Moscow Journal of Combinatorics and Number Theory

2019
vol. 8 no. 1

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msp

Moscow Journal of Combinatorics and Number Theory
Vol. 8, No. 1, 2019
dx.doi.org/10.2140/moscow.2019.8.47

Admissible endpoints of gaps in the Lagrange spectrum

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For any irrational number α define the Lagrange constant $\mu(\alpha)$ by

$$\mu^{-1}(\alpha) = \liminf_{p\in\mathbb{Z},\, q\in\mathbb{N}} |q(q\alpha - p)|.$$

The set of all values taken by $\mu(\alpha)$ as α varies is called the *Lagrange spectrum* $\mathbb{L}$. An irrational α is called attainable if the inequality

$$\left|\alpha - \frac{p}{q}\right| \leqslant \frac{1}{\mu(\alpha)q^2}$$

holds for infinitely many integers p and q. We call a real number $\lambda \in \mathbb{L}$ *admissible* if there exists an irrational attainable α such that $\mu(\alpha) = \lambda$. In a previous paper we constructed an example of a nonadmissible element in the Lagrange spectrum. In the present paper we give a necessary and sufficient condition for admissibility of a Lagrange spectrum element. We also give an example of an infinite sequence of left endpoints of gaps in $\mathbb{L}$ which are not admissible.

1. Introduction

The Lagrange spectrum $\mathbb{L}$ is usually defined as the set of all values of the Lagrange constants

$$\mu(\alpha) = \Bigl(\liminf_{p\in\mathbb{Z},\, q\in\mathbb{N}} |q(q\alpha - p)|\Bigr)^{-1}$$

as α runs through the set of irrational numbers. Consider the continued fraction expansion of α

$$\alpha = [a_0; a_1, a_2, \ldots, a_n, \ldots].$$

For any positive integer i define

$$\lambda_i(\alpha) = [a_i; a_{i+1}, a_{i+2}, \ldots] + [0; a_{i-1}, a_{i-2}, \ldots, a_1].$$

It is well-known fact that

$$\limsup_{i\to\infty} \lambda_i(\alpha) = \mu(\alpha). \tag{1}$$

The equation (1) provides an equivalent definition of the Lagrange constant $\mu(\alpha)$.

The following properties of $\mathbb{L}$ are well known. The Lagrange spectrum is a closed set [Cusick 1975] with minimal point $\sqrt{5}$. All the numbers of $\mathbb{L}$ which are less than 3 form a discrete set. The Lagrange spectrum contains all elements greater than $\sqrt{21}$; see [Freiman 1973; Schecker 1977]. The complement

Research supported by RNF grant No. 14-11-00433. The author is a Young Russian Mathematics award winner and would like to thank its sponsors and jury.
MSC2010: 11J06.
Keywords: Lagrange spectrum, Diophantine approximation, continued fractions.

of $\mathbb{L}$ is a countable union of *maximal gaps* of the spectrum. The maximal gaps are open intervals (a, b) such that $(a, b) \cap \mathbb{L} = \varnothing$, but a and b both lie in the Lagrange spectrum. There are infinitely many gaps in the nondiscrete part of the Lagrange spectrum [Gbur 1976].

Let α be an arbitrary irrational number. If the inequality

$$\left|\alpha - \frac{p}{q}\right| \leqslant \frac{1}{\mu(\alpha) q^2}$$

has infinitely many solutions for integer p and q, we call α *attainable*. This definition was first given in [Malyshev 1977]. One can easily see [Gayfulin 2017] that α is attainable if and only if $\lambda_i(\alpha) \geqslant \mu(\alpha)$ for infinitely many indices i. We also call a real number $\lambda \in \mathbb{L}$ *admissible* if there exists an irrational attainable number α such that $\mu(\alpha) = \lambda$.

Let B denote a doubly infinite sequence of positive integers

$$B = (\dots, b_{-n}, \dots, b_{-1}, b_0, b_1, \dots, b_n, \dots).$$

For an arbitrary integer i define

$$\lambda_i(B) = [b_i; b_{i-1}, \dots] + [0; b_{i+1}, b_{i+2}, \dots].$$

We will call a doubly infinite sequence B purely periodic if there exists a finite sequence P such that $B = (\overline{P})$. A doubly infinite sequence B is called eventually periodic if there exist three finite sequences P_l, R, P_r such that $B = (\overline{P}_l, R, \overline{P}_r)$. One can also consider an equivalent definition of the Lagrange spectrum using the doubly infinite sequences. We use the notation from [Cusick and Flahive 1989]:

$$L(B) = \limsup_{i\to\infty} \lambda_i(B), \qquad M(B) = \sup \lambda_i(B).$$

The Lagrange spectrum $\mathbb{L}$ is exactly the set of values taken by $L(B)$ as B runs through the set of doubly infinite sequences of positive integers. The set of values taken by $M(B)$ is called the Markoff spectrum. We will denote this set by $\mathbb{M}$.

