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Sets of inhomogeneous linear forms can be not isotropically winning

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We give an example of irrational vector $\boldsymbol{\theta} \in \mathbb{R}^2$ such that the set

$$\mathrm{Bad}_{\boldsymbol{\theta}} := \Big\{(\eta_1, \eta_2) : \inf_{x \in \mathbb{N}} x^{1/2} \max_{i=1,2} \|x\theta_i - \eta_i\| > 0\Big\}$$

is not absolutely winning with respect to McMullen's game.

1. Introduction

We consider a problem related to inhomogeneous Diophantine approximation. Given $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^2$ we study the set of pairs $(\eta_1, \eta_2) \in \mathbb{R}^2$ such that the system of two linear forms

$$\|x\theta_1 - \eta_1\|, \quad \|x\theta_2 - \eta_2\|,$$

where $\|\cdot\|$ stands for the distance to the nearest integer, is badly approximable. We prove a statement complementary to our recent result from [Bengoechea et al. 2017]. We construct $\boldsymbol{\theta}$ such that the set

$$\mathrm{Bad}_{\boldsymbol{\theta}} := \Big\{(\eta_1, \eta_2) : \inf_{x \in \mathbb{N}} x^{1/2} \max_{i=1,2} \|x\theta_i - \eta_i\| > 0\Big\}$$

is not isotropically winning.

Our paper is organized as follows. In Section 2 we discuss different games appearing in Diophantine problems. In Section 3 we give a brief survey on inhomogeneous badly approximable systems of linear forms and formulate our main result, Theorem 3.1. Sections 4 and 5 are devoted to some auxiliary observations. In Sections 6, 7, and 8 we give a proof for Theorem 3.1.

2. Schmidt's game and its generalizations

The following game was introduced by Schmidt [1966; 1969; 1980]. Let $0 < \alpha, \beta < 1$. Suppose that two players A and B choose in turn a nested sequence of closed balls:

$$B_1 \supset A_1 \supset B_2 \supset A_2 \supset \cdots$$

with the property that the diameters $|A_i|$, $|B_i|$ of the balls A_i, B_i satisfy

$$|A_i| = \alpha |B_i|, \quad |B_{i+1}| = \beta |A_i| \qquad \text{for all } i = 1, 2, 3, \ldots,$$

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for fixed $0 < \alpha, \beta < 1$. A set $E \subset \mathbb{R}^n$ is called (α, β)-*winning* if player A has a strategy which guarantees that intersection $\bigcap A_i$ meets E regardless of the way B chooses to play. A set $E \supset \mathbb{R}^n$ is called an α-*winning* set if it is (α, β)-winning for all $0 < \beta < 1$.

There are different modifications of Schmidt's game: the strong game and absolute game introduced in [McMullen 2010], the hyperplane absolute game introduced in [Kleinbock and Weiss 2010], the potential game considered in [Fishman et al. 2013], and some others. In [Bengoechea et al. 2017], we introduced isotropically winning sets. Let us describe here some of these generalizations in more detail.

The definition of an absolutely winning set was given in [McMullen 2010]. Consider the following game. Suppose A and B choose in turn a sequence of balls A_i and B_i such that the sets

$$B_1 \supset (B_1 \backslash A_1) \supset B_2 \supset (B_2 \backslash A_2) \supset B_3 \supset \cdots$$

are nested. For fixed $0 < \beta < \frac{1}{3}$ we suppose

$$|B_{i+1}| \geq \beta |B_i|, \quad |A_i| \leq \beta |B_i|.$$

We say E is an *absolute winning* set if for all $\beta \in \left(0, \frac{1}{3}\right)$, player A has a strategy which guarantees that $\cap B_i$ meets E regardless of how B chooses to play. Mcmullen proved that an absolute winning set is α-winning for all $\alpha < \frac{1}{2}$. Several examples of absolute winning sets were exhibited by McMullen [2010]. In particular, a set of badly approximable numbers in $\mathbb{R}$ is absolutely winning. However the set of simultaneously badly approximable vectors in $\mathbb{R}^n$ for $n > 1$ is not absolutely winning.

In [Bengoechea et al. 2017] another strong variant of the winning property was given. We say that a set $E \subset \mathbb{R}^n$ is *isotropically winning* if for each $d \leq n$ and for each d-dimensional affine subspace $\mathcal{A} \subset \mathbb{R}^n$ the intersection $E \cap \mathcal{A}$ is $\frac{1}{2}$-winning for Schmidt's game considered as a game in $\mathcal{A}$. It is clear that an absolute winning set is isotropically winning for each $\alpha \leq \frac{1}{2}$.

