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Correction Terms and the Nonorientable Slice Genus

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Abstract. By considering negative surgeries on a knot K in S^3, we derive a lower bound on the nonorientable slice genus $\gamma_4(K)$ in terms of the signature $\sigma(K)$ and the concordance invariants $V_i(\overline{K})$; this bound strengthens a previous bound given by Batson and coincides with Ozsváth–Stipsicz–Szabó's bound in terms of their υ invariant for L-space knots and quasi-alternating knots. A curious feature of our bound is superadditivity, implying, for instance, that the bound on the stable nonorientable slice genus is sometimes better than that on $\gamma_4(K)$.

1. Introduction

Given a knot K in S^3, it is a very classical problem to determine the minimal genus of an orientable surface F in B^4 whose boundary is K. More recently, some attention has been drawn to the case of nonorientable surfaces instead. Namely, we can define $\gamma_4(K)$ as the minimal nonorientable genus among all such surfaces, where the nonorientable genus of F is defined as $b_1(F)$.

Batson and Ozsváth, Stipsicz, and Szabó gave lower bounds in terms of Heegaard Floer data. More precisely, Batson [2] proved that

$$\gamma_4(K) \geq \frac{\sigma(K)}{2} - d(S^3_{-1}(K)), \tag{1.1}$$

where $d(S^3_{-1}(K))$ is the Heegaard Floer correction term (or d-invariant) of the 3-manifold obtained as (-1)-surgery along K in its unique spin^c structure (which is hence omitted from the notation). Ozsváth, Stipsicz, and Szabó [17, Theorem 1.2] proved that

$$\gamma_4(K) \geq \left|\frac{\sigma(K)}{2} - \upsilon(K)\right|, \tag{1.2}$$

where υ is a concordance invariant defined in terms of the Floer homology package.

The main goal of this paper is to provide a new lower bound that generalizes that of Batson. It is phrased in terms of the concordance invariants $\{V_i(\overline{K})\}_i$ associated with the mirror $\overline{K}$ of K; these invariants were defined by Rasmussen [20]

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and further studied by Ni and Wu [15] (see also Section 2). We further package these invariants into a single integer-valued invariant, which we call φ, $\varphi(K) = \min_{m\geq 0}\{m + 2V_m(\overline{K})\}$.

Theorem 1.1. *For every knot K in S^3,*

$$\gamma_4(K) \geq \frac{\sigma(K)}{2} - \varphi(K). \tag{1.3}$$

The existence of such a bound was indicated, but not made explicit, by Batson in his PhD thesis [3]. Moreover, since $d(S^3_{-1}(K)) = 2V_0(\overline{K}) \geq \varphi(K)$, this is a strengthening of (1.1). Equation (1.3) also implies the existence of a bound in terms of the invariant ν^+ defined by Hom and Wu [11]. By definition, we have $V_{\nu^+(\overline{K})}(\overline{K}) = 0$, so Theorem 1.1 implies at once

$$\gamma_4(K) \geq \frac{\sigma(K)}{2} - \nu^+(\overline{K}). \tag{1.4}$$

Note that this bound is formally identical to (1.1), (1.2), and (1.3); to the best of the authors' knowledge, this bound never appeared in the literature.

We further show that the bound of Theorem 1.1 is sharp (see Remark 5.2) and agrees with that of (1.2) in the case of alternating knots (see Proposition 6.2).

We note here that the bound (1.3) presents the following curious feature: it is superadditive in the knot K in the sense that the bound for $K_1\#K_2$ can be strictly larger than the sum of the two bounds for K_1 and K_2. In fact, the same is true for the bounds (1.1) and (1.4). In particular, the bound for nK can give more information on $\gamma_4(K)$ than the bound for K.

In Proposition 7.1, we exhibit an example where this phenomenon actually occurs.

Using superadditivity, we can optimize the bound as follows:

$$\gamma_4(K) \geq \frac{\sigma(K)}{2} - \omega(K),$$

where

$$\omega(K) := \lim_{n\to\infty} \frac{1}{n}\varphi(nK) \leq \varphi(K).$$

Organization of the Paper

In Section 2, we recall some basic facts about spin^c structures on 3- and 4-manifold and d-invariants, and we state all the results concerning them that we use in this paper. In Section 3, we fix the notation and construct a cobordism W° from a particular 3-manifold Q (defined in that section) to $S^3_{-n}(K)$, which will be crucial to deduce the bound in equation (1.3). In Section 4, we label spin^c structures on W°, compute their Chern classes, and understand their restrictions to ∂W°. In Section 5, we apply a twisted version of the Ozsváth–Szabó's inequality (see Theorem 2.4) to W° to obtain the desired bound on $\gamma_4(K)$. In Section 6, we compare our bound to those of Batson and Ozsváth–Stipsicz–Szabó (see equations (1.1) and (1.2)), and we refine it using superadditivity. Finally, in Section 7, we give an example of a knot K where the bound for nK is actually better than the bound for K.

2. All You Need Is Correction Terms

Given an oriented manifold M of dimension 3 or 4, recall that the set of spin^c structures $\mathrm{Spin}^c(M)$ is an affine space over $H^2(M;\mathbb{Z})$. Given an oriented 4-manifold X with boundary $\partial X = Y$, the restriction map

$$\mathrm{Spin}^c(X) \to \mathrm{Spin}^c(Y) \tag{2.1}$$

is modeled over

$$H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z}).$$

With every spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(M)$ it is possible to associate an element in $H^2(M;\mathbb{Z})$, known as the (*first*) *Chern class* of $\mathfrak{s}$ and usually denoted by $c_1(\mathfrak{s})$. The map

$$c_1\colon\ \mathrm{Spin}^c(M) \to H^2(M;\mathbb{Z})$$

is injective if and only if $H^2(M;\mathbb{Z})$ has no 2-torsion. A spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(M)$ is called *torsion* if $c_1(\mathfrak{s})$ is a torsion element in $H^2(M;\mathbb{Z})$.

Let $-M$ denote the manifold M endowed with the opposite orientation. There is a canonical bijection

$$\iota\colon\ \mathrm{Spin}^c(M) \to \mathrm{Spin}^c(-M),$$

which is modeled over the canonical isomorphism $\iota\colon\ H^2(M;\mathbb{Z}) \to H^2(-M;\mathbb{Z})$ (see [8, Section 1.2.3]). If $\mathfrak{s} \in \mathrm{Spin}^c(M)$, then we will denote by the same letter $\mathfrak{s}$ the corresponding spin^c structure on $-M$. It is worth noting that such a bijection commutes with the restriction map (see equation (2.1)) and that

$$c_1(\iota(\mathfrak{s})) = \iota(c_1(\mathfrak{s})).$$

Remark 2.1. Let X^4 be the trace of the 2-handle cobordism from S^3 to $S^3_n(K)$, where K is a knot in S^3, and $n > 0$ is a positive integer. Then we can label the spin^c structures on X as follows: we denote by $\mathfrak{s}_k$ the unique spin^c structure on X such that

$$\langle c_1(\mathfrak{s}_k), [\Sigma]\rangle = n + 2k,$$

where Σ is a Seifert surface for K pushed into $S^3 \times I$ and capped off with the core of the 2-handle. From this labeling we derive a labeling of spin^c structures over $S^3_n(K)$ by $\mathbb{Z}/n\mathbb{Z}$ by setting

$$\mathfrak{t}_k := \mathfrak{s}_k|_{S^3_n(K)},$$

where we make no distinction between an integer and its class modulo n. Here and in what follows, we refer the reader to [19, Section 2.4] for further details.

We say that a pair $(Y, \mathfrak{t})$, where $\mathfrak{t}$ is a torsion spin^c structure on the 3-manifold Y, is a *torsion* spin^c 3-manifold.

Ozsváth and Szabó [18] introduced a Heegaard Floer theoretical invariant $d(Y, \mathfrak{t})$, called the *correction term* or *d-invariant,* associated with a pair $(Y, \mathfrak{t})$, where Y is a rational homology 3-sphere equipped with a spin^c structure $\mathfrak{t}$. In [18, Section 9], they explain how it is possible to generalize it to invariants d_b and d_t (*bottom* and *top*) associated with a torsion spin^c 3-manifold $(Y, \mathfrak{t})$, where Y is now a 3-manifold with *standard* HF^∞ (which is equivalent to having a trivial triple cup product [14]). See also [13, Section 3] for an introduction to d-invariants of arbitrary 3-manifolds with standard HF^∞. Behrens and the first author [4] used Heegaard Floer homology with twisted coefficients to further generalize this to an invariant $\underline{d}(Y, \mathfrak{t})$ associated with an arbitrary torsion spin^c 3-manifold $(Y, \mathfrak{t})$.

In the case of rational homology 3-spheres, we have

$$d(Y, \mathfrak{t}) = d_b(Y, \mathfrak{t}) = d_t(Y, \mathfrak{t}) = \underline{d}(Y, \mathfrak{t}). \tag{2.2}$$

More generally, we have the following:

THEOREM 2.2 ([18, Proposition 4.2], [13, Proposition 3.7], and [4, Proposition 3.8]). *Let $(Y, \mathfrak{t})$ be a torsion* spin^c *3-manifold, and suppose that Y has standard* HF^∞. *Then, under the canonical identification* $\mathrm{Spin}^c(Y) \cong \mathrm{Spin}^c(-Y)$,

$$d_b(Y, \mathfrak{t}) = -d_t(-Y, \mathfrak{t}) \geq \underline{d}(Y, \mathfrak{t}).$$

In the rest of this section, we state the results that we need about d-invariants.

