neg(\neg\psi \wedge \neg\eta)$, respectively $\neg\forall x \neg\psi$ and inserting $\neg\neg$ before each atomic subformula of φ except $\perp$ (see [7]). We say that a set of axioms T is closed under the negative translation if $\overline{T^{\vdash_c}} \subseteq iT$, that is, for any formula φ, $T \vdash_c \varphi$ implies $T \vdash_i \overline{\varphi}$. We say that a classical theory S is $\forall_2$-conservative over an intuitionistic theory iT if $S^{\vdash_c}_{\forall_2} \subseteq iT$, that is, whenever $S \vdash_c \forall \overline{x} \exists \overline{y} \varphi(\overline{x}, \overline{y})$ for an open formula φ, then $T \vdash_i \forall \overline{x} \exists \overline{y} \varphi(\overline{x}, \overline{y})$. The notion of Π_2-conservativity is similar by requiring the above for all bounded formulas φ. An intuitionistic theory iT is said to be closed under Friedman's translation if whenever it proves a formula φ, then it proves φ^ψ (see Remark 1.3) for all ψ. We abbreviate this as $\cup_\psi (iT)^\psi \subseteq iT$. It is said to be closed under Markov's Rule if whenever it proves $\neg\neg\exists\overline{y}\varphi(\overline{x}, \overline{y})$ for a formula φ decidable in that theory, then it proves $\exists\overline{y}\varphi(\overline{x}, \overline{y})$. We denote the restricted corresponding rule when φ is assumed open by $\mathrm{MR}_{\mathrm{open}}$. By $(iT_{\neg\neg\exists_1})^{\mathrm{dne}} \subseteq iT$ we mean that iT is closed under $\mathrm{MR}_{\mathrm{open}}$. Friedman observed (see [2]) that closure of iT under Friedman's translation implies its closure under Markov's Rule for atomic formulas. In the case of the extended language $\mathcal{L}_{\mathrm{PR}}$, which has an additional symbol for each primitive recursive function, this means closure under MR for primitive recursive predicates denoted $\mathrm{MR}_{\mathrm{PR}}$. At the time it was already known that closure under $\mathrm{MR}_{\mathrm{PR}}$ in conjunction with decidability of atomic formulas and closure under the negative translation implies Π_2-conservativity. These were actually stated for $T = \mathrm{PA}$, in which case $iT = \mathrm{HA}$ (Heyting Arithmetic), considered in the language $\mathcal{L}_{\mathrm{PR}}$. For the language $\mathcal{L}$, we will see in Theorem 2.1 below an $\mathcal{L}$-version of these implications interpolated by a couple of properties in terms of $\mathcal{H}$.

Theorem 2.1 *For any set of axioms T in $\mathcal{L}$,*

(i) If $(iT_{\neg\neg\exists_1})^{\mathrm{dne}} \cup \mathrm{PEM}_{\mathrm{atomic}} \cup \overline{T^{\vdash_c}} \subseteq iT$, then $T^{\vdash_c}_{\forall_2} \subseteq iT$.

(ii) If $(iT_{\neg\neg\exists_1})^{\mathrm{dne}} \not\subseteq iT$, then $T^{\vdash_c}_{\forall_2} \not\subseteq iT$. If $\mathrm{PEM}_{\mathrm{atomic}} \subseteq iT$ and $T^{\vdash_c}_{\forall_2} \not\subseteq iT$, then $\mathcal{H}(T) \not\subseteq iT$.

(iii) If $\overline{T^{\vdash_c}} \subseteq iT$ but $\mathcal{H}(T) \not\subseteq iT$, then $\cup_\varphi (iT)^\varphi \not\subseteq iT$.

(iv) If $\cup_\varphi (iT)^\varphi \cup \mathcal{H}(T) \not\subseteq iT$, then $iT \notin \mathrm{range}(\mathcal{H})$.

