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Smooth Rational Curves on Singular Rational Surfaces

Ziquan Zhuang

Abstract. We classify all complex surfaces with quotient singularities that do not contain any smooth rational curves under the assumption that the canonical divisor of the surface is not pseudo-effective. As a corollary, we show that if X is a log del Pezzo surface such that, for every closed point $p \in X$, there is a smooth curve (locally analytically) passing through p, then X contains at least one smooth rational curve.

1. Introduction

Let X be a projective rationally connected variety defined over $\mathbb{C}$. When X is smooth, it is well known that there are many smooth rational curves on X: if $\dim X = 2$, then X is isomorphic to a blowup of either $\mathbb{P}^2$ or a ruled surface $\mathbb{F}_e$; if $\dim X \geq 3$, then any two points on X can be connected by a very free rational curve, that is, the image of $f : \mathbb{P}^1 \to X$ such that f^*T_X is ample, and a general deformation of f is a smooth rational curve on X (for the definition of rationally connected variety and the mentioned properties, see [Kol96]). It is then natural to ask about the existence of smooth rational curves on X when X is singular. In this paper, we study this problem on rational surfaces.

There are some possible obstructions to the existence of smooth rational curves. It can happen that there is no smooth curve germ passing through the singular points of X (e.g. when X has an E_8 singularity) whereas the smooth locus of X contains no rational curves at all (this could be the case when the smooth locus is of log Calabi–Yau or log general type), and then we will not be able to find any smooth rational curves on X. Hence, to produce smooth rational curves on X, we need some control on the singularities of X and the "negativity" of its smooth locus. We show that these restrictions are also sufficient, in particular, we prove the following theorem, which is one of the main results of this paper.

Theorem. *Let X be a surface with only quotient singularities. Assume that*

(1) *K_X is not pseudo-effective*;
(2) *For every closed point $p \in X$, there is a smooth curve* (*locally analytically*) *passing through p.*

Then X contains at least one smooth rational curve.

In fact, we prove something stronger. By studying various adjoint linear systems on rational surfaces, we show that condition (1), combined with the nonexistence

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of smooth rational curves, has strong implication on the divisor class group of X (Proposition 2.5), which allows us to classify all surfaces with quotient singularities that satisfy condition (1) but do not contain smooth rational curves (Theorem 2.15). It turns out that all such surfaces have an E_8 singularity, which is the only surface quotient singularity that does not admit a smooth curve germ.

This paper is organized as follows. In Section 2, we study the existence of smooth rational curves on rational surfaces with quotient singularities whose anticanonical divisor is pseudo-effective but not numerically trivial and give the proof of the main result. In Section 3, we study some examples and propose a few questions. In particular, we construct some rational surfaces with quotient singularity and numerically trivial canonical divisor that contain no smooth rational curves.

Conventions

We work over the field $\mathbb{C}$ of complex numbers. Unless mentioned otherwise, all varieties in this paper are assumed to be proper and all surfaces normal. A surface X is called log del Pezzo if there is an effective $\mathbb{Q}$-divisor D on X such that (X, D) is klt and $-(K_X + D)$ is ample.

2. Proof of Main Theorem

In this section, we classify all surfaces with quotient singularities containing no smooth rational curves under the assumption that the anticanonical divisor is pseudo-effective but not numerically trivial. As a corollary, we will see that if X is a log del Pezzo surface that has no E_8 singularity (since E_8 is the only surface quotient singularity whose fundamental cycle contains no reduced component, by [GSLJ94] this is equivalent to saying that for every point $p \in X$, there is a smooth curve germ passing through p), then X contains at least one smooth rational curve.

We start by introducing a few results on adjoint linear systems that we frequently use to identify smooth rational curves on a surface.