The following result by Ni and Wu allows us to compute d-invariants for surgeries on a knot $K \subseteq S^3$ in terms of some knot invariants V_i, which were first introduced in [20] with a different notation. They are a sequence of nonnegative integers $\{V_i(K)\}_{i \geq 0}$ satisfying $V_i(K) - 1 \leq V_{i+1}(K) \leq V_i(K)$. We refer to [15, Section 2.2] for their definition.

THEOREM 2.3 ([15, Proposition 1.6 and Remark 2.10]). *Given positive integers $0 \leq k < n$, we have*

$$d(S^3_n(K), \mathfrak{t}_k) = -\frac{n - (2k-n)^2}{4n} - 2\max\{V_k(K), V_{n-k}(K)\}.$$

Correction terms can be used to give restrictions to intersection forms of 4-manifolds bounding a given 3-manifold (compare also with [18, Theorem 9.15]).

THEOREM 2.4 ([4, Theorem 4.1]). *Let $(W, \mathfrak{s})$ be a negative semidefinite* spin^c *cobordism from $(Y, \mathfrak{t})$ to $(Y', \mathfrak{t}')$, two torsion* spin^c *3-manifolds, such that the map $H_1(Y; \mathbb{Q}) \to H_1(W; \mathbb{Q})$ induced by the inclusion is injective. Then*

$$c_1(\mathfrak{s})^2 + b_2^-(W) \leq 4\underline{d}(Y', \mathfrak{t}') + 2b_1(Y') - 4\underline{d}(Y, \mathfrak{t}) - 2b_1(Y).$$

3. Notation and Construction

Let K be a knot in S^3. If we consider S^3 as the boundary of the 4-ball B^4, then the (orientable) slice genus g_4 is defined as the minimum genus of a smooth orientable surface in B^4 whose boundary is K, and it is a well-studied invariant of K. More recently, the nonorientable slice genus γ_4 has been studied. We have the following definition.

DEFINITION 3.1. Given a knot K in S^3, we define its *nonorientable slice genus* as

$$\gamma_4(K) = \min\{b_1(F) \mid F \hookrightarrow B^4 \text{ smooth, nonorientable }, \partial F = K\},$$

where $b_1(F)$ denotes the first Betti number of F.

REMARK 3.2. With this definition of γ_4, we always have $\gamma_4(K) \geq 1$. We could also consider the four-dimensional crosscap number instead; this is the minimal number h such that K bounds a punctured $\#^h \mathbb{RP}^2$ in B^4. The two definitions are indeed equivalent except when K is slice, in which case our definition yields $\gamma_4(K) = 1$, whereas the four-dimensional crosscap number is 0. We note here that, when K is slice, the bound in (1.3) is in any case $\gamma_4(K) \geq 0$, so this is in fact a bound for the four-dimensional crosscap number as well; this is true since, when K is slice, both $\sigma(K)$ and $\varphi(L)$ vanish (see Proposition 6.1(2)). Our proof, however, actually uses Definition 3.1 of γ_4, to which therefore we stick.

Batson [2] proved that the nonorientable slice genus of a knot can be arbitrarily large. More specifically, for a nonorientable surface F as in Definition 3.1, Batson gives the following inequality (see [2, Theorem 1.5]):

$$b_1(F) + 2d(S^3_{-1}(K)) \geq \frac{e(F)}{2}. \tag{3.1}$$

Here $d(S^3_{-1}(K))$ denotes the d-invariant of $S^3_{-1}(K)$ in the unique spin^c structure, whereas $e(F)$ is the *normal Euler number* of F: given a nonvanishing section s of the normal bundle ν_F (which always exists since F deformation retracts on a 1-complex), we let

$$e(F) = -\operatorname{lk}(K, s(K)).$$

Batson [2] combines equation (3.1) and the "signature" inequality from [9]

$$b_1(F) \geq \sigma(K) - \frac{e(F)}{2} \tag{3.2}$$

to derive the bound for the nonorientable slice genus in equation (1.1). The main result of this paper follows from a generalisation of equation (3.1), equation (5.4) below, where instead of the (-1)-surgery along K we consider $(-n)$-surgeries for arbitrary integers $n \geq 1$. Inspired by [2] and [13], we construct a negative semi-definite cobordism from a 3-manifold Q to $S^3_{-n}(K)$, and use Theorem 2.4 to give a lower bound to $b_1(F)$.

We now give the details of the construction illustrated in Figure 1. Let K be a knot in $S^3 = \partial B^4$, and let F denote a smooth nonorientable surface properly embedded in B^4 such that $\partial F = K$. Fix an integer $n > 0$. Let W denote the 4-manifold obtained by attaching a $(-n)$-framed 2-handle to B^4 along $K \subset \partial B^4$. We denote with Y the boundary of W, that is, $Y = S^3_{-n}(K)$. Then the surface F can be capped off with the core of the 2-handle to obtain a closed surface $\widehat{F} \subseteq W$.

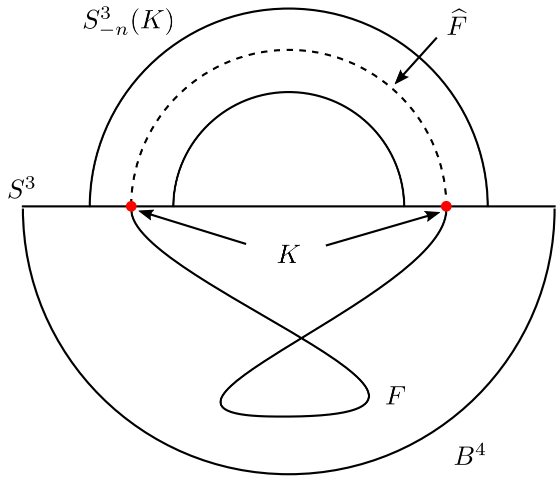

Figure 1 The 4-manifold W obtained by attaching a $(-n)$-framed 2-handle to B^4 along a knot $K \subseteq S^3$. $N = \mathcal{N}_W(\widehat{F})$ denotes a regular neighborhood of $\widehat{F}$ in W, and $Q = \partial N$.

Notice that

$$b_1(\widehat{F}) + 1 = b_1(F) =: h.$$

If $e = e(F)$ denotes the normal Euler number of F, and $e(\widehat{F})$ denotes the Euler number of the closed surface $\widehat{F}$, then we have

$$e(\widehat{F}) = e - n.$$

As already noticed in [2], e is even because the self-intersection of F in B^4 can be computed algebraically over $\mathbb{Z}/2\mathbb{Z}$.

Let $N = \mathcal{N}_W(\widehat{F})$ denote a regular neighborhood of $\widehat{F}$ in W. We define $Q = \partial N$. Notice that Q (resp. N) is a circle (resp. disc) bundle over the closed surface $\widehat{F} \cong (\mathbb{RP}^2)^{\#h}$ of Euler number $e-n$. According to the notation in [13, Section 2], we have $N \cong P_{h,e-n}$ and $Q \cong Q_{h,e-n}$, and moreover Q has standard HF^∞.

The manifold $W^\circ := W \setminus N$ is a cobordism between Q and $S^3_{-n}(K)$. Since the labeling of spin^c structures is better understood for positive surgeries, we consider also the manifold $-W$, obtained from W by reversing the orientation; $-W$ is the 4-manifold obtained by attaching an n-framed 2-handle to B^4 along $\overline{K}$. This allows us to label the spin^c structures on W and on Y; by a slight abuse of notation, we write $\mathfrak{s}_k$ and $\mathfrak{t}_k$, dropping the identifications $\mathrm{Spin}^c(W) \cong \mathrm{Spin}^c(-W)$ and $\mathrm{Spin}^c(Y) = \mathrm{Spin}^c(-Y)$.

Table 1

	W	$W^\circ \sqcup N$	Q
H^0	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}$
H^1	0	$0\oplus\mathbb{Z}^{h-1}$	$\mathbb{Z}^{h-1}$
H^2	$\mathbb{Z}$	$?\oplus\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^{h-1}\oplus T$
H^3	0	$\mathbb{Z}\oplus 0$	$\mathbb{Z}$

4. Labeling Spin^c Structures

4.1. (Co)Homological Computations

The aim of this subsection is to compute $H^2(W^\circ;\mathbb{Z})$ in order to understand spin^c structures on W°. Consider the Mayer–Vietoris long exact sequence in cohomology associated with $W = W^\circ \cup_Q N$. When we do not specify it, we assume that we are using $\mathbb{Z}$ coefficients (see Table 1).

The cohomology of W can be easily obtained by recalling that W is constructed by attaching a 2-handle to B^4. The cohomology of N is also straightforward, since N deformation retracts on $\widehat{F} = (\mathbb{RP}^2)^{\#h}$. As for Q, its cohomology can be deduced from [13, Lemma 2.1], and it is written in the table above, where T is the torsion subgroup of $H_1(Q)$, which is, according to [13, Lemma 2.1],

$$T = \begin{cases} \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & \text{if } e(\widehat{F}) \text{ is even,} \\ \mathbb{Z}/4\mathbb{Z} & \text{if } e(\widehat{F}) \text{ is odd.} \end{cases}$$

In both cases, the map $H^2(N)\cong\mathbb{Z}/2\mathbb{Z}\to T$ is nontrivial. From the cohomology groups that we already know (and the fact that the map $H^1(N)\to H^1(Q)$ is an isomorphism) we can deduce almost all the cohomology groups of W°. $H^2(W^\circ)$ will depend on the parity of $e(\widehat{F})$, according to the following lemma.