Proof: (i) Suppose that T classically but not intuitionistically proves $\forall\overline{x}\exists\overline{y}\varphi(\overline{x}, \overline{y})$, for an open formula φ. From closure under the negative translation we get $T \vdash_i \forall\overline{x}\neg\forall\overline{y}\neg\overline{\varphi}(\overline{x}, \overline{y})$ and therefore $T \vdash_i \neg\neg\exists\overline{y}\, \overline{\varphi}(\overline{x}, \overline{y})$. Now since atomic formulas are decidable in iT, by Lemma 1.1(ii) and $\vdash_i \varphi \longleftrightarrow \overline{\varphi}$ for any open φ we have $T \vdash_i \varphi \longleftrightarrow \overline{\varphi}$. Therefore $T \vdash_i \neg\neg\exists\overline{y}\ \varphi(\overline{x}, \overline{y})$ which contradicts closure of iT under $\mathrm{MR}_{\mathrm{open}}$.

(ii) Note that if for some open formula $\psi(\overline{x}, \overline{y})$, $iT \vdash \neg\neg\exists\overline{y}\psi(\overline{x}, \overline{y})$ but $iT \nvdash \exists\overline{y}\psi$, then T classically but not intuitionistically proves the $\forall_2$-sentence $\varphi : \forall\overline{x}\exists\overline{y}\psi(\overline{x}, \overline{y})$. From this together with decidability of atomic formulas in iT and by Lemma 1.1(iii), one gets $\varphi \in \mathcal{H}(T)$ while by assumption $\varphi \notin iT$.

(iii) We give the following argument due to Buss, which he used to conclude $\mathcal{H}(\mathrm{PA}) \subseteq \mathrm{HA}$ from the facts that HA is closed under both Friedman's and the negative translations. First note that for any semipositive formula φ, $\varphi \vdash_i \overline{\varphi}$. This can be proved by induction on the complexity of φ, using $\neg\neg\overline{\varphi_2} \to \overline{\varphi_2}$ in the induction step $\varphi = \varphi_1 \to \varphi_2$ (where φ_1 is atomic by semipositivity of φ). If φ is not semipositive, the conclusion may fail, as it can be seen, for example, for $\varphi = I_y(2y \leq x)$ in Theorem 2.3 below. Fix any T which is closed under the negative translation. Then for any formula φ, $T \vdash_c \neg\varphi$ implies $T \vdash_i (\overline{\varphi} \to \perp)$. So if φ is semipositive and $T \vdash_c \neg\varphi$, then $T \vdash_i (\varphi \to \perp)$. Assume on the contrary that iT is closed under Friedman's translation. Then for any semipositive φ with $T \vdash_c \neg\varphi$ and formula θ, $T \vdash_i (\varphi \to \perp)^{\theta}$, that is, $T \vdash_i \varphi^{\theta} \to \theta$. This means that iT proves all of Buss's axioms for $\mathcal{H}(T)$, so we get the contradiction $\mathcal{H}(T) \subseteq iT$.

(iv) Assume first that $\mathcal{H}(T) \nsubseteq iT$. By the soundness theorem in [1], for every classical theory S, $\mathcal{H}(S) \subseteq S$ (consider one-node structures and use classical completeness, here S is closed under $\vdash_c$). If $\mathcal{H}(S) = iT$, then $iT \subseteq S$. Now since $T \subseteq iT$, we would have $T \subseteq S$, and therefore $\mathcal{H}(T) \subseteq \mathcal{H}(S)$, proving the contradiction $\mathcal{H}(T) \subseteq iT$.

