

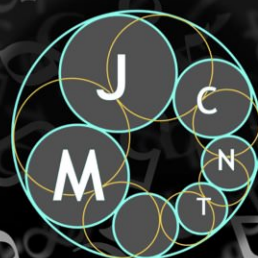
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On the domination number of a graph defined by containment

Peter Frankl



# On the domination number of a graph defined by containment

Peter Frankl

Let  $n > k > 2$  be integers. Define a bipartite graph between all  $k$ -element and all 2-element subsets of an  $n$ -element set by drawing an edge if and only if the first one contains the second. The domination number of this graph is determined up to a factor of  $1 + o(1)$ . The short proof relies on some extremal results concerning hypergraphs.

## 1. Introduction

For a graph  $\mathcal{G} = (V, \mathcal{E})$  a subset  $D \subset V$  is called a *dominating set* if for every vertex  $x \in V \setminus D$  there is an edge  $E \in \mathcal{E}$  satisfying  $x \in E$  and  $E \cap D \neq \emptyset$ . The *domination number*  $\varrho(\mathcal{G})$  is the minimum of  $|D|$  over all dominating sets.

To determine  $\varrho(\mathcal{G})$  for a given graph is very difficult in general. In the present paper we address this problem for a bipartite graph defined via containments of sets.

For  $n$  and  $k$  positive integers, with  $n > k$ , we denote by  $[n] = \{1, 2, \dots, n\}$  the standard  $n$ -element set and by  $\binom{[n]}{k}$  the collection of all  $k$ -element subsets of  $[n]$ . For integers  $n > k > \ell \geq 2$ , we define the bipartite graph  $\mathcal{B} = \mathcal{B}_n(k, \ell)$  on the vertex set  $\binom{[n]}{k} \cup \binom{[n]}{\ell}$  by drawing an edge between  $F \in \binom{[n]}{k}$  and  $G \in \binom{[n]}{\ell}$  if and only if  $G \subset F$ .

The problem of determining or estimating  $\varrho(\mathcal{B})$  was raised in [Badakhshian et al. 2019] by Badakhshian, Katona and Tuza. They determined  $\varrho(\mathcal{B}_n(3, 2))$  up to a factor  $1 + o(1)$ , where  $o(1) \rightarrow 0$  as  $n \rightarrow \infty$ .

In the present paper we extend their work to all  $k \geq 3$ .

**Theorem 1.1.** 
$$\varrho(\mathcal{B}_n(k, 2)) = (1 + o(1)) \binom{n}{2} \frac{k+3}{k^2-1}.$$

To prove the lower bound we use a result from [Erdős et al. 1986] extending the celebrated Ruzsa–Szemerédi theorem [1978]. To obtain the matching upper bound we apply a probabilistic construction based on a result of [Frankl and Rödl 1985]. To prove similar results for  $\varrho(\mathcal{B}_n(k, \ell))$  where  $\ell \geq 3$  appears to be much harder (Section 4).

## 2. Proof of the lower bound

Let  $k \geq 3$  be fixed and  $\varepsilon > 0$  be arbitrarily small. Choose  $\mathcal{G} \subset \binom{[n]}{2}$  and  $\mathcal{F} \subset \binom{[n]}{k}$  such that  $\mathcal{F} \cup \mathcal{G}$  is a dominating set for  $\mathcal{B} = \mathcal{B}_n(k, 2)$ . Our aim is to prove

$$|\mathcal{F}| + |\mathcal{G}| > \binom{n}{2} \left( \frac{k+3}{k^2-1} - \varepsilon \right). \quad (1)$$

This research was done while the author was visiting Academia Sinica in Taipei.

**Keywords:** finite sets, graphs, hypergraphs, Turán's theorem.

Since  $\frac{k+3}{k^2-1} \leq \frac{3}{4}$  for  $k \geq 3$ , we may assume that

$$|\mathcal{F}| \leq \frac{3}{4} \binom{n}{2}. \quad (2)$$

**Proposition 2.1.**  $|\mathcal{G}| > \frac{1-\varepsilon}{k-1} \binom{n}{2}$  for all  $n > n_0(k, \varepsilon)$ .

*Proof of the proposition.* Let  $m$  be an integer (later qualified) and consider an  $m$ -element set  $R \subset [n]$ . If  $R$  contains no  $F \in \mathcal{F}$ , then the assumption on domination is equivalent to the fact that  $\mathcal{G}_R := \mathcal{G} \cap \binom{R}{2}$  has no independent set of  $k$  vertices. By Turán's theorem [1941] (or see [Bollobás 1978]), we have

$$\begin{aligned} \left| \mathcal{G} \cap \binom{R}{2} \right| &> (k-1) \binom{m/(k-1)}{2} = \frac{m(m-k+1)}{2(k-1)} \\ &> \binom{m}{2} \frac{1-\varepsilon/2}{k-1} \text{ for } m > 2k/\varepsilon. \end{aligned} \quad (3)$$

We now assume  $m$  is large enough that (3) is satisfied. Let us choose the set  $P \in \binom{[n]}{m}$  uniformly at random.