We will call a doubly infinite sequence B *weakly associated* with an irrational number $\alpha = [a_0; a_1, \dots, a_n, \dots]$ if the following condition holds:

(1) For any natural i the pattern $(b_{-i}, b_{-i+1}, \dots, b_0, \dots, b_i)$ occurs in the sequence $a_1, a_2, \dots, a_n, \dots$ infinitely many times.

We will call B *strongly associated* with α if, additionally,

(2) $\mu(\alpha) = \lambda_0(B) = M(B)$.

One can easily see that if B is weakly associated with α then $\mu(\alpha) \geqslant M(B)$. As we will show in Lemma 4.1, if α has bounded partial quotients, it has at least one strongly associated sequence.

2. Results of [Gayfulin 2017]

Theorem I. *The quadratic irrationality* $\lambda_0 = [3; 3, 3, 2, 1, \overline{1, 2}] + [0; 2, 1, \overline{1, 2}]$ *belongs to* $\mathbb{L}$, *but if* α *is such that* $\mu(\alpha) = \lambda_0$ *then* α *is not attainable.*

Theorem II. *If* $\lambda \in \mathbb{L}$ *is not a left endpoint of some maximal gap in the Lagrange spectrum then there exists an attainable* α *such that* $\mu(\alpha) = \lambda$.

One can easily formulate these theorems using the concept of admissible numbers, introduced above.

Theorem I′. *The quadratic irrationality* $\lambda_0 = [3; 3, 3, 2, 1, \overline{1, 2}] + [0; 2, 1, \overline{1, 2}]$ *belongs to* $\mathbb{L}$, *but is not admissible.*

Theorem II′. *If* $\lambda \in \mathbb{L}$ *is not a left endpoint of some maximal gap in the Lagrange spectrum then* λ *is an admissible number.*

3. Main results

Our first theorem is a small generalization of Theorem 3 in [Gayfulin 2017]. The proof will be quite similar and use some lemmas from that paper.

Theorem 1. *Let* a *be a left endpoint of a gap* (a, b) *in the Lagrange spectrum and* α *be an irrational number such that* $\mu(\alpha) = a$. *Consider a doubly infinite sequence* B *strongly associated with* α. *Then* B *is an eventually periodic sequence.*

It follows from Theorems I and II that there exist nonadmissible elements in the Lagrange spectrum but all such numbers are left endpoints of some maximal gaps in $\mathbb{L}$. The following theorem gives a necessary and sufficient condition of admissibility of a Lagrange spectrum element.

Theorem 2. *A left endpoint of a gap in the Lagrange spectrum* a *is admissible if and only if there exists a quadratic irrationality* α *such that* $\mu(\alpha) = a$.

Of course, every quadratic irrationality is strongly associated with the unique sequence, which is purely periodic. Therefore Theorem 2 is equivalent to the following statement.

Corollary 3.1. *A left endpoint of a gap in the Lagrange spectrum* a *is not admissible if and only if there does not exist a purely periodic sequence* B *such that* $\lambda_0(B) = M(B) = a$.

Theorem 2 provides an instrument to verify nonadmissible points in $\mathbb{L}$. Define

$$\alpha_n^* = 2 + [0; \underbrace{1, \dots, 1}_{2n-2}, \overline{2, 2, 1, 2}] + [0; \underbrace{1, \dots, 1}_{2n-1}, 2, \underbrace{1, \dots, 1}_{2n-2}, \overline{2, 2, 1, 2}],$$

$$\beta_n = 2 + 2[0; \overline{\underbrace{1, \dots, 1}_{2n}, 2}].$$

The fact that (α_n^*, β_n) is the maximal gap in the Markoff spectrum was proved in [Gbur 1976]. It is easy to show that α_n^* and β_n belong to $\mathbb{L}$; we will do this in Section 6. Hence, as $\mathbb{L} \subset \mathbb{M}$ [Cusick 1975], the interval (α_n^*, β_n) is the maximal gap in $\mathbb{L}$ too.