Next assume $\cup_{\varphi}(iT)^{\varphi} \nsubseteq iT$ and $\mathcal{H}(S) = iT$, for a classical theory S. We prove the contradiction that (the set of Buss's axioms for) $\mathcal{H}(S)$ is closed under Friedman's translation. The argument goes as follows. For a semipositive formula φ and arbitrary formulas ψ and θ, from the fact $\theta \vdash_i \psi^{\theta}$ mentioned in Remark 1.3 and by induction on φ we have $\varphi^{(\psi^{\theta})} \equiv_i (\varphi^{\psi})^{\theta}$ and therefore $(\neg\varphi)^{(\psi^{\theta})} \vdash_i ((\neg\varphi)^{\psi})^{\theta}$. □

Example 2.2 It is immediate from (ii) and (iv) of Theorem 2.1 that if a classical fragment T of PA extending PA^- has a Diophantine- (i.e., $\exists_1$) definable provably total function which is not provably total in iT (see [2] and [3]), then $iT \notin$ range $\mathcal{H}$. We bring here an example of this suggested by one of the referees. Recall that the class Π_1, respectively Σ_1, is the closure of the set Δ_0 of bounded formulas under blocks of $\forall$'s, respectively $\exists$'s, and $I\Pi_1$ is PA^- together with all instances of induction on Π_1-formulas. It is well known that (see [3]) the exponential function is a Diophantine-definable provably total function of $I\Pi_1$. On the other hand, Wehmeier proved in [9] that any provably total function of $i\Pi_1$ which has a Σ_1-definition, is majorized in $\mathbb{N}$ by some polynomial. Hence $i\Pi_1 \notin$ range $(\mathcal{H})$. The reason for bringing in the Diophantine-definability issue is as follows. By Lemma 1.1(iii) if all atomic formulas are decidable in iT, then T is $\forall_2$-conservative over $\mathcal{H}(T)$. However T need not be Π_2-conservative (in fact not even Σ_1-conservative) over $\mathcal{H}(T)$. We give an example for this in Theorem 2.3(iv). On the other hand, for the language $\mathcal{L}_{\mathrm{PR}}$ if all atomic formulas are decidable in iT, then T is Π_2-conservative over $\mathcal{H}(T)$.

To establish $i\Pi_1$ does not prove totality of exponentiation, Wehmeier proved in [9] that a two-node Kripke model of $i\Pi_1$ is obtained if one puts a classical model of $I\Pi_1$ above a Δ_0- elementary submodel of it which is a model of $I\Delta_0$. We put a classical nonstandard model of $\mathrm{Th}(\mathbb{N})$ over the semi-ring generated by an infinitely large element in Theorem 2.3 below to get a model of (e.g.) $i\forall_1$ whose lower node does not decide a $\forall_2$-sentence classically provable over PA^- by a single instance of open induction. Our $\forall_2$-sentence is the statement that the function $\lfloor \frac{x}{2} \rfloor$ is total, that is, the sentence $\forall x \exists y(x = 2y \vee x = 2y + 1)$ which we denote by AEO. We will use the first pruning lemma from van Dalen et al. [8]. It says that if φ and ψ are formulas with possible parameters from the world M_α at some node α of a Kripke structure such that $\alpha \nVdash \psi$, then $\alpha \Vdash \varphi^\psi$ if and only if $\alpha \Vdash^\psi \varphi$, where $\Vdash^\psi$ is forcing in the Kripke structure obtained from the original one by pruning nodes forcing ψ.

Theorem 2.3

(i) $T_1 =: \mathrm{PA}^- + I_y(2y \leq x) \vdash_i \forall x \neg\neg\exists y(x = 2y \vee x = 2y + 1)$.

(ii) $T_2 =: \mathrm{PA}^- + Th_{\exists-\mathrm{free}}(\mathbb{N}) + \neg\neg Th(\mathbb{N}) + \mathcal{H}(Th_{\Sigma_1 \cup \forall_1}(\mathbb{N})) \nvdash_i (I_y(2y \leq x))^{\mathrm{AEO}} \vee L_y(x < 2y)$.

(iii) *If* $T_1 \subseteq T \subseteq T_2$, *then* $(iT_{\neg\neg\exists_1})^{\mathrm{dne}} \nsubseteq iT$ *and* $\cup_\psi (iT)^\psi \nsubseteq iT$.