LEMMA 4.1. *We have*

$$H^2(W^\circ) = \begin{cases} \mathbb{Z}^h\oplus\mathbb{Z}/2\mathbb{Z} & \textit{if } e(\widehat{F}) \textit{ is even,} \\ \mathbb{Z}^h & \textit{if } e(\widehat{F}) \textit{ is odd.} \end{cases}$$

Proof. From the long exact sequence above we have an exact sequence

$$0\to\mathbb{Z}\to H^2(W^\circ)\to\mathbb{Z}^{h-1}\oplus\mathbb{Z}/2\mathbb{Z}\to 0,$$

regardless of the parity of $e(\widehat{F})$. The two possible extensions are $\mathbb{Z}^h\oplus\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}^h$.

In the first case, $H^2(W^\circ;\mathbb{F}_2)\cong H^2(W^\circ)\otimes\mathbb{F}_2$ is isomorphic to $\mathbb{F}_2^{h+1}$ and, in the second case, to $\mathbb{F}_2^h$. Therefore, to understand $H^2(W^\circ)$, it suffices to determine the rank of $H^2(W^\circ;\mathbb{F}_2)$.

Table 2

	Q	$W^\circ \sqcup N$	W
H_3	$\mathbb{F}_2$	$\mathbb{F}_2 \oplus 0$	0
H_2	$H_2(Q;\mathbb{F}_2)$	$? \oplus \mathbb{F}_2$	$\mathbb{F}_2$
H_1	$H_1(Q;\mathbb{F}_2)$	$? \oplus \mathbb{F}_2^h$	0
H_0	$\mathbb{F}_2$	$\mathbb{F}_2 \oplus \mathbb{F}_2$	$\mathbb{F}_2$

Consider the Mayer–Vietoris long exact sequence in homology associated with $W = W^\circ \cup_Q N$, with $\mathbb{F}_2$ coefficients. Since the coefficient ring is a field, homology and cohomology are dual to each other. As in the previous case, the homologies of W and N are quite straightforward to compute. According to [13, Proof of Lemma 2.1], $H_1(Q;\mathbb{F}_2) \cong H_2(Q;\mathbb{F}_2) \cong \mathbb{F}_2^{h+1}$ if $e(\widehat{F})$ is even, and $H_1(Q;\mathbb{F}_2) \cong H_2(Q;\mathbb{F}_2) \cong \mathbb{F}_2^h$ if $e(\widehat{F})$ is odd (see Table 2).

Consider the connecting morphism

$$\partial\colon H_2(W;\mathbb{F}_2) \to H_1(Q;\mathbb{F}_2).$$

Since $H_2(W;\mathbb{F}_2) \cong \mathbb{F}_2$ is generated by $[\widehat{F}]$ and $\widehat{F}$ is disjoint from $Q = \partial N$, ∂ vanishes.

From this we deduce that $H_2(W^\circ;\mathbb{F}_2) \cong \mathbb{F}_2^h$ if $e(\widehat{F})$ is odd and $H_2(W^\circ;\mathbb{F}_2) \cong \mathbb{F}_2^{h+1}$ if $e(\widehat{F})$ is even, and hence we conclude the proof of the lemma. □

4.2. Intersection Form

In this section, we study the intersection forms on $H_2(W)$ and $H^2(W)$.

LEMMA 4.2. *The intersection form Q_W on $H_2(W) \cong \mathbb{Z}$ is given by $Q_W = (-n)$. The intersection form Q^W on $H^2(W) \cong \mathbb{Z}$ is given by $Q^W = (-\frac{1}{n})$.*

Proof. The intersection form on $H_2(W)$ is $(-n)$ because the 4-manifold W is obtained by attaching a $(-n)$-framed 2-handle to B^4.

The intersection form on $H^2(W)$ can be worked out by considering the following portion of the long exact sequence in homology associated with the pair (W, Y), where $Y = S^3_{-n}(K)$:

$$0 \to H_2(W) \to H^2(W) \to H_1(Y) \to 0.$$

Such a short exact sequence is isomorphic to

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

The generator of $H_2(W)$ is mapped to n times the generator of $H^2(W)$, so the intersection form on $H^2(W)$ is represented by the matrix $(-\frac{1}{n})$. □

It is also worth noting that, for each $c \in H^2(W)$, $c|_{W^\circ}$ restricts to a torsion class on both boundary components, and therefore it makes sense to consider its square.

We claim that

$$Q^{W^\circ}(c|_{W^\circ}) = Q^W(c). \tag{4.1}$$

Indeed, the class $nc \in H^2(W) \cong H_2(W, Y)$ is in the image of the map $H_2(W) \to H_2(W, Y)$. Now, the map $\iota\colon H_2(W^\circ) \to H_2(W)$ is surjective: this comes from the Mayer–Vietoris sequence for $W = W^\circ \cup N$, since the connecting morphism $\partial : H_2(W) \to H_1(Q)$ vanishes (see the proof of Lemma 4.1) and $H_2(N) = 0$.

Therefore, there is an element $d \in H_2(W^\circ)$ such that $\iota(d) = nc$. Note that the image of d in $H_2(W^\circ, \partial W^\circ)$ represents the Poincaré dual of $nc|_{W^\circ} \in H^2(W^\circ)$. The elements d and nc can be represented by some copies of a surface $S \subseteq W^\circ$. The squares $Q^{W^\circ}(d)$ and $Q^W(nc)$ can be computed as the algebraic self-intersection $S \cdot S$ of S, which in turn can be computed in an arbitrarily small neighborhood of S.

4.3. *Spin^c Structures*

Recall (see Remark 2.1) that spin^c structures on $-W$ are labeled by integers as follows:

$$\langle c_1(\mathfrak{s}_k), [\Sigma]\rangle = 2k + n.$$

By symmetry we also get a labeling for $\mathrm{Spin}^c(W)$, and we still denote the spin^c structures on W by $\mathfrak{s}_k$. The structures $\mathfrak{s}_k$ and $\mathfrak{s}_{k'}$ restrict to the same spin^c structure on Y if and only if $n \mid (k - k')$. In such a case, we denote the restriction to Y by $\mathfrak{t}_k = \mathfrak{t}_{k'}$.

It is worth noting that we have isomorphisms $H^2(W) \cong \mathbb{Z}$ and $H^2(Y) \cong \mathbb{Z}/n\mathbb{Z}$ such that, under these identifications, the restriction map is the usual projection $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, and $c_1(\mathfrak{t}_k) \equiv 2k \pmod n$.

To apply Theorem 2.4, we need a spin^c structure on the cobordism W° that restricts to a torsion spin^c structure on Q. Therefore, we introduce the following notation.

Definition 4.3. Given a 4-manifold X, we define $\mathrm{Spin}^c_{\mathrm{tor}}(X)$ to be the subset of $\mathrm{Spin}^c(X)$ of elements that restrict to torsion spin^c structures on ∂X.

Notice that, in our case, $\mathrm{Spin}^c_{\mathrm{tor}}(W^\circ)$ is given by all spin^c structures that restrict to torsion spin^c structures on Q, because all spin^c structures on Y are already torsion. We will now give a description of $\mathrm{Spin}^c_{\mathrm{tor}}(W^\circ)$ in the case of $e(\widehat{F})$ odd (or, equivalently, n odd).

4.4. *Case of $e(\widehat{F})$ Odd*

By Lemma 4.1 we have that $H^2(W^\circ) \cong \mathbb{Z}^h$. From the Mayer–Vietoris exact sequence associated with $W = W^\circ \cup_Q N$ we find:

$$\begin{tikzcd}
0 \arrow[r] & H^2(W) \arrow[r] \arrow[d, "\wr"] & H^2(W^\circ) \oplus H^2(N) \arrow[r] \arrow[d, "\wr"] & H^2(Q) \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\alpha"] & \mathbb{Z}^h \oplus \mathbb{Z}/2\mathbb{Z} \arrow[r, "\beta"] & \mathbb{Z}^{h-1} \oplus \mathbb{Z}/4\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

We have that $\alpha(1) = (c, 1)$ for some nonzero $c \in \mathbb{Z}^h$; otherwise, the quotient would contain a $\mathbb{Z}/2\mathbb{Z}$ summand. Then we have that

$$\mathbb{Z}^{h-1} \oplus \mathbb{Z}/4\mathbb{Z} \cong \frac{\mathbb{Z}^h \oplus \mathbb{Z}/2\mathbb{Z}}{\langle (c,1) \rangle} \cong \frac{\mathbb{Z}^h}{\langle 2c \rangle}.$$

This implies that $c = 2d$, where $d \in \mathbb{Z}^h$ is a primitive element. We denote by $x \in H^2(W^\circ)$ the element that corresponds to d, and we let $\mathcal{A} = \langle x \rangle \subseteq H^2(W^\circ)$. Therefore, $\mathrm{Spin}^c_{\mathrm{tor}}(W^\circ)$ is an affine space over $\mathcal{A}$.

It follows from the exact sequence above that the image of the map

$$\mathrm{Spin}^c(W) \to \mathrm{Spin}^c(W^\circ)$$

is contained inside $\mathrm{Spin}^c_{\mathrm{tor}}(W^\circ)$. Moreover, the map is modeled on the map

$$H^2(W) \cong \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \cong \mathcal{A}.$$

It follows from the naturality of the first Chern class that $c_1(\mathfrak{s}_k|_{W^\circ}) = (2n+4k)x$

$$\begin{tikzcd}
\mathrm{Spin}^c(W) \arrow[r] \arrow[d, "c_1"] & \mathrm{Spin}^c_{\mathrm{tor}}(W^\circ) \arrow[d, "c_1"] \\
n + 2\mathbb{Z} \arrow[r, "\cdot 2"] & 2n + 4\mathbb{Z} \subseteq \mathcal{A}
\end{tikzcd}$$

The Chern classes of all spin^c structures in $\mathrm{Spin}^c_{\mathrm{tor}}(W^\circ)$ form the subset $2n+2\mathbb{Z} = 2\mathbb{Z} \subseteq \mathbb{Z} \cong \mathcal{A}$. This motivates the following definition.