Theorem 3. *For any integer* $n \geqslant 2$ *the irrational number* α_n^* *is not admissible.*

One can easily see that $\alpha_1^* = 2 + [0; \overline{2, 2, 1, 2}] + [0; 1, 2, \overline{2, 2, 1, 2}] = \mu([0; \overline{2, 2, 1, 2}]) = M(\overline{2, 2, 1, 2})$. Thus, α_1^* is an admissible number by Theorem 2.

4. Proof of Theorem 1

The following statement is well known. See the proof in [Cusick and Flahive 1989, Chapter 1, Lemma 6].

Lemma 4.1. *Let $A = \dots, a_{-1}, a_0, a_1, \dots$ be any doubly infinite sequence. If $M(A)$ is finite, then there exists a doubly infinite sequence B such that $M(A) = M(B) = \lambda_0(B)$.*

Using the same argument for the sequence $A = (a_1, a_2, \dots, a_n, \dots)$, one can easily show:

Lemma 4.2. *Let $\alpha = [0; a_1, \dots, a_n, \dots]$ be an arbitrary irrational number and $a_i < c$ for all $i \in \mathbb{N}$, for some positive real number c. Then there exists a doubly infinite sequence B which is strongly associated with α.*

As $\alpha \leqslant \sqrt{21}$, all elements of B are bounded by 4. For any natural n define $\varepsilon_n = 2^{-(n-1)}$, $\delta_n = 5^{-2(n+2)}$. We need the following lemmas from [Gayfulin 2017].

Lemma 4.3. *Suppose $\alpha = [a_0; a_1, \dots, a_n, b_1, \dots]$ and $\beta = [a_0; a_1, \dots, a_n, c_1, \dots]$, where $n \geqslant 0$, a_0 is an integer, and $a_1, \dots, a_n, b_1, b_2, \dots, c_1, c_2, \dots$ are positive integers bounded by 4 with $b_1 \neq c_1$. Then for n odd, $\alpha > \beta$ if and only if $b_1 > c_1$; for n even, $\alpha > \beta$ if and only if $b_1 < c_1$. Also,*

$$\delta_n < |\alpha - \beta| < \varepsilon_n.$$

Lemma 4.4. *Let $\gamma = [0; c_1, c_2, \dots, c_N, \dots]$ and $\gamma' = [0; c'_1, c'_2, \dots, c'_N, \dots]$ be two irrational numbers with partial quotients not exceeding 4. Suppose that every sequence of partial quotients of length $2n+1$ which occurs in the sequence $(c'_1, c'_2, \dots, c'_N, \dots)$ infinitely many times also occurs in the sequence $(c_1, c_2, \dots, c_N, \dots)$ infinitely many times. Then $\mu(\gamma') < \mu(\gamma) + 2\varepsilon_n$.*

The following technical lemma was formulated in [Gayfulin 2017] for $N = (2n+1)(4^{2n+1}+1)$ and the proof was incorrect. However, this is not crucial for the results of that paper, as we just need N to be bounded from above by some growing function of n. In this paper, we give a new version of the lemma with correct proof.

Lemma 4.5. *Let n be an arbitrary positive integer. Define $N = N(n) = (2n+2)(4^{2n+2}+1)$. If $b_1, b_2, \dots, b_N$ is an arbitrary integer sequence of length N such that $1 \leqslant b_i \leqslant 4$ for all $1 \leqslant i \leqslant N$, then there exist two integers n_1, n_2 such that $b_{n_1+i} = b_{n_2+i}$ for all $0 \leqslant i \leqslant 2n+1$ and $n_1 \equiv n_2 \pmod 2$.*

Proof. There exist only 4^{2n+2} distinct sequences of length $2n+2$ with elements $1, 2, 3, 4$. Consider $4^{2n+2}+1$ sequences: $(b_1, \dots, b_{2n+2})$, $(b_{2n+3}, \dots, b_{4n+4})$, $\dots$, $(b_{(2n+2)4^{2n+2}+1}, \dots, b_{(2n+2)4^{2n+2}+2n+2})$. Dirichlet's principle implies that there exist two coinciding sequences among them. Denote these sequences by $(b_{n_1}, \dots, b_{n_1+2n+1})$ and $(b_{n_2}, \dots, b_{n_2+2n+1})$. Note that the index of the first element of each sequence is odd; hence $n_1 \equiv n_2 \equiv 1 \pmod 2$, which finishes the proof. $\square$

If $n_1 \equiv n_2 \pmod 2$ then the sequence $(b_{n_1}, b_{n_1+1}, \dots, b_{n_2-1})$ has even length. This fact will be useful in our argument.