**Claim 2.2.** Let  $n > m^3/\varepsilon$ . Then the probability of  $\binom{P}{k} \cap \mathcal{F} \neq \emptyset$  is smaller than  $\varepsilon/2$ .

*Proof.* Since each  $F \in \mathcal{F}$  is contained in  $\binom{n-k}{m-k}$  subsets  $R \in \binom{[n]}{m}$ , (2) implies the upper bound  $\frac{3}{4} \binom{n}{2} \binom{n-k}{m-k}$  on the number of  $R$  in question. Using  $k \geq 3$  we obtain the upper bound

$$\frac{3}{4} \binom{n}{2} \binom{n-3}{m-3} = \binom{n}{m} \cdot \frac{m-2}{n-2} \binom{m}{2} \cdot \frac{3}{4} < \binom{n}{m} \frac{m^3}{2n} < \frac{\varepsilon}{2} \binom{n}{m}.$$

In view of the claim, for  $n > m^3/\varepsilon$  a proportion of more than  $(1-\varepsilon/2)$  of  $R \in \binom{[n]}{m}$  satisfy (3). Now  $(1-\varepsilon/2)^2 > 1-\varepsilon$  implies the inequality in Proposition 2.1, with  $n_0(k, \varepsilon) > (2k/\varepsilon)^3/\varepsilon$ .  $\square$

Let  $\mathcal{H} = \binom{[n]}{2} \setminus \mathcal{G}$  be the graph of those edges  $H \in \binom{[n]}{2}$  that are not in  $\mathcal{G}$ . Since  $\mathcal{F} \cup \mathcal{G}$  is a dominating set for  $\mathcal{B}$ , for each  $H \in \mathcal{H}$  there exists some  $F \in \mathcal{F}$  with  $H \subset F$ . From this we infer

$$|\mathcal{F}| \geq \frac{|\mathcal{H}|}{\binom{k}{2}}. \quad (4)$$

Using (4) together with Proposition 2.1 one can show that

$$|\mathcal{F}| + |\mathcal{G}| \geq \frac{1-\varepsilon}{k-1} \binom{n}{2} + \frac{k-2+\varepsilon}{(k-1)} \binom{n}{2}$$

which is slightly weaker than (1). To prove (1), we would need (4) with  $\binom{k}{2} - 1$  in the denominator.

Our strategy is relatively simple. We try and list (some of) the edges of  $\mathcal{F}$ :  $F_1, F_2, \dots, F_q$  such that  $\binom{F_1}{2} \cap \mathcal{G} \neq \emptyset$ , then  $\binom{F_2}{2} \cap (\mathcal{G} \cup \binom{F_1}{2}) \neq \emptyset$ , etc. That is, we choose sequentially  $F_i$ ,  $1 \leq i \leq q$ , so that  $\binom{F_i}{2} \cap \mathcal{G} \neq \emptyset$  or  $|F_j \cap F_i| \geq 2$  for some  $1 \leq j < i$ . For each  $F_i$  let  $\mathcal{E}(F_i)$  consist of those  $E \in \mathcal{H}$  that  $E \not\subset F_j$  for  $1 \leq j < i$ . From the construction it follows that

$$|\mathcal{E}(F_i)| \leq \binom{k}{2} - 1 \text{ for all } 1 \leq i \leq q. \quad (5)$$

Should  $\mathcal{F} = \{F_1, \dots, F_q\}$  hold, (1) would follow. In the opposite case set  $\mathcal{F}_0 = \{F_1, \dots, F_q\}$  and  $\mathcal{H}_0 = ((\binom{F_1}{2}) \cup \dots \cup (\binom{F_q}{2})) \setminus \mathcal{G}$ .

Choosing  $q$  maximal,  $\binom{F}{2} \cap \mathcal{G} = \emptyset$  and  $|F \cap F_i| \leq 1$  follow for  $F \in \mathcal{F} \setminus \mathcal{F}_0$ ,  $1 \leq i \leq q$ .