DEFINITION 4.4. We define $\mathfrak{s}^\circ_k \in \mathrm{Spin}^c_{\mathrm{tor}}(W^\circ)$ to be the spin^c structure on W° that restricts to a torsion spin^c structure on Q and satisfies

$$c_1(\mathfrak{s}^\circ_k) = (2n + 2k)x.$$

REMARK 4.5. It follows from the computations that

$$\begin{gathered}\mathrm{Spin}^c(W) \to \mathrm{Spin}^c_{\mathrm{tor}}(W^\circ) \\ \mathfrak{s}_k \mapsto \mathfrak{s}^\circ_{2k}\end{gathered}$$

and also that $\mathfrak{s}^\circ_k \in \mathrm{Spin}^c_{\mathrm{tor}}(W^\circ)$ extends to a spin^c structure on W if and only if k is even.

We now want to understand the restriction of the spin^c structure $\mathfrak{s}^\circ_k$ to Y. This is done in the following lemma. Instead of W, we use $W_n = -W$ and $S^3_n(\overline{K}) = -Y$ to label the spin^c structure, so we can stick to the usual positive surgery conventions.

LEMMA 4.6. *For all $k \in \mathbb{Z}$, we have*

$$\mathfrak{s}^\circ_{2k}|_{S^3_n(\overline{K})} = \mathfrak{t}_k \quad \textit{and} \quad \mathfrak{s}^\circ_{n+2k}|_{S^3_n(\overline{K})} = \mathfrak{t}_k.$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} H^2(W_n) \arrow[rr] \arrow[dr, "\pi"'] & & \mathcal{A} \arrow[dl, "r"] \\ & H^2(S^3_n(\overline{K})) & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z} \arrow[rr, "\cdot 2"] \arrow[dr, "\pi"'] & & \mathbb{Z} \arrow[dl, "r"] \\ & \mathbb{Z}/n\mathbb{Z} & \end{tikzcd}$$

Recall that we chose isomorphisms $H^2(W_n) \cong \mathbb{Z}$ and $H^2(S^3_n(\overline{K})) \cong \mathbb{Z}/n\mathbb{Z}$ such that $\pi(1) = 1 \in \mathbb{Z}/n\mathbb{Z}$. Then

$$c_1(\mathfrak{t}_k) = \pi(c_1(\mathfrak{s}_k)) = n + 2k = 2k.$$

Since n is odd, 2 is invertible modulo n, so every spin^c structure on $S^3_n(\overline{K})$ is determined by its first Chern class.

By the naturality of the Chern class we have that, for every $k \in \mathbb{Z}$, the following diagram commutes:

$$\begin{tikzcd} c_1(\mathfrak{s}_k) \arrow[rr, mapsto] \arrow[dr, mapsto, "\pi"'] & & c_1(\mathfrak{s}^\circ_{2k}) \arrow[dl, mapsto, "r"] \\ & c_1(\mathfrak{t}_k) & \end{tikzcd} \qquad \begin{tikzcd} n + 2k \arrow[rr, mapsto, "\cdot 2"] \arrow[dr, mapsto, "\pi"'] & & 2n + 4k \arrow[dl, mapsto, dashed, "r"] \\ & 2k & \end{tikzcd}$$

From this we obtain that $\mathfrak{s}^\circ_{2k}|_{S^3_n(\overline{K})} = \mathfrak{t}_k$.

For the case of $\mathfrak{s}^\circ_{n+2k}$, recall that $c_1(\mathfrak{s}^\circ_{n+2k}) = 4n + 4k$. From the commutativity of the diagram below we deduce that $\mathfrak{s}^\circ_{n+2k}|_{S^3_n(\overline{K})} = \mathfrak{t}_k$.

$$\begin{tikzcd} 2n + 2k \arrow[rr, mapsto] \arrow[dr, mapsto, "\pi"'] & & 4n + 4k \arrow[dl, mapsto, dashed, "r"] \\ & 2k & \end{tikzcd}$$

□

4.5. *Case of $e(\widehat{F})$ Even*

When $e(\widehat{F})$ is even, $H^2(W^\circ) \cong \mathbb{Z}^h \oplus \mathbb{Z}/2\mathbb{Z}$ by Lemma 4.1. We can check that $\mathrm{Spin}^c_{\mathrm{tor}}(W^\circ)$ is an affine space over a submodule

$$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \subseteq \mathbb{Z}^h \oplus \mathbb{Z}/2\mathbb{Z},$$

where the $\mathbb{Z}$ summand is generated by a primitive element x. We can then define

$$\mathfrak{s}^\circ_k := \mathfrak{s}_k|_{W^\circ} \in \mathrm{Spin}^c_{\mathrm{tor}}(W^\circ),$$

and, if γ denotes the generator of the $\mathbb{Z}/2\mathbb{Z}$ summand, then

$$\widetilde{\mathfrak{s}^\circ_k} := \mathfrak{s}^\circ_k + \gamma \in \mathrm{Spin}^c_{\mathrm{tor}}(W^\circ).$$

We can check that $\widetilde{\mathfrak{s}^\circ_k}$ restricts to Q to a nonextendible spin^c structure $\widetilde{\mathfrak{t}}$, and to Y

to the spin^c structure $\mathfrak{t}_{k+\frac{n}{2}}$. Moreover, we have that

$$c_1(\widetilde{\mathfrak{s}^\circ_k})^2 = -\frac{(n+2k)^2}{n}.$$

Note that n is even because so is $e(\widehat{F})$, so $k+\frac{n}{2}$ is an integer.

5. A Bound for the Nonorientable Slice Genus

We now prove Theorem 1.1, which we restate here. Recall that we have defined $\varphi(K)$ to be the quantity $\min_{m\geq 0}\{m+2V_m(\overline{K})\}$.

THEOREM 1.1. *For every knot K in S^3,*

$$\gamma_4(K) \geq \frac{\sigma(K)}{2} - \varphi(K). \tag{1.3}$$

Proof. Choose an odd integer $n>0$, and let k be any integer. We denote by $[k]$ the representative for the residue class of k modulo n such that $0\leq [k] < n$. By Remark 4.5 and Lemma 4.6 the spin^c structure $\mathfrak{s}^\circ_{n+2k}$ restricts to a nonextendible spin^c structure on Q, which we denote by $\widetilde{\mathfrak{t}}$, and to $\mathfrak{t}_k$ on Y.

We apply Theorem 2.4 to the cobordism $(W^\circ, \mathfrak{s}^\circ_{n+2k})$ turned upside down, that is, seen as a cobordism from $(-Y,\mathfrak{t}_k)$ to $(-Q,\widetilde{\mathfrak{t}})$: the assumption of the injectivity of the map $H_1(Y;\mathbb{Q})\to H_1(W^\circ;\mathbb{Q})$ is automatically satisfied, since Y is a rational homology sphere. Moreover, in this case, by (2.2), $\underline{d}(Y,\mathfrak{t}) = d(Y,\mathfrak{t})$ for each $\mathfrak{t}\in \mathrm{Spin}^c(Y)$. The inequality of Theorem 2.4 then reads as follows:

$$c_1(\mathfrak{s}^\circ_{n+2k})^2 + b_2^-(W^\circ) \leq 4\underline{d}(-Q,\widetilde{\mathfrak{t}}) + 2b_1(Q) - 4d(-S^3_{-n}(K),\mathfrak{t}_k). \tag{5.1}$$

We now compute each term of equation (5.1). We have that $b_2^-(W^\circ)=1$ and $b_1(Q)=h-1$. Moreover,

$$c_1(\mathfrak{s}^\circ_{n+2k})^2 = ((4n+4k)x)^2 = -\frac{1}{4n}\cdot(4n+4k)^2 = -\frac{4}{n}\cdot(n+k)^2,$$

where we used the fact that $Q^{W^\circ}(2x,2x) = -\frac{1}{n}$.

As for the d-invariant of $S^3_{-n}(K)$, by Theorems 2.2 and 2.3 we have

$$d(-S^3_{-n}(K),\mathfrak{t}_k) = d(S^3_n(\overline{K}),\mathfrak{t}_k) = -\frac{n-(2[k]-n)^2}{4n} - 2\max\left\{\overline{V}_{[k]}, \overline{V}_{n-[k]}\right\},$$

where we set $\overline{V}_i := V_i(\overline{K})$.