Lemma 4.6. *Let B be an arbitrary integer sequence of even length. Let A be an arbitrary finite integer sequence and C an arbitrary nonperiodic infinite sequence. Then*

$$\min([0; A, B, B, C], [0; A, C]) < [0; A, B, C] < \max([0; A, B, B, C], [0; A, C]). \quad (2)$$

Proof. As the sequence C is nonperiodic, the continued fractions in (2) are not equal. Without loss of generality, one can say that the sequence A is empty. Suppose that

$$[0;\, B, C] > [0;\, B, B, C].$$

As the length of B is even, one can see that $[0;\, C] > [0;\, B, C]$, which is exactly the right-hand side of (2). The case when $[0;\, B, C] < [0;\, B, B, C]$ is treated in exactly the same way. □

Lemma 4.7. *Let $\gamma = [0;\, b_1, b_2, \ldots, b_N, \ldots]$ be an arbitrary irrational number, not a quadratic irrationality. Consider the sequence $B_N = (b_1, b_2, \ldots, b_N)$ and define two numbers n_1 and n_2 from Lemma 4.5. Define two new sequences of positive integers*

$$B_N^1 = (b_1, b_2, \ldots, b_{n_1-1}, b_{n_2}, b_{n_2+1}, \ldots, b_N),$$
$$B_N^2 = (b_1, b_2, \ldots, b_{n_1-1}, b_{n_1}, \ldots, b_{n_2-1}, b_{n_1}, \ldots, b_{n_2-1}, b_{n_2}, b_{n_2+1}, \ldots, b_N).$$

Let us also define two new irrational numbers:

$$\gamma^1 = [0;\, b_1, b_2, \ldots, b_{n_1-1}, b_{n_2}, b_{n_2+1}, \ldots, b_N, b_{N+1}, \ldots] = [0;\, B_N^1, b_{N+1}, \ldots],$$
$$\gamma^2 = [0;\, b_1, b_2, \ldots, b_{n_1-1}, b_{n_1}, \ldots, b_{n_2-1}, b_{n_1}, \ldots, b_{n_2-1}, b_{n_2}, b_{n_2+1}, \ldots, b_N, \ldots] = [0;\, B_N^2, b_{N+1}, \ldots].$$

Then $\max(\gamma^1, \gamma^2) > \gamma$.

Proof. We apply Lemma 4.6 for $A = (b_1, b_2, \ldots, b_{n_1-1})$, $B = (b_{n_1}, b_{n_1+1}, \ldots, b_{n_2-1})$, $C = (b_{n_2}, b_{n_2+1}, \ldots)$. Here $\gamma = [0;\, A, B, C]$, $\gamma^1 = [0;\, A, C]$, and $\gamma^2 = [0;\, A, B, B, C]$. Note that as γ is not a quadratic irrationality, the sequence C is not periodic. □

Now we are ready to prove Theorem 1.

Proof. Suppose that B is not periodic on the right side. Consider an increasing sequence of indices $k(j)$ such that for any natural j the sequence $(a_{k(j)-j}, \ldots, a_{k(j)}, \ldots, a_{k(j)+j})$ coincides with the sequence $(b_{-j}, \ldots, b_0, \ldots, b_j)$. Of course,

$$\lim_{j\to\infty} \lambda_{k(j)}(\alpha) = \lambda_0(B) = \mu(\alpha).$$

Without loss of generality, one can say that $k(j+1) - k(j) \to \infty$ as $j \to \infty$. Consider an even n such that $\varepsilon_n < \frac{1}{2}(b-a)$ and $N = N(n)$ as defined in Lemma 4.5. Define $n_1 < n_2$ from Lemma 4.5 for the sequence $(b_1, \ldots, b_N)$. As B is not periodic to the right, define a minimal positive integer r such that $b_{n_1+r} \neq b_{n_2+r}$. Consider the sequences B_N^1, B_N^2 and the continued fractions γ_1, γ_2 from Lemma 4.7 applied to the continued fraction $[0;\, b_1, \ldots, b_n \ldots] = \gamma$. If $\gamma_2 > \gamma$, define $g = 2$; otherwise we put $g = 1$. Consider the doubly infinite sequence $B' = (\ldots, b_{-n}, b_0, B_N^g, b_{N+1}, \ldots)$. Note that