LEMMA 2.1. *Let X be a smooth rational surface, and D a reduced divisor on X. Then $|K_X + D| = \emptyset$ if and only if every connected component of D is a rational tree (i.e. every irreducible component of D is a smooth rational curve, and the dual graph of D is a disjoint union of trees).*

Proof. We have an exact sequence $0 \to \omega_X \to \omega_X(D) \to \omega_D \to 0$ that induces a long exact sequence

$$\cdots \to H^0(X, \omega_X) \to H^0(X, \omega_X(D)) \to H^0(D, \omega_D) \to H^1(X, \omega_X) \to \cdots .$$

Since X is a smooth rational surface, $H^0(X, \omega_X) = H^1(X, \omega_X) = 0$; hence $H^0(X, \omega_X(D)) = 0$ if and only if $H^0(D, \omega_D) = 0$. We now show that the latter condition holds if and only if every connected component of D is a rational tree. By doing this we may assume that D is connected. Since D is reduced,

$H^0(D, \omega_D) = 0$ is equivalent to $p_a(D) = 0$. Let D_i $(i = 1, \ldots, k)$ be the irreducible components of D. Then we have $0 = p_a(D) = \sum_{i=1}^k p_a(D_i) + e - v + 1$, where e and v are the numbers of edges and vertices in the dual graph of D. Since each $p_a(D_i) \geq 0$ and $e - v + 1 \geq 0$, we have equality everywhere, and hence the lemma follows. □

We also need an analogous result when X is not smooth.

LEMMA 2.2. *Let X be a projective normal Cohen–Macaulay variety of dimension at least 2, and D a Weil divisor on X. Then we have an exact sequence*

$$0 \to \omega_X \to \mathcal{O}_X(K_X + D) \to \omega_D \to 0, \tag{2.1}$$

where ω_X and ω_D are the dualizing sheaves of X and D, and K_X is the canonical divisor of X.

Proof. See [Kol13, 4.1]. □

COROLLARY 2.3. *Let X be a rational surface with only rational singularities, and D an integral curve on X. Then D is a smooth rational curve if and only if $|K_X + D| = \emptyset$.*

Proof. Since X is a normal surface, it is CM, so we can apply the previous lemma to get the exact sequence (2.1), which induces the long exact sequence

$$H^0(X, \omega_X) \to H^0(X, \mathcal{O}_X(K_X + D)) \to H^0(D, \omega_D) \to H^1(X, \omega_X) \to \cdots.$$

Since X is a rational surface with only rational singularities, we have $H^0(X, \omega_X) = H^1(X, \omega_X) = 0$, and hence D is a smooth rational curve iff $H^0(D, \omega_D) = 0$ iff $|K_X + D| = \emptyset$. □

We may notice that the Lemmas 2.1 and 2.2 only apply to rational surfaces whereas our main theorem is stated for arbitrary surfaces. This is only a minor issue, as illustrated by the following lemma.

LEMMA 2.4. *Let X be a surface. Assume that X does not contain any smooth rational curves. Then either K_X is nef, or $-K_X$ is numerically ample and $\rho(X) = 1$.*

Here since $-K_X$ is in general not $\mathbb{Q}$-Cartier, its nefness or numerical ampleness is understood in the sense of [Sak87]. In particular, if we further assume that X has rational singularities (which implies that X is $\mathbb{Q}$-factorial) and K_X is not pseudo-effective (as we do in our main theorem), then $-K_X$ is ample, and X is a rational surface of Picard number one by [KT09, Lemma 3.1].

Proof of Lemma 2.4. First, suppose X is not relatively minimal. By [Sak87, Theorem 1.4] we may run the K_X-MMP on X. Let $f : X \to Y$ be the first step in the MMP. Since $-K_X$ is f-ample, by [Sak85, Theorem 6.3] we have $R^1 f_* \mathcal{O}_X = 0$. Let $C \subseteq X$ be an irreducible curve contracted by f, and $\mathcal{I}_C$ its ideal sheaf. Since the fibers of f have dimension ≤ 1, we have $R^2 f_* \mathcal{I}_C = 0$ by the theorem of formal functions. It then follows from the long exact sequence associated with

$0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$ that $H^1(C, \mathcal{O}_C) = R^1 f_* \mathcal{O}_C = 0$, and hence C is a smooth rational curve on X, contrary to our assumption.

We may therefore assume that X is relatively minimal. If K_X is not nef, then by [Sak87, Theorem 3.2], either $-K_X$ is numerically ample and $\rho(X) = 1$, or X admits a fibration $g : X \to B$ whose general fiber is $\mathbb{P}^1$. However, the latter case cannot occur since X contains no smooth rational curves. This proves the lemma. □

Now we come to a useful criterion for whether a surface contains at least one smooth rational curve.