Finally, by [13, Theorem 5.1] and Theorem 2.2 we have that

$$\underline{d}(-Q,\widetilde{\mathfrak{t}}) \leq -d_t(Q,\widetilde{\mathfrak{t}}) = -\left(\frac{e(\widehat{F})-2}{4}+a\right) \leq -\frac{e-n-2}{4}.$$

Therefore, equation (5.1) becomes

$$\begin{aligned}-\frac{4}{n}\cdot(n+k)^2+1 \leq{}& \frac{n-(2[k]-n)^2}{n}\\ &+8\max\left\{\overline{V}_{[k]},\overline{V}_{n-[k]}\right\} - (e-n-2) + 2h - 2,\end{aligned}$$

which can be re-written as follows:

$$2h + 8\max\left\{\overline{V}_{[k]}, \overline{V}_{n-[k]}\right\} \geq e - n - \frac{4(n+k)^2 - (2[k]-n)^2}{n}. \tag{5.2}$$

Given a fixed integer $m \geq 0$, it is not difficult to check that the best bound for h coming from equation (5.2) and involving $\overline{V}_m$ is obtained by setting $n = 2m + 2j + 1$ and $k = -n \pm m$ (where j is an arbitrary non-negative integer). The bound for h that we obtain in this case is then:

$$h \geq \frac{e}{2} - 2m - 4\overline{V}_m. \tag{5.3}$$

By taking the maximum over $m \geq 0$, we obtain the generalisation of (3.1), namely

$$b_1(F) \geq \frac{e(F)}{2} - 2\varphi(K). \tag{5.4}$$

By combining it with equation (3.2) as in [2], we obtain

$$\gamma_4(K) \geq \frac{\sigma(K)}{2} - \varphi(K). \tag{5.5}$$

□

REMARK 5.1. By definition, $\varphi(K) = \min\{m + 2V_m(\overline{K})\}$, hence we obtain:

$$\gamma_4(K) \geq \frac{\sigma(K)}{2} - m - 2V_m(\overline{K}). \tag{5.6}$$

By setting $m = 0$, we obtain exactly Batson's inequality (1.1).

REMARK 5.2. For every $m \geq 0$, the bound in equation (5.6) is sharp in the sense that, for each m, there exists a knot K_m such that $\gamma_4(K_m) = \frac{\sigma(K_m)}{2} - m - 2V_m(\overline{K_m})$. The knot $K_0 = T_{3,-4}$ exhibits that the inequality is sharp for $m = 0$, as already shown by Batson [2].

For $m \geq 1$, consider the torus knot $K = T_{3,-5}$, whose signature is 8. Since $\overline{K} = T_{3,5}$ is a *positive* torus knot, hence an L-space knot, the invariants $V_i(T_{3,5})$ coincide with the torsion coefficients [18, Corollary 7.5]:

$$V_i(\overline{K}) = \sum_{j>0} j a_{j+i},$$

where

$$\Delta_{\overline{K}}(t) = a_0 + \sum_{j>0} a_j(t^j + t^{-j})$$

is the Alexander polynomial of $\overline{K}$. We can explicitly compute that, for $\overline{K} = T_{3,5}$,

$$\Delta_{T_{3,5}}(t) = t^4 - t^3 + t - 1 + t^{-1} - t^{-3} + t^{-4}.$$

It follows that $V_1(\overline{K}) = 1$ and that equation (5.6) for $m = 1$ gives

$$\gamma_4(K) \geq \frac{8}{2} - (1+2) = 1.$$

Since $\overline{K}$ bounds a Moebius band in B^4, as shown in Figure 2 (see also [2, Section 5]), it follows that (5.6) is sharp for $m = 1$.

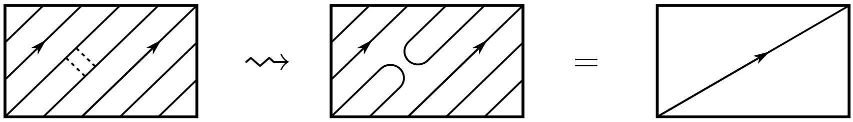

Figure 2 A nonorientable cobordism of genus 1 from $T_{3,5}$ to the unknot. The rectangle represents a torus, which is embedded in S^3 in the standard way. The (nonoriented) band surgery above carries $T_{3,5}$ to the unknot.

For all $m > 1$, consider the knot mK, the connected sum of m copies of K. Recall from [5, Proposition 6.1] that the sequence $\{V_i(\cdot)\}_i$ satisfies the following subadditivity property: $V_{j+l}(J\#L) \leq V_j(J)\#V_l(L)$ for each pair (j,l) of nonnegative integers and each pair (J,L) of knots. By subadditivity of γ_4, subadditivity of the V_i, and additivity of the signature we obtain

$$\begin{aligned} m = m\gamma_4(K) \geq \gamma_4(mK) &\geq \frac{\sigma(mK)}{2} - (m + 2V_m(m\overline{K})) \\ &\geq m\left(\frac{\sigma(K)}{2} - (1 + 2V_1(\overline{K}))\right) = m. \end{aligned}$$

It follows that here all the inequalities are in fact *equalities* and that therefore (5.6) is sharp for every $m \geq 1$.

REMARK 5.3. In the proof of Theorem 1.1, we only considered surgery with some odd framing $n > 0$. If we considered the case of even n and applied Theorem 2.4 to the torsion spin^c structure $\widetilde{\mathfrak{s}_k^\circ}$ (defined in Section 4.5), we would have obtained exactly the same bound as equation (5.6) for all $m \geq 0$.

6. Comparison to Other Bounds

In this section, we study some properties of the functions φ and ω defined in the Introduction and discuss the relationship between the bounds given by (1.1), (1.2), and (1.3).

PROPOSITION 6.1. *The invariant φ is a concordance invariant with values in the nonnegative integers. It has the following properties*:

(1) $0 \leq \varphi(K) \leq \min\{\nu^+(\overline{K}), 2V_0(\overline{K})\}$;
(2) $\varphi(K) = 0$ *if and only if* $V_0(\overline{K}) = \nu^+(\overline{K}) = 0$; *in particular, if K is slice, then* $\varphi(K) = 0$;
(3) *if there is an orientable genus-g cobordism from K_1 to K_2, then* $|\varphi(K_1) - \varphi(K_2)| \leq g$;
(4) *if K_+ is obtained from K_- by performing a crossing change from negative to positive, then* $\varphi(K_-) - 1 \leq \varphi(K_+) \leq \varphi(K_-)$;
(5) *for every two knots* K_1, K_2, $\varphi(K_1\#K_2) \leq \varphi(K_1) + \varphi(K_2)$.

We remark here that, in particular, φ, much like ν^+, and by contrast with σ and υ, does *not* induce a homomorphism from the concordance group to the integers.

Proof of Proposition 6.1. The sequence $\{V_i(K)\}_i$ is a concordance invariant, and hence so is φ; moreover, the quantity $m + 2V_m(\overline{K})$ is a nonnegative integer for each m, and hence so is $\varphi(K)$.

(1) When $m = 0$, $m + 2V_m(\overline{K}) = 2V_0(\overline{K})$, whereas for $m = \nu^+(\overline{K})$, $m + 2V_m(\overline{K}) = \nu^+(\overline{K})$. By definition, $\varphi(K) \le 2V_0(\overline{K})$ and $\varphi(K) \le \nu^+(\overline{K})$.

(2) Observe that $m + 2V_m(\overline{K})$ is always strictly positive if $m > 0$; hence, if $\varphi(K) = \min_m\{m + 2V_m(\overline{K})\} = 0$, then the minimum can only be attained at $m = 0$, and, in that case, $V_0(\overline{K}) = 0$, which implies $\nu^+(\overline{K}) = 0$. The converse follows from point (1).

   When K is slice, $\nu^+(\overline{K}) = 0$, and hence $\varphi(K)$ vanishes, too.

(3) By [5, Lemma 5.1] we have that, under the given assumptions, $V_{m+g}(\overline{K_1}) \le V_m(\overline{K_2})$ for each nonnegative integer m. It follows that $m + g + 2V_{m+g}(\overline{K_1}) \le m + 2V_m(\overline{K_2}) + g$, and hence, minimizing over m, we have

$$\varphi(K_1) \le \min_{m' \ge g}\{m' + 2V_{m'}(\overline{K_2})\} \le \varphi(K_2) + g.$$

   Exchanging the roles of K_1 and K_2, we obtain the symmetric inequality.

(4) Observe that there is a genus-1 cobordism from K_- to K_+, obtained by smoothing the double point in the trace of the crossing change homotopy. Thus, point (3) shows that $\varphi(K_-) - 1 \le \varphi(K_+)$. Using [6, Theorem 6.1] we also obtain

$$V_m(\overline{K_+}) \le V_m(\overline{K_-}),$$

   from which, for each $m \ge 0$,

$$m + 2V_m(\overline{K_+}) \le m + 2V_m(\overline{K_-}),$$

   and minimizing over all values of m yields the desired inequality.

(5) For all nonnegative integers k, l, $V_{k+l}(\overline{K_1 \# K_2}) \le V_k(\overline{K_1}) + V_l(\overline{K_2})$ by [5, Proposition 6.1], and hence

$$\begin{aligned}
\varphi(K_1 \# K_2) &= \min_n\{n + 2V_n(\overline{K_1 \# K_2})\} \\
&\le \min_n \min_{k+l=n}\{k + l + 2V_k(\overline{K_1}) + 2V_l(\overline{K_2})\} \\
&= \min_k\{k + 2V_k(\overline{K_1})\} + \min_l\{l + 2V_l(\overline{K_2})\} \\
&= \varphi(K_1) + \varphi(K_2).
\end{aligned}$$

□

We will compare our bound with (1.2) obtained by Ozsváth–Stipsicz–Szabó, and in order to do so, we need to compare $\upsilon(K)$ with $\varphi(K)$. We say that a knot is *Floer-thin* if its knot Floer homology is supported on the diagonal $i - j = -\tau(K)$.

PROPOSITION 6.2. *When K is a Floer-thin knot with $\tau(K) \ge 0$ or an L-space knot, then $\varphi(\overline{K}) = -\upsilon(K)$ and $\varphi(K) = 0$.*

In particular, the bound given by (5.6) for both K and $\overline{K}$ is at most as strong as that given by υ when K is an L-space knot or an alternating knot.

Proof. Recall that, for a Floer-thin knot K with $\tau(K) = \pm n$, we have $V_i(K) = V_i(T_{2,\pm(2n+1)})$ [1, equation (8)], and hence $\varphi(K) = \varphi(T_{2,\pm(2n+1)})$. Analogously, it follows from [16, Theorem 1.14] that $\upsilon(K) = \upsilon(T_{2,\pm(2n+1)})$. It follows that it suffices to prove the statement for L-space knots.