$$a = \lambda_0(B) < \lambda_0(B') < a + \varepsilon_n < b.$$

Consider the corresponding continued fraction α' which is obtained from the continued fraction α by replacing every segment $(a_{k(j)}, \ldots, a_{k(j)+N}) = (a_{k(j)}, B_N)$ by the segment $(a_{k(j)}, B_N^g)$ for every $j \geqslant n_2 + r$. One can easily see that α' and α satisfy the condition of Lemma 4.4 and hence $\mu(\alpha') < \mu(\alpha) + 2\varepsilon_n$. But as $\mu(\alpha) + 2\varepsilon_n < b$ and (a, b) is the gap in $\mathbb{L}$, we have

$$\mu(\alpha') \leqslant \mu(\alpha) = a. \tag{3}$$

On the other hand, one can easily see that the sequence B' is weakly associated with α'. This means that

$$\mu(\alpha') \geqslant M(B) \geqslant \lambda_0(B') > \lambda_0(B) = a.$$

We obtain a contradiction with (3). The case when B is not periodic on the left side is considered in exactly the same way. □

5. Proof of Theorem 2

The following lemma from [Gayfulin 2017] immediately implies the "$\Leftarrow$" part of the statement of Theorem 2.

Lemma 5.1. *Consider an arbitrary point a in the Lagrange spectrum. If there exists a quadratic irrationality γ such that $\mu(\gamma) = a$, then a is admissible.*

Now it is sufficient to prove that if a is an admissible left endpoint of a gap in the Lagrange spectrum, then there exists a quadratic irrationality α such that $\mu(\alpha) = a$.

Proof. Let a be an admissible left endpoint of some gap in the Lagrange spectrum. Let $\alpha = [a_0; a_1, \dots, a_n, \dots]$ be an irrational number such that $\mu(\alpha) = a$. Suppose that α is attainable, but not a quadratic irrationality. Let $k(j)$ be a growing sequence of indices such that

$$\lambda_{k(j)}(\alpha) \geqslant \mu(\alpha). \tag{4}$$

Of course,

$$\lim_{j\to\infty} \lambda_{k(j)}(\alpha) = \mu(\alpha).$$

Consider a sequence $B = (\dots, b_{-n}, \dots, b_{-1}, b_0, b_1, \dots, b_n, \dots)$ strongly associated with α having the following property: the sequence $(b_{-i}, \dots, b_0, \dots, b_i)$ coincides with the sequence $(a_{k(j)-i}, \dots, a_{k(j)}, \dots, a_{k(j)+i})$ for infinitely many j's. Theorem 1 implies that B is eventually periodic. That is, there exist a positive integer m and two finite sequences L and R such that

$$B = (\bar{L}, b_{-m}, \dots, b_0, \dots, b_m, \bar{R}).$$

It follows from (4) that one of the inequalities

$$[a_{k(j)}; a_{k(j+1)}, \dots] \geqslant [b_0; b_1, \dots, b_m, \bar{R}],$$
$$[0; a_{k(j-1)}, \dots, a_1] \geqslant [0; b_{-1}, \dots, b_{-m}, \bar{L}]$$

holds for infinitely many j's. Note that $[a_{k(j)}; a_{k(j+1)}, \dots] \neq [b_0; b_1, \dots, b_m, \bar{R}]$, as α is not a quadratic irrationality and, of course, $[0; a_{k(j-1)}, \dots, a_1] \neq [0; b_{-1}, \dots, b_{-m}, \bar{L}]$. Suppose that

$$[a_{k(j)}; a_{k(j+1)}, \dots] > [b_0; b_1, \dots, b_m, \bar{R}] \tag{5}$$

for infinitely many j's. Denote by p the length of period R. Denote by $r(j)$ the minimal positive number such that $a_{k(j)+r(j)} \neq b_{r(j)}$. Without loss of generality, one can say that:

(1) $k(j+1) - k(j) - r(j) \to \infty$ as $j \to \infty$.

(2) $[a_{k(j)}; a_{k(j+1)}, \dots] > [b_0; b_1, \dots, b_m, \bar{R}]$ for every $j \in \mathbb{N}$.

(3) $[a_{k(j)}; a_{k(j+1)}, \dots, a_{k(j)+m}] = [b_0; b_1, \dots, b_m]$ for every $j \in \mathbb{N}$.

(4) The sequence $(a_{k(j)-j},\dots,a_{k(j)},\dots,a_{k(j)+j})$ coincides with the sequence $(b_{-j},\dots,b_0,\dots,b_j)$ for every $j\in\mathbb{N}$.

(5) Period length p is even.