PROPOSITION 2.5. *Let X be a surface with only rational singularities. Assume that K_X is not pseudo-effective. Then the following are equivalent*:

(1) *X contains no smooth rational curves*;
(2) *The class group* $\mathrm{Cl}(X)$ *is infinite cyclic and is generated by some effective divisor D linearly equivalent to $-K_X$.*

Proof. First, assume that (2) holds. By [KT09, Lemma 3.1], X is necessarily a rational surface. If X contains a smooth rational curve C, then by Corollary 2.3, $|K_X + C| = \emptyset$, but by (2), we may write $C \sim kD$ for some integer $k \geq 1$, and $K_X + C \sim (k-1)D$ is effective, a contradiction, so (1) follows.

Now assume that (1) holds. By Lemma 2.4 and its subsequent remark, X is a rational surface with ample anticanonical divisor. Let H be an ample divisor on X and assume that there exists some effective divisor C on X that is not an integral multiple of $-K_X$ in $\mathrm{Cl}(X)$. Among such divisors, we may choose C so that $(H.C)$ is minimal. Clearly, C is integral, and by (1) it is not a smooth rational curve; hence by Corollary 2.3, $K_X + C$ is effective. Since $-K_X$ is ample, we have $(K_X + C.H) < (C.H)$, so by our choice of C, $K_X + C$ is an integral multiple of K_X, and hence so is C, a contradiction. It follows that every effective divisor on X is linearly equivalent to a multiple of $-K_X$. Since $\mathrm{Cl}(X)$ is generated by the class of effective divisors, we see that it is infinite cyclic and generated by $-K_X$. Now let m be the smallest positive integer such that $-mK_X$ is effective. Write $-mK_X \sim \sum a_i D_i$, where $a_i > 0$, and D_i is integral. As m is minimal and each D_i is also a multiple of $-K_X$, we have indeed $-mK_X \sim D$ an integral curve. Since D is not smooth rational by (1), again by Corollary 2.3, $K_X + D$ is effective, but $K_X + D \sim -(m-1)K_X$, so by the minimality of m we have $m = 1$, and thus all the assertions in (2) are proved. □

From now on, X will always be a normal surface that satisfies the assumptions and the equivalent conditions (1)–(2) of Proposition 2.5. In particular, X is rational, $\mathbb{Q}$-factorial, $-K_X$ is ample, and $\mathrm{Pic}(X) \cong \mathbb{Z}$ is generated by $-rK_X$, where r is the smallest positive integer such that rK_X is Cartier (i.e. the index of X). We further assume that X has at worst quotient singularities (or equivalently, klt singularities, as we are in the surface case). Let X^0 be the smooth locus of X, $\pi : Y \to X$ the minimal resolution, and $E \subset Y$ the reduced exceptional locus.

LEMMA 2.6. *With the notation as before, we have an exact sequence*

$$0 \to \mathrm{Cl}(X)/\mathrm{Pic}(X) \to H^2(E,\mathbb{Z})/H_2(E,\mathbb{Z}) \to H_1(X^0,\mathbb{Z}) \to 0$$

and an isomorphism $H^2(E,\mathbb{Z})/H_2(E,\mathbb{Z}) \cong \mathbb{Z}/r\mathbb{Z}$.

Here we identify $H_2(E,\mathbb{Z})$ as a subgroup of $H^2(E,\mathbb{Z})$ by the composition $H_2(E,\mathbb{Z}) \to H_2(Y,\mathbb{Z}) \to H^2(Y,\mathbb{Z}) \to H^2(E,\mathbb{Z})$, where the first and last maps are induced by the inclusion $E \subset Y$, and the second by the Poincaré duality. In other words, the intersection pairing on Y induces a nondegenerate pairing $H_2(E,\mathbb{Z}) \times H_2(E,\mathbb{Z}) \to \mathbb{Z}$, and hence we may view $H_2(E,\mathbb{Z})$ as a subgroup of $H^2(E,\mathbb{Z})$. Note that the intersection numbers between irreducible components of E only depend on the singularities of X, so the quotient $H^2(E,\mathbb{Z})/H_2(E,\mathbb{Z})$ should be considered as a local invariant of the singularities of X.