When K is an L-space knot, then a direct computation from the knot Floer complex shows that $V_i(\overline{K}) = 0$ for every i; hence, $\varphi(K) = 0$. On the other hand, Borodzik and Hedden [6, Proposition 4.6] have shown that

$$\upsilon(K) = \Upsilon_K(1) = -\min_n \{n + 2V_n(K)\} = -\varphi(\overline{K}),$$

as desired. □

In the case of Floer-thin knots, we can actually say more about φ.

PROPOSITION 6.3. *If K is a Floer-thin knot with $\tau(K) \geq 0$, then we have*

$$\varphi(\overline{K}) = \nu^+(K) = \tau(K) = -\upsilon(K).$$

If, additionally, K is quasi-alternating, then $\varphi(\overline{K}) = -\sigma(K)/2$, and in this case the bounds (1.2) and (1.3), applied to K and $\overline{K}$, yield

$$\gamma_4(K) \geq 0.$$

Proof. By [1, equation (8)] we know that the minimum of $\{m + 2V_m(K)\}$ is attained at $m = \tau(K) = \nu^+(K)$. This implies at once that $\varphi(\overline{K}) = \tau(K)$. The equality with $-\upsilon(K)$ follows from Proposition 6.2.

When K is quasi-alternating, then $\tau(K) = -\sigma(K)/2$, and the second part of the statement readily follows. □

In many instances, the bound given by υ is better than that given by φ; this is true, for example, for many knots of the form $K_1\#\overline{K_2}$, where K_1 and K_2 are L-space knots.

EXAMPLE 6.4. Consider the two knots $K_1 = T_{2,3}$, $K_2 = T_{5,6}$, and let $K = K_1\#\overline{K}_2$. We compute $\sigma(K_1) = -2$, $\sigma(K_2) = -16$, $\upsilon(K_1) = -1$, and $\upsilon(K_2) = -6$. Using the techniques from [12] as in [5], we can also compute $\varphi(K) = 6$ and $\varphi(\overline{K}) = 0$.

It follows that the bound given by (1.3), applied to both K and $\overline{K}$, gives $\gamma_4(K) \geq 1$, whereas the bound given by (1.2) is $\gamma_4(K) \geq 2$.

As a consequence of Proposition 6.1, we deduce the following interesting feature of φ.

COROLLARY 6.5. *The invariant $\varphi(K)$ is subadditive. In particular, the following identity holds:*

$$\lim_{n\to\infty} \frac{1}{n}\varphi(nK) = \inf_n \frac{1}{n}\varphi(nK).$$

Proof. By property (5) of Proposition 6.1 the function $n \mapsto \varphi(nK)$ is subadditive in the sense that $\varphi(aK + bK) \leq \varphi(aK) + \varphi(bK)$ for every $a, b \geq 0$. The existence of the limit follows from Fekete's lemma [10]. □

DEFINITION 6.6. We call $\omega(K) = \lim_n \frac{1}{n}\varphi(nK)$.

We now introduce the *stable* nonorientable 4-genus $\gamma_4^{\mathrm{st}}(K)$ of K, that is, the limit $\lim_{n\to\infty} \frac{1}{n}\gamma_4(nK)$. Notice that the limit exists since the sequence $\{\gamma_4(nK)\}_n$ is subadditive and that $\gamma_4^{\mathrm{st}}(K) \leq \gamma_4(K)$.

THEOREM 6.7. *The invariant $\omega(K)$ is a concordance invariant of K, and it descends to a subadditive homogeneous function $\omega\colon \mathcal{C} \to \mathbb{R}_{\geq 0}$. Additionally*:

(1) $\gamma_4^{\mathrm{st}}(K) \geq \frac{\sigma(K)}{2} - \omega(K)$;
(2) *if there is an orientable genus-g cobordism between K_1 and K_2, then* $|\omega(K_1) - \omega(K_2)| \leq g$;
(3) *if there is a crossing change* (*from negative to positive*) *from K_- to K_+, then* $\omega(K_-) - 1 \leq \omega(K_+) \leq \omega(K_-)$.

REMARK 6.8. It follows immediately from subadditivity that if the inequality in Theorem 1.1 is an equality for each of the two knots K and L, then the inequality is also an equality for $K\#L$, and $\gamma_4(K\#L) = \gamma_4(K) + \gamma_4(L)$. In particular, if the inequality in Theorem 1.1 is an equality for a knot K, then $\gamma_4(nK) = n\gamma_4(K)$ for each n, and $\gamma_4^{\mathrm{st}}(K) = \gamma_4(K)$. Analogously, if the inequality in Theorem 6.7(1) is an equality for both K and L, then it is also sharp for $K\#L$, and $\gamma_4^{\mathrm{st}}(K\#L) = \gamma_4^{\mathrm{st}}(K) + \gamma_4^{\mathrm{st}}(L)$.

As remarked for φ before, ω is not a homomorphism, since it takes only nonnegative values. Note also that ω is not identically 0, since, by Proposition 6.3 applied to nK for all $n \geq 0$, $\omega(K)$ coincides with $\sigma(K)/2$ for Floer-thin knots with positive signature.

Also, by definition, $\omega(K) \leq \varphi(K)$, and in particular the bound for $\gamma_4^{\mathrm{st}}(K)$ given by ω can be *better* than that given by φ on $\gamma_4(K)$ (see Proposition 7.1 for an example). This is in contrast with the bound given, for example, by τ, s, or ν^+ on the stable orientable slice genus: the first two are linear, whereas the third is *sub*linear in K [5, Theorem 1.4].

Proof of Theorem 6.7. The invariant ω is a concordance invariant, since φ is, and it takes nonnegative values, since φ does. Moreover, it is subadditive by construction:

$$\begin{aligned}\omega(K\#L) &= \lim_n \left\{ \frac{1}{n}\varphi(n(K\#L)) \right\} \leq \lim_n \left\{ \frac{1}{n}(\varphi(nK) + \varphi(nL)) \right\} \\ &= \lim_n \left\{ \frac{1}{n}\varphi(nK) \right\} + \lim_n \left\{ \frac{1}{n}\varphi(nL) \right\} = \omega(K) + \omega(L),\end{aligned}$$

where the inequality follows from the subadditivity of φ (Property (5) of Proposition 6.1).

It is also homogeneous, in the sense that $\omega(nK) = n\omega(K)$:

$$\begin{aligned}\omega(nK) &= \lim_m \frac{1}{m}\varphi(mnK)\\ &= n\lim_m \frac{1}{mn}\varphi(mnK)\\ &= n\lim_{m'} \frac{1}{m'}\varphi(m'K) = n\omega(K).\end{aligned}$$

(1) Applying (5.6) for nK, we obtain, for each $n \geq 1$:

$$\gamma_4(nK) \geq \frac{\sigma(nK)}{2} - \varphi(nK) = n\frac{\sigma(K)}{2} - \varphi(nK),$$

from which

$$\gamma_4^{\mathrm{st}}(K) = \lim_n \frac{\gamma_4(nK)}{n} \geq \frac{\sigma(K)}{2} - \lim_n \frac{\varphi(nK)}{n} = \frac{\sigma(K)}{2} - \omega(K).$$

Properties (2) and (3) follow immediately from the corresponding properties of φ stated in Proposition 6.1. □

7. An Example

An interesting feature of ω is that – by contrast with φ – it can attain noninteger values, as we shall see presently. To this end, we study an example in detail: we show that $\omega(T_{2,3} - T_{5,6}) = \frac{26}{5}$. Before doing so, we recall some facts about Krcatovich's reduced knot Floer complex.

Krcatovich [12] associates with each knot $J \subset S^3$ a reduced version of the knot Floer complex, denoted by $\underline{\mathrm{CFK}}^-(J)$. The reduced knot Floer complex for L-space knots is of a particularly simple form in that it only consists of a single *tower,* that is, it is isomorphic to $\mathbb{F}_2[U]$ as an $\mathbb{F}_2[U]$-module, but *not* as a graded module (see [12, Corollary 4.2]).

Krcatovich also observed that if we are only concerned with correction terms, then the connected sum of two L-space knots behaves as an L-space knot [12, Example 2]; more specifically, he showed that if K and K' are L-space knots, then $\underline{\mathrm{CFK}}^-(K\#K')$ fits in a short exact sequence of complexes:

$$0 \to T \to \underline{\mathrm{CFK}}^-(K\#K') \to A \to 0,$$

where T is a tower, and A is acyclic. In this case, we write $\underline{\mathrm{CFK}}^-(K\#K') \approx T$; moreover, if C is another chain complex such that $C \approx T$, then we also write $\underline{\mathrm{CFK}}^-(K\#K') \approx C$. In Krcatovich's terminology, $\underline{\mathrm{CFK}}^-(K\#K')$ has a *representative staircase,* which is determined by T; conversely, the staircase determines T and the collection $\{V_i(K\#K')\}_i$. Moreover, for any other knot L, we can use T as a substitute for $\underline{\mathrm{CFK}}^-(K\#K')$ to compute $\mathrm{CFK}^-(K\#K'\#L)$ in the sense that there is a filtered quasi-isomorphism

$$T \otimes \mathrm{CFK}^-(L) \cong \underline{\mathrm{CFK}}^-(K\#K') \otimes \mathrm{CFK}^-(L).$$

PROPOSITION 7.1. *Let $K = T_{2,3} - T_{5,6}$. Then $\omega(K) = \frac{26}{5} < \varphi(K) = 6$. Moreover, $\omega(K) < \frac{\varphi(nK)}{n}$ for all $n \in \mathbb{Z}_{>0}$, so the limit in Definition 6.6 is not attained at any n.*

Before proving the proposition, recall that, in the case of torus knots $T_{p,q}$, the representative staircase is determined by the arithmetics of p and q (compare also with [12, Example 2] and [7, Section 5]). In what follows, we will be concerned with the connected sum $nT_{5,6}$ of n copies of $T_{5,6}$, and in this case the result reads:

$$\underline{\mathrm{CFK}}^-(nT_{5,6}) \approx \underline{\mathrm{CFK}}^-(T_{5,5n+1}),$$

that is, the representative staircase for $nT_{5,6}$ is the staircase of $T_{5,5n+1}$.