Denote by $t(j)$ the number of periods P in the sequence $(b_{m+1},\dots,b_{r(j)})$. Of course,

$$t(j)=\left[\frac{r(j)-m}{p}\right]$$

and $t(j)$ tends to infinity as $j\to\infty$. Lemma 4.3 implies that since (5) holds, we have

$$[a_{k(j)};a_{k(j+1)},\dots,a_{k(j)+m},\underbrace{R,\dots,R}_{t(j)\text{ times}},\dots,a_{k(j)+r(j)},\dots]>[b_0;b_1,\dots,b_m,\bar{R}].$$

Denote by α_n a continued fraction obtained from the continued fraction $\alpha=[a_0;a_1,\dots,a_n,\dots]$ as follows: for any $j\in\mathbb{N}$ if $t(j)>n$, then every pattern

$$a_{k(j)},a_{k(j+1)},\dots,a_{k(j)+m},\underbrace{R,\dots,R}_{t(j)\text{ times}},\dots,a_{k(j)+r(j)}$$

is replaced by the pattern

$$a_{k(j)},a_{k(j+1)},\dots,a_{k(j)+m},\underbrace{R,\dots,R}_{n\text{ times}},\dots,a_{k(j)+r(j)}.$$

As the length of the period R is even, by Lemma 4.3 one has

$$[a_{k(j)};a_{k(j+1)},\dots,a_{k(j)+m},\underbrace{R,\dots,R}_{n\text{ times}},\dots,a_{k(j)+r(j)},\dots]-[b_0;b_1,\dots,b_m,\bar{R}]>\delta_{m+(n+1)p}. \tag{6}$$

On the other hand, as the sequence $(a_{k(j)-j},\dots,a_{k(j)})$ coincides with the sequence $(b_{-j},\dots,b_0)$ for all $j\in\mathbb{N}$, by Lemma 4.3 one has

$$\big|[0;a_{k(j)-1},\dots,a_{k(j)-j},\dots,a_1]-[0;b_{-1},\dots,b_{-m},\bar{L}]\big|<\varepsilon_j. \tag{7}$$

For any positive integers n,m,p there exists J such that for all $j>J$ one has $\varepsilon_j<\frac{1}{2}\delta_{m+(n+1)p}$. Now, from (6) and (7) we have for $j>J$

$$\begin{aligned}\big([0;a_{k(j)-1},\dots,a_{k(j)-j},\dots,a_1]&\\ +[a_{k(j)};a_{k(j+1)},\dots,a_{k(j)+m},\underbrace{R,\dots,R}_{n\text{ times}},\dots,a_{k(j)+r(j)},\dots]\big)&\\ -\big([0;b_{-1},\dots,b_{-m},\bar{L}]+[b_0;b_1,\dots,b_m,\bar{R}]\big)&>\tfrac{1}{2}\delta_{m+(n+1)p}.\end{aligned} \tag{8}$$

Considering the limit in (8) as $j\to\infty$, we can easily see that $\mu(\alpha_n)\geqslant\mu(\alpha)+\frac{1}{2}\delta_{m+(n+1)p}$.

Note that

$$\lim_{n\to\infty}\mu(\alpha_n)=\mu(\alpha)=a. \tag{9}$$

Indeed, every pattern of length np which occurs in the sequence of partial quotients of α infinitely many times occurs in the sequence of partial quotients of α_n infinitely many times. Similarly, every pattern of length np which occurs in the sequence of partial quotients of α_n infinitely many times, occurs in the sequence of partial quotients of α infinitely many times too. Then, by Lemma 4.4,

$$|\mu(\alpha) - \mu(\alpha_n)| < 2\varepsilon_{np} = 2^{-np+2} \to 0$$

as $n \to \infty$. We obtain a contradiction with the fact that a is the left endpoint of the gap (a, b) in the Lagrange spectrum. Indeed, (8) implies that $\mu(\alpha_n) > \mu(\alpha)$ for all $n \in \mathbb{N}$. In addition, (9) implies that there exists a positive integer N such that for any $n > N$ one has $a = \mu(\alpha) < \mu(\alpha_n) < b$.