3. Inhomogeneous approximations

The first important result on inhomogeneous approximations in the one-dimensional case is due to Khinchine [1926]. He proved that there exists an absolute constant γ such that for every $\theta \in \mathbb{R}$ there exists $\eta \in \mathbb{R}$ such that

$$\inf_{q \in \mathbb{Z}} q \|q\theta - \eta\| > \gamma.$$

Later (see [Khinchin 1937; 1948]) he proved that for given positive numbers $n, m \in \mathbb{Z}$ there exists a positive constant γ_{nm} such that for any $m \times n$ real matrix $\boldsymbol{\theta}$ there exists a vector $\boldsymbol{\eta} \in \mathbb{R}^n$ such that

$$\inf_{\boldsymbol{x} \in \mathbb{Z}^m \backslash \{0\}} (\|\boldsymbol{\theta x} - \boldsymbol{\eta}\|_{\mathbb{Z}^n})^n \|\boldsymbol{x}\|^m > \gamma_{nm}$$

(here $\|\cdot\|_{\mathbb{Z}^n}$ stands for the distance to the nearest integral point in sup-norm). These results are presented in a wonderful book by Cassels [1957].

Jarník [1941], proved a generalization of this statement. Suppose $\psi(t)$ is a function decreasing to zero as $t \to +\infty$. Let $\rho(t)$ be the function inverse to the function $t \mapsto 1/\psi(t)$. Suppose that for all $t > 1$ one has $\psi_\theta(t) \leq \psi(t)$. Then there exists a vector $\boldsymbol{\eta} \in \mathbb{R}^n$ such that

$$\inf_{\boldsymbol{x} \in \mathbb{Z}^m \backslash \{0\}} (\|\boldsymbol{\theta x} - \boldsymbol{\eta}\|_{\mathbb{Z}^n}) \cdot \rho(8m \cdot \|\boldsymbol{x}\|) > \gamma$$

with appropriate $\gamma = \gamma(n, m)$.

Denote by

$$\mathrm{Bad}_\theta = \bigl\{\alpha \in [0,1) : \inf_{q\in\mathbb{N}} q \cdot \|q\theta - \alpha\| > 0\bigr\}.$$

It happens that the winning property of this inhomogeneous Diophantine set was considered quite recently. Tseng [2009] showed that Bad_θ is winning for all real numbers θ in classical Schmidt's sense. For the corresponding multidimensional sets

$$\mathrm{Bad}(n,m) = \bigl\{\boldsymbol{\theta} \in \mathrm{Mat}_{n\times m}(\mathbb{R}) : \inf_{q\in\mathbb{Z}^m_{\neq 0}} \max_{1\le i\le n} (|q|^{m/n} \|\boldsymbol{\theta}_i(q)\|) > 0\bigr\}.$$

the winning property is shown, for example, in [Einsiedler and Tseng 2011; Moshchevitin 2011]. In [Broderick et al. 2013] it was shown that the set $\mathrm{Bad}(n,m)$ is hyperplane absolutely winning. The methods used in [Broderick et al. 2013] come from [Broderick et al. 2011].

Further generalizations deal with the *twisted* sets

$$\mathrm{Bad}(i,j) = \bigl\{(\theta_1,\theta_2) \in \mathbb{R}^2 : \inf_{q\in\mathbb{N}} \max(q^i\|q\theta_1\|, q^j\|q\theta_2\|) > 0\bigr\},$$

where i, j are real positive numbers satisfying $i + j = 1$, introduced by Schmidt. In [An 2016] it was proved that $\mathrm{Bad}(i,j)$ is winning for the standard Schmidt game. In higher dimension, we fix an n-tuple $\boldsymbol{k} = (k_1, \ldots, k_n)$ of real numbers satisfying

$$k_1, \ldots, k_n > 0 \quad \text{and} \quad \sum_{i=1}^n k_i = 1, \tag{1}$$

and define

$$\mathrm{Bad}(\boldsymbol{k}, n, m) = \bigl\{\boldsymbol{\theta} \in \mathrm{Mat}_{n\times m}(\mathbb{R}) : \inf_{q\in\mathbb{Z}^m_{\neq 0}} \max_{1\le i\le n} (|q|^{mk_i} \|\boldsymbol{\theta}_i(q)\|) > 0\bigr\}.$$

Here, $|\cdot|$ denotes the supremum norm, $\boldsymbol{\theta} = (\boldsymbol{\theta}_{ij})$, and $\boldsymbol{\theta}_i(q)$ is the product of the i-th line of $\boldsymbol{\theta}$ with the vector q, i.e.,

$$\boldsymbol{\theta}_i(q) = \sum_{j=1}^m q_j \boldsymbol{\theta}_{ij}.$$

In the twisted setting, much less is known. In particular up to now the winning property of the set $\mathrm{Bad}(\boldsymbol{k}, n, m)$ in dimension greater that two is not proved.

Given $\boldsymbol{\theta} \in \mathrm{Mat}_{n\times m}(\mathbb{R})$, we define

$$\mathrm{Bad}_{\boldsymbol{\theta}}(\boldsymbol{k}, n, m) = \bigl\{x \in \mathbb{R}^n : \inf_{q\in\mathbb{Z}^m_{\neq 0}} \max_{1\le i\le n} (|q|^{mk_i} \|\boldsymbol{\theta}_i(q) - x_i\|) > 0\bigr\}.$$

Harrap and Moshchevitin [2017] showed that this set is winning provided that $\boldsymbol{\theta} \in \mathrm{Bad}(\boldsymbol{k}, n, m)$. In [Bengoechea et al. 2017] it was proved that if we suppose that $\boldsymbol{\theta} \in \mathrm{Bad}(\boldsymbol{k}, n, m)$, the set $\mathrm{Bad}_{\boldsymbol{\theta}}(\boldsymbol{k}, n, m)$ is isotropically winning.[1]

We should note that even in the case $n = 2$, $m = 1$ it is not known if the set $\mathrm{Bad}_{\boldsymbol{\theta}}(\boldsymbol{k}, 2, 1)$ is α-winning for some positive α without the condition $\boldsymbol{\theta} \in \mathrm{Bad}(\boldsymbol{k}, 2, 1)$.