Proof of Lemma 2.6. The existence of the exact sequence follows from [MZ88, Lemma 2]. If $\mathrm{Cl}(X) \cong \mathbb{Z}\cdot[-K_X]$, then from what we just said $\mathrm{Pic}(X) \cong \mathbb{Z}\cdot[-rK_X]$, and hence $\mathrm{Cl}(X)/\mathrm{Pic}(X) \cong \mathbb{Z}/r\mathbb{Z}$. It remains to prove that $H_1(X^0,\mathbb{Z}) = 0$. Since the intersection matrix of E is nondegenerate, $H_1(X^0,\mathbb{Z})$ is finite. If it is not zero, X^0 admits a nontrivial étale cyclic covering of degree $d > 1$, and hence $\mathrm{Pic}(X^0) \cong \mathrm{Cl}(X)$ contains a d-torsion, a contradiction. □

If $p \in X$ is a singular point, then let r_p be the local index of p, that is, the smallest positive integer m such that mK_X is Cartier at p, and define $\mathrm{Cl}_p = H^2(E_p,\mathbb{Z})/H_2(E_p,\mathbb{Z})$ in the same way as in Lemma 2.6 with $E_p = \pi^{-1}(p)_{\mathrm{red}}$. As explained in the next lemma, it can be viewed as the "local class group" of X at p. Since (X,p) has quotient singularities, it is locally (in the analytic topology) isomorphic to a neighborhood in $\mathbb{C}^2/G$ of the image of the origin, where G is a finite subgroup of $GL(2,\mathbb{C})$, and then $r_p = |H|$, where H is the image of G under the determinant map $\det : G \subset GL(2,\mathbb{C}) \to \mathbb{C}^*$, and Cl_p is isomorphic to the abelianization of G:

LEMMA 2.7. *In the same notation,* $\mathrm{Cl}_p \cong G/G'$.

Proof. By definition, Cl_p only depends on the intersection matrix of E_p, and hence we may replace X by an étale neighborhood of p; in particular, we may assume that $(X,p) \cong (\mathbb{C}^2/G, 0)$. As before, $\pi : Y \to X$ is the minimal resolution, and thus E_p is a deformation retract of Y. As X is affine and has rational singularities, $H^i(Y,\mathcal{O}_Y) = H^i(X,\mathcal{O}_X) = 0$ for all $i > 0$, so by the long exact sequence associated with the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_Y \to \mathcal{O}_Y^* \to 0$ we have $\mathrm{Pic}(Y) \cong H^2(Y,\mathbb{Z}) \cong H^2(E_p,\mathbb{Z})$ and hence the following commutative diagram (where $U = X \setminus p = Y \setminus E_p$ and $E_{p,i}$ are the irreducible components of E_p):

$$\begin{tikzcd}
\oplus\mathbb{Z}[E_{p,i}] \arrow[r] \arrow[d, "\cong"] & \mathrm{Pic}(Y) \arrow[r] \arrow[d, "\cong"] & \mathrm{Pic}(U) \arrow[r] \arrow[d] & 0 \\
H_2(E_p,\mathbb{Z}) \arrow[r] & H^2(E_p,\mathbb{Z}) \arrow[r] & \mathrm{Pic}(U) \arrow[r] & 0
\end{tikzcd}$$

It follows that $\mathrm{Cl}_p \cong \mathrm{Pic}(U)$. Let $V = \mathbb{C}^2 \setminus 0$. Then $\mathrm{Pic}(V) = 0$, and giving a line bundle on U is equivalent to giving a G-action on the trivial line bundle on V that is compatible with the G-action on V. Such objects are classified by $H^1(G, \mathcal{O}_V^*) = H^1(G, \mathbb{C}^*) \cong G/G'$, so the lemma follows. □

In particular, $r_p \le |\,\mathrm{Cl}_p\,|$. Since r is the lowest common multiple of all r_p and

$$H^2(E, \mathbb{Z})/H_2(E, \mathbb{Z}) \cong \mathbb{Z}/r\mathbb{Z}$$

is the direct sum of all Cl_p, we obtain the following:

COROLLARY 2.8. $\mathrm{Cl}_p \cong \mathbb{Z}/r_p\mathbb{Z}$ *for all* $p \in \mathrm{Sing}(X)$.