If (5) does not hold infinitely many times, then the inequality

$$[0; a_{k(j-1)}, \ldots, a_1] > [0; b_{-1}, \ldots, b_{-m}, \bar{L}]$$

holds infinitely many times. As α is not a quadratic irrationality, for any positive integer s there exists an integer $N(s) > s$ such that for all $n \geqslant N(s)$ the continued fraction $[0; a_n, a_{n-1}, \ldots, a_s]$ is not convergent to the continued fraction $[0; b_{-1}, \ldots, b_{-m}, \bar{L}]$. Without loss of generality, one can say that $k(1) > N(1)$, $k(j+1) > N(2k(j))$. Denote by $r(j)$ the minimal positive number such that $a_{k(j)-r(j)} \neq b_{-r(j)}$. It is easy to see that the number $r(j)$ is well-defined and

$$r(j+1) \leqslant k(j+1) - N(2k(j)) < k(j+1) - 2k(j).$$

Therefore $k(j+1) - r(j+1) - k(j) \to \infty$ as $j \to \infty$. Now one can easily complete the proof using exactly the same argument as we used in the first case. $\square$

6. Proof of Theorem 3

First of all, let us show that (α_n^*, β_n) is the maximal gap in $\mathbb{L}$. As

$$\beta_n = 2 + 2[0; \overline{\underbrace{1, \ldots, 1}_{2n}, 2}] = \mu([0; \overline{\underbrace{1, \ldots, 1}_{2n}, 2}]),$$

we have $\beta_n \in \mathbb{L}$. The proof of the fact that $\alpha_n^* \in \mathbb{L}$ when $n \geqslant 2$ is a little more complicated. Recall that

$$\alpha_n^* = 2 + [0; \underbrace{1, \ldots, 1}_{2n-2}, \overline{2, 2, 1, 2}] + [0; \underbrace{1, \ldots, 1}_{2n-1}, 2, \underbrace{1, \ldots, 1}_{2n-2}, \overline{2, 2, 1, 2}].$$

Denote by $C_n(k)$ the finite sequence of integers

$$C_n(k) = (\underbrace{2, 1, 2, 2}_{k}, \underbrace{1, \ldots, 1}_{2n-2}, 2, \underbrace{1, \ldots, 1}_{2n-1}, 2, \underbrace{1, \ldots, 1}_{2n-2}, \underbrace{2, 2, 1, 2}_{k}).$$

A little calculation shows that

$$L(\overline{C_n(k)}) = 2 + [0; \underbrace{1, \ldots, 1}_{2n-2}, \underbrace{2, 2, 1, 2}_{k}, \ldots] + [0; \underbrace{1, \ldots, 1}_{2n-1}, 2, \underbrace{1, \ldots, 1}_{2n-2}, \underbrace{2, 2, 1, 2}_{k}, \ldots].$$

Therefore $\lim_{k\to\infty} L(\overline{C_n(k)}) = \alpha_n^*$. As $\mathbb{L}$ is closed set, we obtain that $\alpha_n^* \in \mathbb{L}$.

By [Gbur 1976, Lemma 4], α_n^* is a growing sequence. One can easily see that

$$\lim_{n\to\infty} \alpha_n^* = 2 + 2[0; \bar{1}] = \sqrt{5} + 1 \approx 3.236.$$

Thus, we have

$$\alpha_2^* \leqslant \alpha_n^* < 1+\sqrt{5}, \quad \text{where } n \geqslant 2.$$

The following lemma is a compilation of Lemmas 3 and 4 from [Gbur 1976].

Lemma 6.1. *Consider a doubly infinite sequence $B = (\dots, b_{-n}, \dots, b_{-1}, b_0, b_1, \dots, b_n, \dots)$ such that $M(B) < \sqrt{5}+1$. Then all elements of B are bounded by 2 and B does not contain patterns of the form* $(2, 1, 2, 1)$ *and* $(1, 2, 1, 2)$.

By Lemma 4.1, without loss of generality one can say that $M(B) = \lambda_0(B)$. Denote the continued fractions $[0; b_1, \dots, b_n, \dots]$ and $[0; b_{-1}, \dots, b_{-n}, \dots]$ by x and y respectively. Then

$$M(B) = b_0 + x + y.$$

Without loss of generality one can say that $x \leqslant y$. Now we need the following lemma from [Gbur 1976, Theorem 4(i)].