[1] In fact, the approach from [Bengoechea et al. 2017] gives a little bit more. Instead of property that for any subspace $\mathcal{A}$ the intersection $E \cap \mathcal{A}$ is $\frac{1}{2}$-winning in $\mathcal{A}$, one can see that it is α-winning for all $\alpha \in \bigl(0, \frac{1}{2}\bigr]$. It is not completely clear for the author if these two properties are equivalent. (For a closely related problem, see [Dremov 2002].)

In this article we show that the condition $\boldsymbol{\theta}$ be from $\mathrm{Bad}(\boldsymbol{k}, n, m)$ is essential for the isotropically winning property, and prove the following theorem.

Theorem 3.1. *There exists a vector* $\boldsymbol{\theta} = (\theta_1, \theta_2)$ *such that*:

(1) $1, \theta_1, \theta_2$ *are linearly independent over* $\mathbb{Z}$.
(2) $\mathrm{Bad}_{\boldsymbol{\theta}} := \{(\eta_1, \eta_2) : \inf_{x\in\mathbb{N}} x^{1/2} \max_{i=1,2} \|x\theta_i - \eta_i\| > 0\}$ *is not isotropically winning.*

4. Some more remarks

In the sequel, $\boldsymbol{x} = (x_0, x_1, x_2)$ is a vector in $\mathbb{R}^3$, $|\cdot|$ stands for the Euclidean norm of the vector, and by $(\boldsymbol{w}, \boldsymbol{t})$ we denote the inner product of vectors $\boldsymbol{w}$ and $\boldsymbol{t}$.

The proof of Theorem 3.1 we will give in Section 6. There we will construct a special $\boldsymbol{\theta}$ and a one-dimensional affine subspace $\mathcal{P}$ such that $\boldsymbol{\theta} \in \mathcal{P}$ and for the segment $\mathcal{D} = \mathcal{P} \cap \{|\boldsymbol{z} - \boldsymbol{\theta}| \le 1\}$ one has $\mathcal{D} \cap \mathrm{Bad}_{\boldsymbol{\theta}} = \varnothing$. Moreover, given an arbitrary positive function $\omega(t)$ monotonically (slowly) increasing to infinity we can ensure that for all $\boldsymbol{\eta} = (\eta_1, \eta_2) \in \mathcal{D}$ there exist infinitely many $x \in \mathbb{Z}$ such that

$$\max_{i=1,2} \|x\theta_i - \eta_i\| < \frac{\omega(x)}{x}.$$

To explain the construction of the proof it is useful to consider the case when $\theta_1, \theta_2, 1$ are linearly dependent. This case we will discuss in Section 5.

Remark 4.1. From the result of the paper [Bengoechea et al. 2017] it follows that the vector $\boldsymbol{\theta}$ constructed in Theorem 3.1 does not belong to the set

$$\mathrm{Bad} = \left\{(\theta_1, \theta_2) \mid \inf_{x\in\mathbb{N}} x^{1/2} \max(\|\theta_1 x\|, \|\theta_2 x\|) > 0\right\}.$$

Remark 4.2. Let $\boldsymbol{\theta} = (a_1/q, a_2/q)$ be rational. Let $\boldsymbol{\eta} = (\eta_1, \eta_2) \notin \frac{1}{q} \cdot \mathbb{Z}^2$; then for any $x \in \mathbb{Z}$,

$$\max_{i=1,2} \left\| x\frac{a_i}{q} - \eta_i \right\| \ge \mathrm{dist}\big(\boldsymbol{\eta}, \tfrac{1}{q} \cdot \mathbb{Z}^2\big) > 0.$$

So the set

$$\mathcal{B} = \left\{ \boldsymbol{\eta} : \inf_{x\in\mathbb{Z}} \max_{i=1,2} \left\| x\frac{a_i}{q} - \eta_i \right\| > 0 \right\}$$

contains $\mathbb{R}^2 \setminus \frac{1}{q} \cdot \mathbb{Z}^2$ and is trivially winning. It is clear that for any one-dimensional affine subspace ℓ we have $\mathcal{B} \cap \ell \supset \big(\mathbb{R}^2 \setminus \frac{1}{q} \cdot \mathbb{Z}^2\big) \cap \ell$. So obviously $\mathcal{B} \cap \ell$ is also winning in ℓ.