We will also need a lemma about $nT_{2,3}$. We expect that this is true in wider generality.

LEMMA 7.2. *For each positive integer n, the complex $\mathrm{CFK}^\infty(\pm nT_{2,3})$ is filtered chain homotopy equivalent to $\mathrm{CFK}^\infty(\pm T_{2,2n+1}) \oplus A_{\pm n}$, where $A_{\pm n}$ is an acyclic complex over $\mathbb{F}_2[U]$.*

Proof. It suffices to prove the statement for $\mathrm{CFK}^\infty(nT_{2,3})$, since the corresponding statement for $\mathrm{CFK}^\infty(-nT_{2,3})$ follows by taking duals: in fact, $\mathrm{CFK}^\infty(\overline{K})$ is isomorphic to the dual of $\mathrm{CFK}^\infty(K)$, and taking duals preserves direct sums and acyclicity.

We will now prove the statement for $\mathrm{CFK}^\infty(nT_{2,3})$ by induction on n: recall that $\mathrm{CFK}^\infty((n+1)T_{2,3})$ is filtered quasi-isomorphic to $\mathrm{CFK}^\infty(nT_{2,3}) \otimes \mathrm{CFK}^\infty(T_{2,3})$ and that $\mathrm{CFK}^\infty(T_{2,3})$ is filtered quasi-isomorphic to $(\mathbb{F}_2[U, U^{-1}]a \oplus \mathbb{F}_2[U, U^{-1}]b \oplus \mathbb{F}_2[U, U^{-1}]c, \partial_1)$, where $\partial_1 b = Ua + c$ and a and c are cycles; moreover, the Alexander gradings of the generators are $A(a) = 1$, $A(b) = 0$, $A(c) = -1$.

By induction we can assume that $\mathrm{CFK}^\infty(nT_{2,3}) = \mathrm{CFK}^\infty(T_{2,2n+1}) \oplus A_n$, where $\mathrm{CFK}^\infty(T_{2,2n+1})$ is generated over $\mathbb{F}_2[U, U^{-1}]$ by $x_1, \dots, x_{2n+1}$ and is equipped with the differential ∂_n defined by

$$\partial_n x_{2i} = Ux_{2i-1} + x_{2i+1}, \qquad \partial_n x_{2i+1} = 0,$$

and the Alexander grading is $A(x_i) = n + 1 - i$.

We observe that, whenever A is acyclic, $A \otimes C$ is acyclic for every other complex C. Therefore, to prove the theorem, it suffices to show that $\mathrm{CFK}^\infty(T_{2,2n+1}) \otimes \mathrm{CFK}^\infty(T_{2,3}) \cong \mathrm{CFK}^\infty(T_{2,2n+3}) \oplus A$, where A is acyclic.

To this end, consider the subspace V of $\mathrm{CFK}^\infty(T_{2,2n+1}) \otimes \mathrm{CFK}^\infty(T_{2,3})$ spanned by

$$V = \mathrm{Span}_{\mathbb{F}_2[U,U^{-1}]}\{x_1 a, x_1 b, x_i c\},$$

where we drop the $\otimes$ between generators to ease readability, so that $x_1 a$ really means $x_1 \otimes a$. It is easy to check that V is in fact a subcomplex of $\mathrm{CFK}^\infty(T_{2,2n+1}) \otimes \mathrm{CFK}^\infty(T_{2,3})$ and that V is indeed isomorphic to $\mathrm{CFK}^\infty(T_{2,2n+3})$. In fact, an explicit isomorphism is given by $x_1 a \mapsto x_1$, $x_1 b \mapsto x_2$, $x_i c \mapsto x_{i+2}$.

We claim that V has a complement, which is the direct sum of copies of rank-4 subspaces W_{2i} for $i = 1, \dots, n$:

$$W_{2i} = \operatorname{Span}_{\mathbb{F}_2[U,U^{-1}]}\{x_{2i}b, x_{2i-1}b + x_{2i}a, x_{2i+1}b + x_{2i}c, x_{2i+1}a + x_{2i-1}c\}.$$

It is easy to prove that W_{2i} is in fact an acyclic subcomplex for each i and that the W_{2i}, together with V, spans all of $\mathrm{CFK}^\infty(T_{2,2n+1}) \otimes \mathrm{CFK}^\infty(T_{2,3})$.

Moreover, since the ranks of V and W_{2i} add up to the rank of $\mathrm{CFK}^\infty(T_{2,2n+1}) \otimes \mathrm{CFK}^\infty(T_{2,3})$, this is in fact a direct sum decomposition of complexes. Since the W_{2i} are acyclic, we have exhibited the desired decomposition. $\square$

We can now turn to the proof of Proposition 7.1.

Proof of Proposition 7.1. Let $K_1 = T_{2,3}$, $K_2 = T_{5,6}$, and $K = K_1 - K_2$. Note that $\varphi(K) = 6$ was already observed in Example 6.4. Let now $L_n = nK = nK_1 - nK_2$, and $n = 5\ell$. We will prove that, for all positive integers ℓ, we have

$$\varphi(L_{5\ell}) = 26\ell + 1.$$

This implies at once that $\omega(K) = \lim_n \frac{\varphi(L_n)}{n} = \frac{26}{5}$, and that $\varphi(L_{5\ell}) > \omega(K) \cdot 5\ell$ for each ℓ. Moreover, by definition, for each n,

$$\varphi(L_n) \geq \frac{26}{5} n$$

for all positive integers n; since the right-hand side is an integer only if n is a multiple of 5, the inequality is strict also for all n not divisible by 5, and hence the limit is never attained.

We now set out to prove that $\varphi(L_{5\ell}) = 26\ell + 1$.

Since $\underline{\mathrm{CFK}}^-(nK_2) \approx \underline{\mathrm{CFK}}^-(T_{5,5n+1})$, we can use Lemma 7.2 and results from [5] to compute the invariants $V_i(nK_2 - nK_1)$, treating nK_2 as $T_{5,5n+1}$ and $-nK_1$ as $-T_{2,2n+1}$. Indeed, let $J_i = 5\ell K_i$ for $i = 1, 2$.

Given a semigroup $\Gamma \subseteq \mathbb{N} = \{0, 1, \dots\}$, we denote by $\Gamma(\cdot)$ its *enumerating function,* that is, the unique strictly increasing function

$$\Gamma \colon \mathbb{N} \to \mathbb{N}$$

that is surjective on Γ. Note that $\Gamma(0) = 0$. Given an integer x, we denote $(x)_+ = \max\{0, x\}$. Since $\mathrm{CFK}^\infty(-nT_{2,3})$ is, up to an acyclic summand, $\mathrm{CFK}^\infty(-T_{2,2n+1})$, we can apply [5, Theorem 3.1 and Remark 3.3] and obtain

$$\begin{aligned} \nu_v^+(5\ell\overline{K}) &:= \min\left\{i \mid V_i(5\ell\overline{K}) \leq v\right\} \\ &= \left(\max_{k\geq 0}\left\{g(J_2) - g(J_1) + \Gamma_{J_1}(k) - \Gamma_{J_2}(k+v)\right\}\right)_+, \end{aligned}$$

where $\Gamma_{J_1}(\cdot)$ and $\Gamma_{J_2}(\cdot)$ are the enumerating functions associated with the semigroups

$$\Gamma_{J_1} = \langle 2, 10\ell + 1\rangle \quad \text{and} \quad \Gamma_{J_2} = \langle 5, 25\ell + 1\rangle.$$

The genera of the knots J_1 and J_2 are respectively 5ℓ and 50ℓ, so the formula for ν_v^+ becomes

$$\nu_v^+(\overline{L_{5\ell}}) = \left(45\ell - \min_{k\geq 0}\left\{\Gamma_{J_2}(k+v) - \Gamma_{J_1}(k)\right\}\right)_+. \tag{7.1}$$

Note that, with this notation, we have

$$\varphi(L_{5\ell}) = \min_{v\geq 0}\left\{\nu_v^+(\overline{L_{5\ell}}) + 2v\right\}, \tag{7.2}$$

which we are now going to compute.