(iv) $\mathcal{H}(\mathrm{PA}^- + \exists x \forall y \leq x(2y \leq x \to 2y + 2 \leq x)) \nvdash_i \exists x \forall y \leq x(2y \leq x \to 2y + 2 \leq x)$.

Proof: (i) We have $I_y(2y \leq x) \equiv_{i\mathrm{PA}^-} \forall x(\forall y(2y \leq x \to 2y + 2 \leq x) \to \forall y 2y \leq x) \vdash_i \forall x(\neg\forall y 2y \leq x \to \neg\forall y(2y \leq x \to 2y + 2 \leq x)) \equiv_{i\mathrm{PA}^-} \forall x \neg\forall y(2y \leq x \to 2y + 2 \leq x) \vdash_{i\mathrm{PA}^-} \forall x \neg\forall y \neg(x = 2y \vee x = 2y + 1) \vdash_i \forall x \neg\neg\exists y(x = 2y \vee x = 2y + 1)$.

(ii) Consider the two-node Kripke model $\mathcal{K}$ based on the frame $\{0 < 1\}$, where $M_0 = \mathbb{Z}[t]^{\geq 0}$ (polynomials in t over $\mathbb{Z}$ with nonnegative leading coefficient) equipped with the usual $+, \cdot$, and the compatible order determined by making t positive and infinitely large (for more information see [3]) and $\mathcal{M}_1$ is a nonstandard model of $\mathrm{Th}(\mathbb{N})$. Note that, up to an isomorphism of $\mathcal{L}$-structures which sends t to a nonstandard element, $\mathbb{Z}[t]^{\geq 0}$ is an initial segment of any nonstandard model of PA^-. So we may assume that $\mathcal{M}_0$ is a substructure of $\mathcal{M}_1$. Certainly $\mathbb{Z}[t]^{\geq 0} \subsetneqq M_1$, since for instance $\mathcal{M}_0 \not\models \mathrm{AEO}$ (the element t is neither even nor odd in $\mathbb{Z}[t]^{\geq 0}$).

The node 1 is terminal, hence classical (i.e., all formulas are decidable at 1). So as remarked in [8], $1 \Vdash \mathrm{Th}(\mathbb{N})$. On the other hand by Lemma 1.1(i) the lower node, 0, forces every $\exists$-free formula forced at the upper one. This shows that 0 forces $\mathrm{Th}_{\exists-\mathrm{free}}(\mathbb{N})$. Also any PA^--normal Kripke structure forces $i\mathrm{PA}^-$ regardless of whether it is linear or not. For an arbitrary $\tau \in \mathrm{Th}(\mathbb{N})$, $1 \nVdash \neg\tau$, and therefore $0 \Vdash \neg\neg\tau$. As mentioned in [3], $\mathrm{PA}^- \vdash_c \mathrm{Th}_{\Sigma_1}(\mathbb{N})$ and $\mathbb{Z}[t]^{\geq 0} \models \mathrm{Th}_{\Sigma_1 \cup \forall_1}(\mathbb{N})$. So $\mathcal{K}$ is $\mathrm{Th}_{\Sigma_1 \cup \forall_1}(\mathbb{N})$-normal. So $\mathcal{K} \Vdash iT_2$.

Note that the AEO-pruning of $\mathcal{K}$ results in the single-node classical model $\mathcal{M}_0$ and $\mathcal{M}_0 \not\models I_y(2y \leq x)$ (e.g., since $\mathcal{M}_0 \models \mathrm{PA}^-$ but not AEO). Besides telling us that $\mathcal{K}$ is not T_2-normal, this shows $0 \nVdash^{\mathrm{AEO}} I_y(2y \leq x)$ and so by the first pruning lemma, (the lower node of) $\mathcal{K}$ does not force $(I_y(2y \leq x))^{\mathrm{AEO}}$. Also observe that $\mathcal{M}_0 \not\models L_y(x < 2y)$ either as the set $\{2t - 2n \ : \ n \in \mathbb{N}\}$ has no minimum in $\mathbb{Z}[t]^{\geq 0}$. So by *(if)* in the proof of Theorem 1.4, (the node 0 of) $\mathcal{K}$ does not force $L_y(x < y)$. Now by soundness of Kripke semantics for intuitionistic predicate logic, we get $T_2 \nvdash_i (I_y(2y \leq x))^{\mathrm{AEO}} \vee L_y(x < 2y)$. In particular $T_2 \nvdash_i lop$.