5. Linearly dependent case

Let $1, \theta_1, \theta_2$ be linearly dependent and at least one of θ_j is irrational. This means that there exists $\boldsymbol{z} = (z_0, z_1, z_2) \in \mathbb{Z}^3$ such that $(\boldsymbol{z}, \boldsymbol{\theta}) = 0$. Let us consider the two-dimensional rational subspace

$$\pi = \{\boldsymbol{x} \in R^3 : (\boldsymbol{x}, \boldsymbol{z}) = 0\},$$

so $\boldsymbol{\theta} \in \pi$.

Let us define the one-dimensional subspace $\mathcal{P} = \{(x_1, x_2) : (1, x_1, x_2) \in \pi\} \subset \mathbb{R}^2$.

We will prove that there exists a constant γ such that for any $\eta = (\eta_1, \eta_2) \in \mathcal{P}$ the inequality

$$\max_{i=1,2} \|\theta_i x - \eta_i\| < \frac{\gamma}{x}$$

has infinitely many solutions in $x \in \mathbb{N}$. (This statement is similar to Chebyshev's theorem [Khinchin 1964, Theorem 24, Chapter 2].)

Denote by $\Lambda = \pi \cap \mathbb{Z}^3$ the integer lattice with the determinant $d := \det \Lambda = |\boldsymbol{z}|$. Denote by $\{\boldsymbol{g}_\nu = (q_\nu, a_{1\nu}, a_{2\nu})\}_{\nu=1,2,3,\dots} \subset \Lambda$ the sequence of the best approximations of $\boldsymbol{\theta}$ by the lattice Λ and the corresponding parallelograms

$$\Pi_\nu = \left\{\boldsymbol{x} = (x_0, x_1, x_2) \in \pi : 0 \le x_0 \le q_\nu,\ \operatorname{dist}(\boldsymbol{x}, l(\boldsymbol{\theta})) \le \operatorname{dist}(\boldsymbol{g}_{\nu-1}, l(\boldsymbol{\theta}))\right\},$$

which contains a fundamental domain of the two-dimensional Λ. Obviously, $\operatorname{vol} \Pi_\nu \le 4d$. So,

$$\operatorname{dist}(\boldsymbol{g}_{\nu-1}, l(\theta)) \ll \frac{d}{q_\nu}, \tag{2}$$

with an absolute constant in the sign $\ll$. It is clear that for any point $\boldsymbol{\eta} \in \pi$, the shift $\boldsymbol{\eta} + \Pi_\nu$ contains a point of Λ.

For any $\boldsymbol{\eta} = (\eta_1, \eta_2) \in \mathcal{P}$ and for any positive integer ν the planar domain $\bar{\boldsymbol{\eta}} + \Pi_\nu$, $\bar{\boldsymbol{\eta}} = (1, -\eta_1, -\eta_2)$ contains an integer point $\boldsymbol{y} = (x, y_1, y_2) \in \Lambda$.

It is clear that

$$1 \le x \le 1 + q_\nu \tag{3}$$

and

$$\max_{i=1,2} \|\theta_i x - \eta_i\| \ll \operatorname{dist}(\boldsymbol{y}, l(\boldsymbol{\theta}) + \bar{\boldsymbol{\eta}}) \ll \operatorname{dist}(l(\boldsymbol{\theta}), \boldsymbol{g}_{\nu-1}),$$

and by (2),

$$\max_{i=1,2} \|\theta_i x - \eta_i\| \ll \frac{d}{q_\nu}. \tag{4}$$

From (3), (4) it follows that the inequality

$$\max_{i=1,2} \|\theta_i x - \eta_i\| \ll \frac{d}{x}$$

has infinitely many solutions and everything is proved.

6. Inductive construction of integer points

Let $\omega(t)$ be arbitrary positive function monotonically (slowly) increasing to infinity. Here we describe the inductive construction of integer points $\boldsymbol{z}_\nu = (q_\nu, z_{1\nu}, z_{2\nu})$. The base of the induction process is trivial. One can take an arbitrary primitive pair of integer vectors that can be completed to a basis of $\mathbb{Z}^3$.

Suppose that we have two primitive integer vectors

$$\boldsymbol{z}_{\nu-1} = (q_{\nu-1}, z_{1\,\nu-1}, z_{2\,\nu-1}) \in \mathbb{Z}^3, \quad \boldsymbol{z}_\nu = (q_\nu, z_{1\,\nu}, z_{2\,\nu}) \in \mathbb{Z}^3.$$

Now we explain how to construct the next integer vector $\boldsymbol{z}_{\nu+1}$.