Lemma 6.2. *Let B be a doubly infinite sequence such that $M(B) = \lambda_0(B)$. Then for all $n \geqslant 1$ we have*

$$\beta_n \leqslant M(B) = 2+x+y \leqslant \alpha_{n+1}^* \iff x = [0; \underbrace{1, \dots, 1}_{2n}, 2, \dots] \text{ and } y = [0; \underbrace{1, \dots, 1}_{2n}, \dots].$$

It also follows from [Gbur 1976, Theorem 4(ii)] that

$$2 + [0; \underbrace{1, \dots, 1}_{2n+1}, \dots] + [0; \underbrace{1, \dots, 1}_{2n+1}, 2, \dots] < \sqrt{5}+1.$$

Define

$$w_0 = [0; \overline{2, 1, 2, 2}],$$

$$x_0 = [0; \underbrace{1, \dots, 1}_{2n}, \overline{2, 2, 1, 2}] = [0; \underbrace{1, \dots, 1}_{2n}, 2 + w_0],$$

$$y_0 = [0; \underbrace{1, \dots, 1}_{2n+1}, 2, \underbrace{1, \dots, 1}_{2n}, \overline{2, 2, 1, 2}] = [0; \underbrace{1, \dots, 1}_{2n+1}, 2, \underbrace{1, \dots, 1}_{2n}, 2 + w_0].$$

Lemma 6.3. *Let $w = [0; a_1, a_2, \dots, a_n, \dots]$ be a continued fraction with elements equal to 1 or 2. Suppose that the sequence $(a_1, a_2, \dots, a_n, \dots)$ does not contain the pattern $(2, 1, 2, 1)$. Then $w \geqslant w_0$.*

Proof. Denote the elements of the continued fraction $w_0 = [0; \overline{2, 1, 2, 2}]$ by $[0; a'_1, \dots, a'_m, \dots]$. Denote by r the minimal index such that $a_r \neq a'_r$. Suppose that $w < w_0$. Then either r is odd, $a_r = 2$, $a'_r = 1$ or r is even, $a_r = 1$, $a'_r = 2$. However $a'_r = 2$ for any odd r; thus the first case leads to a contradiction. Consider the second case. Of course, $r \geqslant 4$. Then $a'_{r-3} = a_{r-3} = 2$, $a'_{r-2} = a_{r-2} = 1$, $a'_{r-1} = a_{r-1} = 2$. This means that $(a_{r-3}, a_{r-2}, a_{r-1}, a_r) = (2, 1, 2, 1)$ and we obtain a contradiction. □

Now we prove Theorem 3.

Proof. Suppose that α_n^* is admissible for some $n \geqslant 2$. Consider an attainable number α such that $\mu(\alpha) = \alpha_n^*$. Let $B = (\dots, b_{-n}, \dots, b_{-1}, b_0, b_1, \dots, b_n, \dots)$ be any sequence strongly associated with α. Denote by f the increasing function

$$f(t) = [0; \underbrace{1, \dots, 1}_{2n+1}, 2 + t].$$

By Lemma 6.2, there exist $0 < v, w < 1$ such that

$$x = [0; \underbrace{1, \dots, 1}_{2n}, 2+v] \quad \text{and} \quad y = [0; \underbrace{1, \dots, 1}_{2n-1}, 1+w].$$

Note that $x \leqslant x_0$. Indeed,

$$x = [0; \underbrace{1, \dots, 1}_{2n}, 2+v] \leqslant [0; \underbrace{1, \dots, 1}_{2n}, 2+w_0] = x_0 \quad \Longleftrightarrow \quad v \geqslant w_0.$$

The last equality follows from Lemmas 6.3 and 6.1. Therefore

$$y = [0; \underbrace{1, \dots, 1}_{2n-1}, 1+w] \geqslant y_0 = f(x_0).$$

In particular, $b_{-2n-1} = 1$, $b_{-2n-2} = 2$ and there exists $0 < v \leqslant x_0$ such that $y = f(v) \geqslant f(x_0)$. Hence $v \geqslant x_0$. On the other hand,

$$2 + v + f(x) = \lambda_{-2n-2}(B) \leqslant M(B) = \lambda_0(B) = 2 + x + f(v).$$

As $|f(y) - f(z)| < |y - z|$ for any $0 < y, z < 1$, one can easily see that $v \leqslant x$. Thus, $v = x = x_0$ and $y = y_0$. Hence the sequence B satisfies

$$B = (\overline{2, 1, 2, 2}, \underbrace{1, \dots, 1}_{2n}, 2, \underbrace{1, \dots, 1}_{2n+1}, 2, \underbrace{1, \dots, 1}_{2n}, \overline{2, 2, 1, 2})$$

and is not purely periodic. We obtain a contradiction with Corollary 3.1, as we supposed B to be an arbitrary sequence strongly associated with α. ☐

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Received 10 Jan 2018.

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Moscow Journal of Combinatorics and Number Theory

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Moscow Journal of Combinatorics and Number Theory (ISSN 2220-5438 electronic, 2220-5438 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

MJCNT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

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Moscow Journal of Combinatorics and Number Theory

vol. 8 no. 1 2019