(iii) By (ii), $T \nvdash_i (I_y(2y \leq x))^{\mathrm{AEO}}$ and therefore $T \nvdash_i$ AEO (since as mentioned in Remark 1.3, AEO $\vdash_i \varphi^{\mathrm{AEO}}$ for any φ). Combining the latter with (i) we see that iT is not closed under $\mathrm{MR}_{\mathrm{open}}$ (so by Theorem 2.1 and Lemma 1.1, T is not $\forall_2$-conservative over iT and $\mathcal{H}(T) \nsubseteq iT \notin \mathrm{range}(\mathcal{H})$). On the other hand, since T includes $I_y(2y \leq x)$, we get from the former that iT is not closed under Friedman's translation (even in the cases when T is not closed under the negative translation).

(iv) Consider the two-node Kripke model obtained by putting $\mathbb{Z}[\frac{t}{2}]^{\geq 0}$ over $\mathbb{Z}[t]^{\geq 0}$ (using soundness of the Kripke semantics again). Therefore the *if* parts of the Σ_1-version of Lemma 1.1(ii) and of the Π_2-version of Lemma 1.1(iii) fail. □

Example 2.4 There is no classical theory S such that $\mathcal{H}(S) = iop$ or $\mathcal{H}(S) = i\forall_1$. The theory iop is not complete with respect to Iop-normal Kripke structures (it is sound though, as we saw in Section 1) and $i\forall_1 \nvdash_i lop$. The function $\lfloor \frac{x}{2} \rfloor$ is a Diophantine-definable provably total function of $(\mathrm{PA}^- + I_y(2y \leq x))^{\vdash_c}$ but not of $i\forall_1$. The theories iop and $i\forall_1$ also satisfy the other four negative statements in Theorem 2.1 as they are not closed under $\mathrm{MR}_{\mathrm{open}}$. Let us mention in passing, however, that iop and $i\forall_1$ (as any other fragment of HA of the form $i\Gamma$, that is, the intuitionistic theory axiomatized by PA^- plus instances of induction on formulas in Γ) have the Disjunction Property and Explicit Definability (see Smorynski [6]) and are therefore closed under Markov's Rule (for decidable, in particular open, formulas) with *one* free variable. We finally note that both Iop and $I\forall_1$ are closed under the negative translation. For any set of axioms T and formula φ we have $T \vdash_c \varphi \Longrightarrow \overline{T} \vdash_i \overline{\varphi}$, (see [7]). So it is enough to show that iop and $i\forall_1$ prove the negative translations of their axioms. Note that for any instance of open, respectively $\forall_1$-, induction, its negative translation is again such an instance ($\overline{I_x\varphi} = I_x\overline{\varphi}$, $\overline{I_z\forall\overline{x}\varphi(\overline{x},\overline{y},z)} = I_z\forall\overline{x}\overline{\varphi(\overline{x},\overline{y},z)}$ and $\overline{\varphi}$ is open if φ is). As for the axioms in PA^-, one may treat them one by one or note that they have intuitionistically equivalent forms $\forall\overline{x}(P(\overline{x}) \wedge Q(\overline{x}) \to \exists y R(\overline{x}, y))$, where P, Q, and R are atomic and use $\forall\overline{x}(P(\overline{x}) \wedge Q(\overline{x}) \to \exists y R(\overline{x}, y)) \vdash_i \forall\overline{x}(\neg\neg P(\overline{x}) \wedge \neg\neg Q(\overline{x}) \to \neg\forall y \neg R(\overline{x}, y))$.

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