We consider the two-dimensional subspace

$$\pi_\nu = \langle \boldsymbol{z}_{\nu-1}, \boldsymbol{z}_\nu \rangle_{\mathbb{R}}.$$

The pair of vectors $z_{\nu-1}$ and z_ν is primitive, so the lattice spanned by them is

$$\Lambda_\nu := \langle z_{\nu-1}, z_\nu \rangle_{\mathbb{Z}} = \pi_\nu \cap \mathbb{Z}^3.$$

By $d_\nu = \det \Lambda_\nu$ we denote the two-dimensional fundamental volume of the lattice Λ_ν. Now we define the vector $\boldsymbol{n}_\nu = (n_{0\nu}, n_{1\nu}, n_{2\nu}) \in \mathbb{R}^3$ from the conditions

$$\pi_\nu = \{\boldsymbol{x} \in \mathbb{R}^3 : (\boldsymbol{x}, \boldsymbol{n}_\nu) = 0\}, \quad |\boldsymbol{n}_\nu| = 1.$$

Put

$$\sigma_\nu = \operatorname{dist}(z_{\nu-1}, l(z_\nu)). \tag{5}$$

Obviously, $|z_\nu| \asymp q_\nu$ and

$$\sigma_\nu \asymp \frac{d_\nu}{q_\nu}. \tag{6}$$

We define a vector $\boldsymbol{e}_\nu$ from the conditions

$$\boldsymbol{e}_\nu \in \pi_\nu, \quad |\boldsymbol{e}_\nu| = 1, \quad (\boldsymbol{e}_\nu, z_\nu) = 0, \tag{7}$$

so $\boldsymbol{e}_\nu$ is parallel to π_ν and orthogonal to z_ν.

Define the rectangle

$$\Pi_\nu = \Big\{ \boldsymbol{x} = (x_0, x_1, x_2) : \boldsymbol{x} = t z_\nu + r \boldsymbol{e}_\nu, \ 0 \le t \le |z_\nu|, \ |r| \le \sigma_\nu \Big\}.$$

It is clear that rectangle $\Pi_\nu \subset \pi_\nu$ contains a fundamental domain of the lattice Λ_ν. We need two axillary vectors z_ν^a and z_ν^b defined as

$$z_\nu^a = z_\nu + a_\nu \boldsymbol{e}_\nu, \quad z_\nu^b = z_\nu^a + b_\nu \boldsymbol{n}_\nu,$$

where positive a_ν is chosen in such a way that

$$a_\nu d_\nu^2 \le \nu^{-1} \omega\left(\frac{q_\nu^2}{d_\nu^2} \cdot \frac{1}{a_\nu} \right) \tag{8}$$

and

$$b_\nu = a_\nu \min\left(1, \frac{d_\nu}{q_\nu}\right). \tag{9}$$

From the construction, it follows that

$$|z_\nu^a| \asymp |z_\nu^b| \asymp |z_\nu| \asymp q_\nu. \tag{10}$$

The integer lattice $\mathbb{Z}^3$ splits into levels with respect to the two-dimensional sublattice Λ_ν in such a way that

$$\mathbb{Z}^3 = \bigsqcup_{i \in \mathbb{Z}} \Lambda_{\nu,i},$$

where $\Lambda_{\nu,j} = \Lambda_\nu + j z'$, $j \in \mathbb{Z}$ and integer vector z' completes the couple $z_{\nu-1}$, z_ν to the basis in $\mathbb{Z}^3$. We consider the affine subspace $\pi_\nu^1 = \pi_\nu + z' \supset \Lambda_{\nu,1}$, which is parallel to π_ν. It is clear that $\operatorname{dist}(\pi_\nu, \pi_\nu^1) = 1/d_\nu$.

We need to determine the next integer point $z_{\nu+1}$. Denote by $\mathfrak{P}$ the central projection with center 0 onto the affine subspace π_ν^1. We consider the triangle Δ with vertices z_ν, z_ν^a, z_ν^b and its image $\mathfrak{P}\Delta$ under

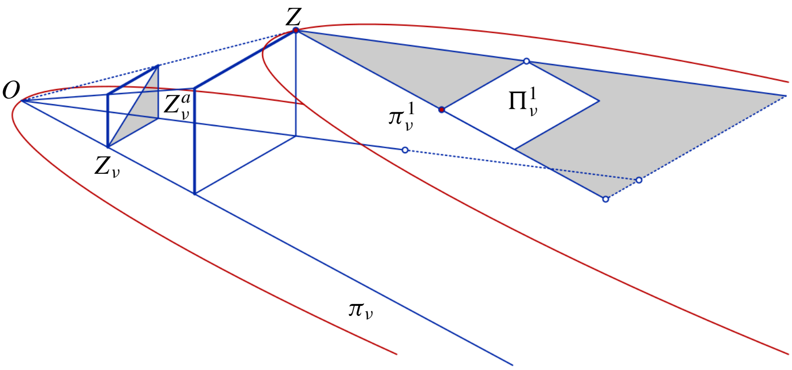

Figure 1. The central projection $\mathfrak{P}$.

the projection $\mathfrak{P}$ (Figure 1).