We define  $\mathcal{F}_1 = \{F_1, \dots, F_{q_1}\}$  similarly. We choose  $F_1 \in \mathcal{F} \setminus \mathcal{F}_0$  arbitrarily and once  $F_1, \dots, F_{s-1} \in \mathcal{F} \setminus \mathcal{F}_0$  are fixed, we choose an arbitrary  $F_s \in \mathcal{F} \setminus \mathcal{F}_0$  from the rest, satisfying  $|F_i \cap F_s| \geq 2$  for some  $1 \leq i < s$ . Now let  $\mathcal{F}_1$  be a maximal collection obtained in this way. This choice guarantees  $|F \cap F'| \leq 1$  for all  $F \in \mathcal{F} \setminus (\mathcal{F}_0 \cup \mathcal{F}_1)$ ,  $F' \in \mathcal{F}_1$ .

Set  $\mathcal{H}_1 = \bigcup_{F \in \mathcal{F}_1} \binom{F}{2}$ . Our procedure guarantees

$$|\mathcal{H}_1| \leq 1 + |\mathcal{F}_1| \left( \binom{k}{2} - 1 \right). \quad (6)$$

We iterate this procedure. Once  $\mathcal{F}_1, \dots, \mathcal{F}_p$  and thereby  $\mathcal{H}_i = \bigcup_{F \in \mathcal{F}_i} \binom{F}{2}$ ,  $1 \leq i \leq p$  are chosen we have

$$|F \cap F'| \leq 1 \quad \text{for all } F \in \mathcal{G} \cup \mathcal{F}_0 \cup \dots \cup \mathcal{F}_p \text{ and } F' \in \mathcal{F} \setminus (\mathcal{F}_0 \cup \dots \cup \mathcal{F}_p).$$

As long as there are sets remaining in  $\mathcal{F}$  we can define  $\mathcal{F}_{p+1}$  and  $\mathcal{H}_{p+1}$  in the above way.

Eventually we obtain a partition,

$$\mathcal{F} = \mathcal{F}_0 \sqcup \dots \sqcup \mathcal{F}_t$$

such that

$$\mathcal{H}_0 \sqcup \dots \sqcup \mathcal{H}_t = \binom{[n]}{2} \setminus \mathcal{G}$$

(here we used that  $\mathcal{G} \cup \mathcal{F}$  is a dominating set). Moreover (6) holds for 1 replaced by  $i$ :

$$|\mathcal{H}_i| \leq 1 + |\mathcal{F}_i| \left( \binom{k}{2} - 1 \right), \quad 1 \leq i \leq t. \quad (7)$$

Since for  $i = 0$  we do not need the extra 1, we infer

$$\binom{n}{2} - |\mathcal{G}| \leq t + |\mathcal{F}| \left( \binom{k}{2} - 1 \right),$$

or equivalently

$$|\mathcal{G}| + |\mathcal{F}| \geq \frac{\binom{n}{2}}{\binom{k}{2} - 1} + |\mathcal{G}| \frac{\binom{k}{2} - 2}{\binom{k}{2} - 1} - \frac{t}{\binom{k}{2} - 1}.$$

Substituting  $|\mathcal{G}| > \frac{1-\varepsilon}{k-1} \binom{n}{2}$  we obtain

$$\begin{aligned} |\mathcal{G}| + |\mathcal{F}| &> \frac{\binom{n}{2}}{\binom{k}{2} - 1} \left( 1 + \frac{\binom{k}{2} - 2}{k-1} - \frac{\varepsilon}{k-1} \right) - \frac{t}{\binom{k}{2} - 1} \\ &= \binom{n}{2} \left( \frac{k+3}{k^2-1} - \frac{2\varepsilon}{(k^2-1)(k-2)} \right) - \frac{t}{\binom{k}{2} - 1}. \end{aligned}$$

To conclude the proof of the lower bound it is clearly more than sufficient to show that  $t = o\left(\binom{n}{2}\right)$ . To achieve this we will need the following extension of a celebrated result from [Ruzsa and Szemerédi 1978]:

**Theorem 2.3** (Erdős, Frankl, Rödl [Erdős et al. 1986]). *Suppose that  $\mathcal{T} \subset \binom{[n]}{k}$  satisfies  $|T \cap T'| \leq 1$  for all distinct  $T, T' \in \mathcal{T}$ , moreover one cannot find a  $k$ -set  $\{x_1, \dots, x_k\} \subset [n]$  and  $\binom{k}{2}$  distinct members  $T(i, j) \in \mathcal{T}$ ,  $1 \leq i < j \leq k$ , such that  $\{x_i, x_j\} \subset T(i, j)$ . Then*

$$|\mathcal{T}| = o\left(\binom{n}{2}\right). \quad (8)$$

To apply (8) we choose  $F(i)$  as an arbitrary member of  $\mathcal{F}_i$  for  $1 \leq i \leq t$  and define

$$\mathcal{T} = \{F(i) : 1 \leq i \leq t\}.$$

The condition  $|T \cap T'| \leq 1$  is automatically satisfied. To prove the second condition we argue indirectly.

Suppose that we found  $F = \{x_1, \dots, x_k\}$  and  $\binom{k}{2}$  members  $T(i, j) \in \mathcal{T}$  such that  $\{x_i, x_j\} \subset T(i, j)$ . Since  $\mathcal{F} \cup \mathcal{G}$  is a dominating set for  $B$ , either  $F \in \mathcal{F}$  or  $G \subset F$  for some  $G \in \mathcal{G}$ . In the latter case  $G = \{x_i, x_j\}$  for some  $1 \leq i < j \leq k$ . I.e.,  $G \subset T(i, j)$ . But this is impossible since we put all such  $T(i, j)$  into  $\mathcal{F}_0$ . Suppose next  $F \in \mathcal{F}$ . Assume by symmetry  $T(1, 2) \in \mathcal{F}_1$ ,  $T(1, 3) \in \mathcal{F}_2$ . From  $|T(1, \ell) \cap F| \geq 2$  we infer  $F \in \mathcal{F}_{\ell-1}$  for  $\ell = 2, 3$ . This is impossible because of  $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ , giving the desired contradiction.  $\square$

### 3. The proof of the upper bound

We give a probabilistic construction based on the following old result.

Let  $r \geq 2$  be an integer and consider an  $r$ -uniform hypergraph  $\mathcal{H} \subset \binom{X}{r}$ , where  $|X| = m$ . For  $x \in X$  let  $d(x)$  be the degree of  $x$  in  $\mathcal{H}$ , that is, the number of  $H \in \mathcal{H}$  containing  $x$ . The *double degree*  $d(x, y)$  is defined analogously.

The *covering index*  $b(\mathcal{H})$  is defined as the minimal number  $b$  such that there exist  $b$  edges in  $\mathcal{H}$  whose union is equal to  $X$ . Obviously,  $b(\mathcal{H}) \geq m/r$ .

**Theorem 3.1** [Frankl and Rödl 1985]. *Let  $\beta, \varepsilon$  be positive constants,  $r \geq 2$  fixed. There exists  $\delta = \delta(r, \beta, \varepsilon)$  such that, for every  $\mathcal{H} \subset \binom{X}{r}$  satisfying*

- (i)  $|d(x) - |\mathcal{H}|r/m| < \delta|\mathcal{H}|/m$  **or**
- (ii)  $d(x, y) < |\mathcal{H}|r/m^{1+\beta}$ ,

*one has  $b(\mathcal{H}) < (1 + \varepsilon)m/r$ .*

Now we are ready to explain the construction of a nearly optimal dominating set for  $\mathcal{B}_n(k, 2)$ ,  $k \geq 3$ . (Badakhshian et al. [2019] use the same construction for the case  $k = 3$ .)

Let  $n = p(k-1) + q$ ,  $0 \leq q < k-1$  and let  $[n] = X_1 \sqcup \dots \sqcup X_{k-1}$  be a partition with  $p \leq |X_i| \leq p+1$ . Let  $\mathcal{G} := \bigcup_{1 \leq i < k} \binom{X_i}{2}$  be the so-called *Turán graph*. By the pigeonhole principle,  $\mathcal{G}$  dominates all  $k$ -sets in  $\mathcal{B}_n(k, 2)$ .