The enumerating functions $\Gamma_{J_1}(\cdot)$ and $\Gamma_{J_2}(\cdot)$ can be expressed in the following equations:

$$\Gamma_{J_1}(k) = \begin{cases} 2k, & 0\leq k\leq 5\ell, \\ 5\ell + k, & k\geq 5\ell, \end{cases}$$

$$\Gamma_{J_2}(k) = \begin{cases} 5k, & 0\leq k\leq 5\ell, \\ 25\ell + 5\lfloor\frac{k-5\ell}{2}\rfloor + [k-5\ell]_2, & 5\ell\leq k\leq 15\ell, \\ 50\ell + 5\lfloor\frac{k-15\ell}{3}\rfloor + [k-15\ell]_3, & 15\ell\leq k\leq 30\ell, \\ 75\ell + 5\lfloor\frac{k-30\ell}{4}\rfloor + [k-30\ell]_4, & 30\ell\leq k\leq 50\ell, \\ 50\ell + k, & k\geq 50\ell. \end{cases}$$

Note that in equation (7.1) we can in fact take the minimum over $0\leq k\leq 5\ell$, because for $k\geq 5\ell$ the function $\Gamma_{J_1}(k)$ increases at a lesser or equal rate than any translate of Γ_{J_2}: specifically, $\Gamma_{J_1}(k+j) - \Gamma_{J_1}(k) = j \leq \Gamma_{J_2}(k+v+j) - \Gamma_{J_2}(k+v)$. Therefore

$$\nu_v^+(\overline{L_{5\ell}}) = \left(45\ell - \min_{0\leq k\leq 5\ell}\left\{\Gamma_{J_2}(k+v) - \Gamma_{J_1}(k)\right\}\right)_+.$$

Now we return to the proof of Proposition 7.1. Recall that we want to prove that $\varphi(L_{5\ell}) = 26\ell + 1$. By (7.2) we have

$$\varphi(L_{5\ell}) = \min_{v\geq 0}\left\{\nu_v^+(\overline{L_{5\ell}}) + 2v\right\}.$$

As further shown in Lemma 7.3, the choice $v = 13\ell$ gives $\nu_v^+(\overline{L_{5\ell}}) + 2v = 26\ell + 1$. Moreover, it also follows from Lemma 7.3 that $V_0(\overline{L_{5\ell}}) = 13\ell + 1$, and hence choosing $v\geq 13\ell + 1$ yields $2v \geq 26\ell + 2 > 26\ell + 1$.

We now distinguish between $v\leq 5\ell - 1$ and $v\geq 5\ell$. By Lemma 7.4, for $v\in [0, 5\ell - 1]$, we have

$$\nu_v^+(\overline{L_{5\ell}}) + 2v = 45\ell - 3v \geq 45\ell - 15\ell + 3 > 26\ell + 1;$$

by Lemma 7.5, on the other hand, for $v\in[5\ell, 13\ell - 1]$, we have

$$\nu_v^+(\overline{L_{5\ell}}) + 2v \geq 2(13\ell - v) + 1 + 2v = 26\ell + 1.$$

This shows that $\varphi(L_{5\ell}) = 26\ell + 1$, as desired. □

LEMMA 7.3. $\nu_{13\ell}^+(\overline{L_{5\ell}}) = 1$.

Proof. Note that $k+13\ell \in [13\ell, 18\ell]$ since $k \leq 5\ell$. Therefore the difference of the enumerating functions is

$$\begin{aligned} f(k) &:= \Gamma_{J_2}(k+13\ell) - \Gamma_{J_1}(k) \\ &= \begin{cases} 45\ell + 5\lfloor \frac{k}{2} \rfloor + [k]_2 - 2k, & 0 \leq k \leq 2\ell, \\ 50\ell + 5\lfloor \frac{k-2\ell}{3} \rfloor + [k-2\ell]_3 - 2k, & 2\ell \leq k \leq 5\ell. \end{cases} \end{aligned}$$

In the first interval, $f(k+2) \geq f(k)$, whereas in the second interval, $f(k+3) \leq f(k)$. It follows that the minimum is attained for some $k \in \{0, 1, 5\ell-2, 5\ell-1, 5\ell\}$. A direct computation for these five values shows that the minimum is $45\ell - 1$, attained both at $k = 1$ and at $k = 5\ell - 1$. It follows that $\nu^+_{13\ell}(\overline{L_{5\ell}}) = 45\ell - (45\ell - 1) = 1$. $\square$

LEMMA 7.4. *For each $v = 0, \ldots, 5\ell - 1$, $\nu^+_v(\overline{L_{5\ell}}) = 45\ell - 5v$.*

Proof. Note that, since we only need to test $k \leq 5\ell$ when computing the minimum in (7.1), we can assume that, for each value of v in the statement, $k + v \leq 10\ell - 1$. Therefore the difference of the enumerating functions is

$$\begin{aligned} f(k) &:= \Gamma_{J_2}(k+v) - \Gamma_{J_1}(k) \\ &= \begin{cases} 5v + 3k, & 0 \leq k \leq 5\ell - v, \\ 25\ell + 5\lfloor \frac{k+v-5\ell}{2} \rfloor + [k+v-5\ell]_2 - 2k, & 5\ell - v \leq k \leq 5\ell. \end{cases} \end{aligned}$$

Such a function is increasing on the interval $0 \leq k \leq 5\ell - v$, and, on the second interval, it satisfies the condition $f(k+2) - f(k) \geq 1$. It follows that the minimum is attained for some $k = 0, 5\ell - v$ or $5\ell - v + 1$. A direct computation for these values shows that the minimum is $5v$, attained at $k = 0$. Therefore, $\nu^+_v(\overline{L_{5\ell}}) = 45\ell - 5v$. $\square$

LEMMA 7.5. *Let $v = 13\ell - s$ for some $0 < s \leq 8\ell$. Then $\nu^+_v(\overline{L_{5\ell}}) \geq 2s + 1$.*

Proof. Choosing $k = 0$ in equation (7.1), we obtain

$$\nu^+_v(\overline{L_{5\ell}}) \geq 45\ell - \Gamma_{J_2}(13\ell - s).$$

Since $13\ell - s \in [5\ell, 13\ell] \subseteq [5\ell, 15\ell]$, we have

$$\Gamma_{J_2}(13\ell - s) = 45\ell + 5\lfloor -\frac{s}{2} \rfloor + [s]_2.$$

If $s \geq 2$ is even, then $\Gamma_{J_2}(13\ell - s) = 45\ell - \frac{5}{2}s \leq 45\ell - 2s - 1$. If s is odd, then $\Gamma_{J_2}(13\ell - s) = 45\ell - \frac{5}{2}(s+1) + 1 \leq 45\ell - 2s - 1$. In both cases, we have $\Gamma_{J_2}(13\ell - s) \leq 45\ell - 2s - 1$, so we obtain

$$\nu^+_v(\overline{L_{5\ell}}) \geq 45\ell - \Gamma_{J_2}(13\ell - s) \geq 2s + 1. \qquad \square$$

With techniques similar to those used in Proposition 7.1, we can show that ω attains many other positive noninteger values. We conclude with a couple of questions concerning the image of ω and the stable nonorientable slice genus.

QUESTION 7.6. Is $\mathbb{Q}_{\geq 0} \subseteq \mathrm{im}(\omega)$? Can ω take irrational values? Can γ_4^{st} attain noninteger values? What is the image of γ_4^{st}?

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References

[1] P. Aceto and M. Golla, *Dehn surgeries and rational homology balls,* Alg. Geom. Topol. 17 (2017), 487–527.

[2] J. Batson, *Nonorientable slice genus can be arbitrarily large,* Math. Res. Lett. 21 (2014), no. 3, 423–436.

[3] ———, *Obstructions to slicing knots and splitting links,* Ph.D. thesis, MIT, 2014.

[4] S. Behrens and M. Golla, *Heegaard Floer correction terms, with a twist,* Quantum Topol. 9 (2018), no. 1, 1–37.

[5] J. Bodnár, D. Celoria, and M. Golla, *A note on cobordisms of algebraic knots,* Algebr. Geom. Topol. 17 (2017), no. 4, 2543–2564.

[6] M. Borodzik and M. Hedden, *The υ function of L-space knots is a Legendre transform,* Math. Proc. Cambridge (2017, to appear), arXiv:1505.06672.

[7] M. Borodzik and C. Livingston, *Heegaard Floer homology and rational cuspidal curves,* Forum Math. Sigma 2 (2014), e28.

[8] F. Deloup and G. Massuyeau, *Quadratic functions and complex spin structures on three-manifolds,* Topology 44 (2005), no. 3, 509–555.

[9] C. McA. Gordon, R. A. Litherland, *On the signature of a link,* Invent. Math. 47 (1978), no. 1, 53–69.

[10] M. Fekete, *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten,* Math. Z. 17 (1923), no. 1, 228–249.

[11] J. Hom and Z. Wu, *Four-ball genus and a refinement of the Ozsváth–Szabó tau-invariant,* J. Symp. Geom. 14 (2016), 305–323.

[12] D. Krcatovich, *The reduced knot Floer complex,* Topology Appl. 194 (2015), 171–201.

[13] A. S. Levine, D. Ruberman, and S. Strle, *Non-orientable surfaces in homology cobordisms,* Geom. Topol. 19 (2015), no. 1, 439–494, with an appendix by Ira M. Gessel.

[14] T. Lidman, *On the infinity flavor of Heegaard Floer homology and the integral cohomology ring,* Comment. Math. Helv. 88 (2013), no. 4, 875–898.

[15] Y. Ni and Z. Wu, *Cosmetic surgeries on knots in S^3,* J. Reine Angew. Math. 2015 (2015), no. 706, 1–17.

[16] P. S. Ozsváth, A. I. Stipsicz, and Z. Szabó, *Concordance homomorphisms from knot Floer homology,* Adv. Math. 315 (2017), 366–426.

[17] ———, *Unoriented knot Floer homology and the unoriented four-ball genus,* Int. Math. Res. Notices 17 (2017), 5137–5181.

[18] P. S. Ozsváth and Z. Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary,* Adv. Math. 173 (2003), no. 2, 179–261.

[19] ———, *Knot Floer homology and integer surgeries,* Algebr. Geom. Topol. 8 (2008), no. 1, 101–153.

[20] J. A. Rasmussen, *Lens space surgeries and a conjecture of Goda and Teragaito,* Geom. Topol. 8 (2004), no. 3, 1013–1031.

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