Define

$$\boldsymbol{Z} = \mathfrak{P} \boldsymbol{z}_\nu^b. \tag{11}$$

One can see that

$$|\boldsymbol{Z}| \asymp \frac{q_\nu}{d_\nu b_\nu}. \tag{12}$$

Define rays

$$\mathcal{R}_1 = \{\boldsymbol{z} = \boldsymbol{Z} + t\boldsymbol{z}_\nu : t \geq 0\} \quad \text{and} \quad \mathcal{R}_2 = \{\boldsymbol{z} = \boldsymbol{Z} + t\boldsymbol{z}_\nu^a : t \geq 0\}.$$

It is clear that $\mathcal{R}_1 \cap \mathcal{R}_2 = \{\boldsymbol{Z}\}$ and $\mathcal{R}_1, \mathcal{R}_2 \subset \pi_\nu^1$. Moreover, the whole convex angle bounded by rays $\mathcal{R}_1, \mathcal{R}_2$ form the image of the triangle Δ under the projection $\mathfrak{P}$:

$$\mathfrak{P}\Delta = \mathrm{conv}(\mathcal{R}_1 \cup \mathcal{R}_2).$$

The affine subspace π_ν^1 contains the affine lattice $\Lambda_\nu^1 = \Lambda_\nu + \boldsymbol{z}'$ which is congruent to the lattice Λ_ν. Thus, for any $\boldsymbol{\zeta} \in \pi_\nu^1$, the shift $\Pi_\nu + \boldsymbol{\zeta}$ contains an integer point from Λ_ν^1.

Put

$$\tau_\nu = \frac{2\sigma_\nu |\boldsymbol{z}_\nu|}{a_\nu}. \tag{13}$$

Consider the point

$$\boldsymbol{\zeta}_\nu = \boldsymbol{Z} + \tau_\nu \boldsymbol{z}_\nu + \sigma_\nu \boldsymbol{e}_\nu \in \pi_\nu^1,$$

and the rectangle

$$\Pi_\nu^1 = \Pi_\nu + \boldsymbol{\zeta}_\nu \subset \pi_\nu^1.$$

It is clear that

$$\Pi_\nu^1 \subset \mathfrak{P}\Delta$$

(here $\boldsymbol{Z}$ was defined in (11), $\boldsymbol{e}_\nu$ was defined in (7), and the parameters σ_ν, τ_ν come from (5) and (13)).

Now we take the integer point

$$\boldsymbol{z}_{\nu+1} = (q_{\nu+1}, z_{1\,\nu+1}, z_{2\,\nu+1}) \in \Lambda_\nu^1 \cap \Pi_\nu^1.$$

From the construction it follows that

$$q_{\nu+1} \asymp |z_{\nu+1}| \asymp |z| + \tau_\nu |z_\nu| + |z_\nu| \asymp q_\nu \left(1 + \frac{1}{d_\nu b_\nu} + \frac{\sigma_\nu}{a_\nu}\right) \asymp q_\nu \left(1 + \frac{1}{d_\nu b_\nu}\right) + \frac{d_\nu}{a_\nu} \asymp \frac{q_\nu}{d_\nu b_\nu}.$$

(Here we use (6), (9), (10), (12), and (13).) From (9) we see that

$$q_{\nu+1} \gg \left(\frac{q_\nu}{d_\nu}\right)^2 \frac{1}{a_\nu}. \tag{14}$$

Now we are able to define the next two-dimensional lattice

$$\Lambda_{\nu+1} = \langle z_\nu, z_{\nu+1} \rangle_{\mathbb{Z}}.$$

Let $d_{\nu+1}$ be its fundamental volume. We will estimate the value of $d_{\nu+1}$ taking into account (9) as

$$d_{\nu+1} \ll q_\nu \cdot \operatorname{dist}(z_{\nu+1}, l(z_\nu)) \ll \frac{q_\nu}{d_\nu} \cdot \frac{a_\nu}{b_\nu} \ll \left(\frac{q_\nu}{d_\nu}\right)^2 \ll q_\nu^2. \tag{15}$$

From (14) and (15), we deduce that

$$d_{\nu+1} \ll a_\nu d_\nu^2 q_{\nu+1}.$$

By the choice of a_ν (by formula (8)) we have

$$d_{\nu+1} \leq \frac{\omega(q_{\nu+1})}{\nu}. \tag{16}$$

7. The vector $\boldsymbol{\theta}$

Now we define

$$\boldsymbol{\theta}_\nu = (\theta_{1\nu}, \theta_{2\nu}), \quad \theta_{j\nu} = \frac{q_{j\nu}}{q_\nu}.$$

We consider the angles between the successive vectors $\boldsymbol{n}_\nu$ and $\boldsymbol{n}_{\nu+1}$:

$$\alpha_\nu = \operatorname{angle}(\boldsymbol{n}_\nu, \boldsymbol{n}_{\nu+1}) \asymp \tan \operatorname{angle}(\boldsymbol{n}_\nu, \boldsymbol{n}_{\nu+1}).$$

Since $z_{\nu+1} \in \mathfrak{P}\Delta$ (see Figure 2), we have

$$\tan \operatorname{angle}(\boldsymbol{n}_\nu, \boldsymbol{n}_{\nu+1}) \leq \frac{b_\nu}{a_\nu},$$

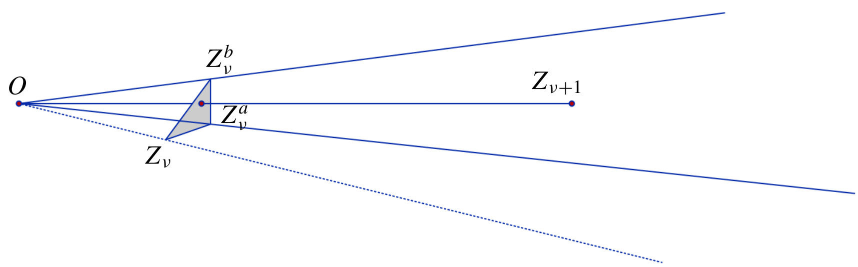

Figure 2. The vector $z_{\nu+1}$ intersects the interior of the triangle $\Delta = z_\nu z_\nu^a z_\nu^b$.