Set  $r = \binom{k}{2} - 1$ . We define an  $r$ -uniform hypergraph  $\mathcal{H}$  on the partite set  $\binom{[n]}{2}$  from  $\mathcal{B}_n(k, 2)$ . Note that for every  $k$ -set  $F \subset [n]$  satisfying  $F \cap X_i \neq \emptyset$  for  $1 \leq i < k$  there is exactly one  $j = j(F)$  such



that  $|F \cap X_j| = 2$ . With such an  $F$  we associate the  $r$ -set  $H(F) = \binom{F}{2} \setminus \{F \cap X_j\}$ . Let  $\mathcal{H}$  be the  $r$ -graph formed by these  $H(F)$ . The actual vertex set of  $\mathcal{H}$  is

$$X = \binom{[n]}{2} \setminus \left( \binom{X_1}{2} \cup \dots \cup \binom{X_{k-1}}{2} \right);$$

that is, the number of vertices is  $m \sim \frac{k-2}{k-1} \binom{n}{2}$ .

If  $|X_1| = \dots = |X_{k-1}|$ , then  $\mathcal{H}$  is regular but even in the general case it is nearly regular. That is, (i) holds for  $m > m(\delta)$ .

Since  $|\mathcal{H}| = (k-1+o(1))p^k/2$  and  $|\mathcal{H}(x, y)| < p^{k-3}$ , (ii) is satisfied with e.g.  $\beta = \frac{1}{3}$  if  $m > m_0(k, \beta)$ .

Applying [Theorem 3.1](#) we obtain a covering of  $X$  which is, say, formed by the edges  $H(F_1), \dots, H(F_b)$ ,  $b < (1+\varepsilon)m/r$ .

Let  $\mathcal{F} = \{F_1, \dots, F_b\}$  be the corresponding family in  $\binom{[n]}{k}$ . Then  $\mathcal{G} \cup \mathcal{F}$  is a dominating set for  $\mathcal{B}_n(k, 2)$ . Substituting  $m = (1+o(1))\frac{k-2}{k-1} \binom{n}{2}$ ,  $r = \binom{k}{2} - 1$ , we infer

$$|\mathcal{G} \cup \mathcal{F}| \leq \binom{n}{2} \left( \frac{1}{k-1} + \frac{k-2}{k-1} \cdot \frac{1}{\binom{k}{2} - 1} + \varepsilon \right) = \binom{n}{2} \left( \frac{k+3}{k^2-1} + \varepsilon \right).$$

Since  $\varepsilon > 0$  was arbitrary, this concludes the proof of the upper bound in [Theorem 1.1](#).  $\square$

#### 4. The general problem

Let us say a few words about  $\varrho(\mathcal{B}_n(k, \ell))$  in the case  $\ell \geq 3$ . One would imagine that to find a small dominating set imitating the strategy used for  $\ell = 2$  should be the best. However, that means that first we choose  $\mathcal{G} \subset \binom{[n]}{\ell}$  covering the whole of  $\binom{[n]}{k}$ , that is, for every  $F \in \binom{[n]}{k}$  there exists  $G \in \mathcal{G}$  with  $G \subset F$ .

The problem is that we do not know the minimal size,  $|\mathcal{G}|$  for such families. It is the famous Turán's Problem (cf. [\[Turán 1961\]](#)) which is still open for all pairs  $(k, \ell)$ ,  $k > \ell \geq 3$ .

At the same time there are some plausible conjectures. For example Turán [\[Turán 1961\]](#) conjectured that in the case  $k = 5$ ,  $\ell = 3$  and  $n > n_0(k, \ell)$  the best construction is  $\mathcal{G} = \binom{X}{3} \cup \binom{Y}{3}$  where  $X \cup Y = [n]$  is a partition and  $|X| = \lfloor \frac{n}{2} \rfloor$ . Using this  $\mathcal{G}$  one can use the approach of [Section 3](#) and show that

$$\varrho(\mathcal{B}_n(5, 3)) \leq (1+o(1)) \left( \frac{1}{4} + \frac{3}{4} \frac{1}{\binom{5}{3} - 1} \right) \binom{n}{3} = \left( \frac{1}{3} + o(1) \right) \binom{n}{3}. \quad (9)$$

Using the results of [\[Frankl and Rödl 2002\]](#) one can prove the matching lower bound assuming that Turán's conjecture is true.

The situation is pretty much the same for other pairs  $(k, \ell)$  whenever the conjectured optimal family for Turán's Problem is a "highly regular"  $\ell$ -graph.

Let us close this paper with a conjecture.

**Conjecture 4.1.**  $\varrho(\mathcal{B}_n(2\ell-1, \ell)) = (1+o(1)) \left( \frac{1}{2^{\ell-1}} + \left( 1 - \frac{1}{2^{\ell-1}} \right) / \left( \binom{2\ell-1}{\ell} - 1 \right) \right) \binom{n}{3}.$

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