and so

$$\alpha_\nu \ll \frac{b_\nu}{a_\nu}. \tag{17}$$

As $z_{\nu+1} \in \mathfrak{P}\Delta$, we have

$$|\boldsymbol{\theta}_\nu - \boldsymbol{\theta}_{\nu+1}| \ll \frac{\sqrt{a_\nu^2 + b_\nu^2}}{q_\nu} \ll \frac{a_\nu}{q_\nu} \tag{18}$$

by the same argument. There exist limits

$$\lim_{\nu\to\infty} \boldsymbol{\theta}_\nu = \boldsymbol{\theta} = (\theta_1, \theta_2) \quad \text{and} \quad \lim_{\nu\to\infty} \boldsymbol{n}_\nu = \boldsymbol{n},$$

and from (17) and (18) we deduce that

$$0 < |\boldsymbol{\theta} - \boldsymbol{\theta}_\nu| \ll \frac{a_\nu}{q_\nu} \tag{19}$$

and

$$\operatorname{angle}(\boldsymbol{n}, \boldsymbol{n}_\nu) \ll \frac{b_\nu}{a_\nu}. \tag{20}$$

It is clear that $\boldsymbol{\theta} \notin \mathbb{Q}^2$. A slight modification[2] of the procedure of choosing vectors z_ν ensures the condition that $1, \theta_1, \theta_2$ are linearly independent over $\mathbb{Z}$. Define $\pi = \{\boldsymbol{x} \in \mathbb{R}^3 : (\boldsymbol{x}, \boldsymbol{n}) = 0\}$. Then $\boldsymbol{\theta} \in \pi$ by continuity and we can assume that $\boldsymbol{n} \notin \mathbb{Q}^3$.

8. Winning property

Consider the one-dimensional affine subspaces

$$\mathcal{P}_\nu = \{(x_1, x_2) \in \mathbb{R}^2 : (1, x_1, x_2) \in \pi_\nu\} \subset \mathbb{R}^2$$

and

$$\mathcal{P} = \{(x_1, x_2) \in \mathbb{R}^2 : (1, x_1, x_2) \in \pi\} \subset \mathbb{R}^2,$$

where π was defined at the end of the previous section. Let

$$B_1(\boldsymbol{\theta}) = \{\boldsymbol{\xi} \in \mathbb{R}^2 : \operatorname{dist}(\boldsymbol{\xi}, \boldsymbol{\theta}) < 1\}.$$

We will show that for any $\boldsymbol{\eta} = (\eta_1, \eta_2) \in \mathcal{P} \cap B_1(\boldsymbol{\theta})$ there exists infinitely many solutions of the inequality

$$\max_{i=1,2} \|\theta_i x - \eta_i\| < \frac{\omega(x)}{x}$$

in integers x. Denote by $\boldsymbol{\eta}_\nu = (\eta_{1\nu}, \eta_{2\nu})$ the orthogonal projection of $\boldsymbol{\eta}$ onto $\mathcal{P}_\nu$. From (20) we see that

$$|\boldsymbol{\eta} - \boldsymbol{\eta}_\nu| \ll \frac{b_\nu}{a_\nu}. \tag{21}$$

[2] A similar procedure was explained in [Moshchevitin 2012]. There, the author provides the linear independence of coordinates of the limit vector by "going away from all rational subspaces" (the beginning of the proof of Theorem 1 in the case $k = 1$, p. 132 and the beginning of §5, p. 146).

For any $\boldsymbol{\eta}_\nu = (\eta_{1\nu}, \eta_{2\nu}) \in \mathcal{P}_\nu$ the planar domain $\bar{\boldsymbol{\eta}}_\nu + \Pi_\nu$, $\bar{\boldsymbol{\eta}}_\nu = (1, -\eta_{1\nu}, -\eta_{2\nu})$ contains an integer point $\boldsymbol{y}_\nu = (x_\nu, y_{1\nu}, y_{2\nu}) \in \Lambda_\nu$. It is clear that

$$|x_\nu| \ll q_\nu \tag{22}$$

and

$$\max_{i=1,2} |\theta_{i\nu} x_\nu - \eta_{i\nu} - y_{i\nu}| \ll \frac{d_\nu}{q_\nu}. \tag{23}$$

By (19), (21), (22), and (23) we have

$$\max_{i=1,2} \|\theta_i x_\nu - \eta_i\| \le |x_\nu| \max_{i=1,2} |\theta_i - \theta_{i\nu}| + \max_{i=1,2} \|\theta_{i\nu} x_\nu - \eta_{i\nu}\| + \max_{i=1,2} |\eta_i - \eta_{i\nu}| \ll a_\nu + \frac{d_\nu}{q_\nu} + \frac{b_\nu}{a_\nu} \ll \frac{d_\nu}{q_\nu}.$$

In the last inequality we use (9). By (16) we have

$$\max_{i=1,2} \|\theta_i x_\nu - \eta_i\| \le \frac{\omega(q_\nu)}{q_\nu}$$

for large ν. As $\bar{\boldsymbol{\eta}} \in \pi$ and $\boldsymbol{y}_\nu \in \pi_\nu$, $\max_{i=1,2} \|\theta_i x_\nu - \eta_i\| \ne 0$ infinitely often (in fact for all large ν).

References

[An 2016] J. An, "2-dimensional badly approximable vectors and Schmidt's game", *Duke Math. J.* **165**:2 (2016), 267–284. MR Zbl

[Bengoechea et al. 2017] P. Bengoechea, N. Moshchevitin, and N. Stepanova, "A note on badly approximable linear forms on manifolds", *Mathematika* **63**:2 (2017), 587–601. MR Zbl

[Broderick et al. 2011] R. Broderick, L. Fishman, and D. Kleinbock, "Schmidt's game, fractals, and orbits of toral endomorphisms", *Ergodic Theory Dynam. Systems* **31**:4 (2011), 1095–1107. MR Zbl

[Broderick et al. 2013] R. Broderick, L. Fishman, and D. Simmons, "Badly approximable systems of affine forms and incompressibility on fractals", *J. Number Theory* **133**:7 (2013), 2186–2205. MR Zbl

[Cassels 1957] J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge Tracts in Mathematics and Mathematical Physics **45**, Cambridge University Press, 1957. MR Zbl

[Dremov 2002] V. A. Dremov, "On domains of (α, β)-winnability", *Dokl. Akad. Nauk* **384**:3 (2002), 304–307. In Russian; translated in *Dokl. Math.* **65**:3 (2002), 365–368. MR Zbl

[Einsiedler and Tseng 2011] M. Einsiedler and J. Tseng, "Badly approximable systems of affine forms, fractals, and Schmidt games", *J. Reine Angew. Math.* **660** (2011), 83–97. MR Zbl

[Fishman et al. 2013] L. Fishman, D. S. Simmons, and M. Urbański, "Diophantine approximation and the geometry of limit sets in Gromov hyperbolic metric spaces", 2013. arXiv

[Harrap and Moshchevitin 2017] S. Harrap and N. Moshchevitin, "A note on weighted badly approximable linear forms", *Glasg. Math. J.* **59**:2 (2017), 349–357. MR Zbl

[Jarník 1941] V. Jarník, "O lineárních nehomogenních diofantických aproximacích", *Rozpr. II. Třídy České Akad.* **51**:29 (1941), 1–21. Translated in French as "Sur les approximations diophantiques linéaires non homogènes", *Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat.* **47** (1946), 145-160. MR Zbl

[Khinchin 1926] A. Khintchine, "Über eine Klasse linearer diophantischer Approximationen", *Rend. Circ. Mat. Palermo* **50** (1926), 170–195. Zbl

[Khinchin 1937] A. Khintchine, "Über die angenäherte Auflösung linearer Gleichungen in ganzen Zahlen", *Acta Arith.* **2** (1937), 161–172. Zbl

[Khinchin 1948] A. Y. Khinchin, "Regular systems of linear equations and a general problem of Chebyshev", *Izvestiya Akad. Nauk SSSR. Ser. Mat.* **12** (1948), 249–258. In Russian. MR

[Khinchin 1964] A. Y. Khinchin, *Continued fractions*, The University of Chicago Press, 1964. MR Zbl

[Kleinbock and Weiss 2010] D. Kleinbock and B. Weiss, "Modified Schmidt games and Diophantine approximation with weights", *Adv. Math.* **223**:4 (2010), 1276–1298. MR Zbl

[McMullen 2010] C. T. McMullen, "Winning sets, quasiconformal maps and Diophantine approximation", *Geom. Funct. Anal.* **20**:3 (2010), 726–740. MR Zbl

[Moshchevitin 2011] N. G. Moshchevitin, "A note on badly approximable affine forms and winning sets", *Mosc. Math. J.* **11**:1 (2011), 129–137. MR Zbl

[Moshchevitin 2012] N. G. Moshchevitin, "Proof of W. M. Schmidt's conjecture concerning successive minima of a lattice", *J. Lond. Math. Soc.* (2) **86**:1 (2012), 129–151. MR Zbl

[Schmidt 1966] W. M. Schmidt, "On badly approximable numbers and certain games", *Trans. Amer. Math. Soc.* **123** (1966), 178–199. MR Zbl

[Schmidt 1969] W. M. Schmidt, "Badly approximable systems of linear forms", *J. Number Theory* **1** (1969), 139–154. MR Zbl

[Schmidt 1980] W. M. Schmidt, *Diophantine approximation*, Lecture Notes in Mathematics **785**, Springer, 1980. MR Zbl

[Tseng 2009] J. Tseng, "Badly approximable affine forms and Schmidt games", *J. Number Theory* **129**:12 (2009), 3020–3025. MR Zbl

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