Quotient surface singularities are classified in [Bri68, Satz 2.11], using the table there together with the well-known classification of Du Val singularities (see e.g. [Dur79]), we see that each singularity of X has to be one of the following: the cyclic singularity $\frac{1}{n}(1, q)$ where $(q, n) = (q + 1, n) = 1$, type $\langle b; 2, 1; 3, 1; 3, 2\rangle$ (recall from [Bri68, Satz 2.11] that a type $\langle b; n_1, q_1; n_2, q_2; n_3, q_3\rangle$ singularity is the one whose dual graph is a fork such that the central vertex represents a curve with self intersection number $-b$ and the three branches are dual graph of the cyclic singularity $(1/n_i)(1, q_i)$ $(i = 1, 2, 3)$), or $\langle b; 2; 3; 5\rangle$ (meaning that it is of type $\langle b; 2, r; 3, s; 5, t\rangle$ for some r, s, t). In particular, E_8 is the only Du Val singularity that appears in the list.

We now turn to the classification of surfaces without smooth rational curves.

LEMMA 2.9. *X has at most one non-Du Val singular point.*

Proof. Since X satisfies (2) of Proposition 2.5, there is an effective divisor $D \in |-K_X|$ (which is necessarily an integral curve). Let $\tilde{D}$ be its strict transform on Y. We may write

$$K_Y + \tilde{D} + \sum a_i E_i = \pi^*(K_X + D) \sim 0, \tag{2.2}$$

where the E_i are the irreducible components of E, and $a_i \in \mathbb{Z}$ (as $K_X + D$ is Cartier on X). Since Y is the minimal resolution, $K_Y + \tilde{D}$ is π-nef, and thus by the negativity lemma [KM98, Lemma 3.39], all $a_i \ge 0$, and we have $a_i \ge 1$ if D passes $p = \pi(E_i)$ or X is not Du Val at p. In the latter case, since K_X is not Cartier at p, D must pass through p.

We claim that D contains at most one singular point of X, and hence at most one singular point of X is not Du Val. Suppose that this is not the case and $p_1, p_2 \in D \cap \mathrm{Sing}(X)$. Let $\Delta_j = \sum_{\pi(E_i)=p_j} a_i E_i$ $(j = 1, 2)$. Then we have $\Delta_j > 0$ and $(\tilde{D}.\Delta_j) \ge 1$. On the other hand, by (2.2) we have $2p_a(\tilde{D}) - 2 + (\Delta_1 + \Delta_2.\tilde{D}) = (K_Y + \tilde{D} + \Delta_1 + \Delta_2.\tilde{D}) \le 0$, and hence $p_a(\tilde{D}) = 0$, $\tilde{D} \cong \mathbb{P}^1$, and $(\tilde{D}.\Delta_j) = 1$ $(j = 1, 2)$. Since $K_Y + \tilde{D} + \Delta_j \equiv_\pi 0$ over p_j, we can apply [KM98, Proposition 5.58] to see that $(Y, \tilde{D} + \Delta_j)$ is lc (hence every curve in Δ_j appears with coefficient one) and the dual graph of $\tilde{D} + \Delta_j$ is a loop, which contradicts the fact that $(\tilde{D}.\Delta_j) = 1$. □

If X is Gorenstein, then by the previous discussion it has only E_8-singularities, and hence by the classification of Gorenstein log del Pezzo surfaces, X is one of the two types of $S(E_8)$ as discussed in [KM99, Lemma 3.6], and it is straightforward to verify that neither of them contains smooth rational curves (e.g. using Proposition 2.5). So from now on we assume that X is not Gorenstein, and by Lemma 2.9 we may denote by p the unique non-Du Val singular point of X and let $\Delta = \pi^{-1}(p)_{\mathrm{red}}$. We also get the following immediate corollary from the proof of Lemma 2.9.

COROLLARY 2.10. *With the same notation, every effective divisor $D \sim -K_X$ passes through p and no other singular points of X.*

In some cases, the curve D constructed in the previous proof turns out to be already a smooth rational curve on X. To be precise, we have the following:

PROPOSITION 2.11. *Let $D \in |-K_X|$, and $\tilde{D}$ its strict transform on Y. Then either X has a cyclic singularity at p and $K_Y + \tilde{D} + \Delta \sim 0$, or (X, p) is a singular point of type $\langle b; 2, 1; 3, 1; 3, 2\rangle$ or $\langle b; 2; 3; 5\rangle$ with $b = 2$.*

Proof. We have $K_Y + \tilde{D} + \sum a_i E_i \sim 0$ as in (2.2), where $a_i \in \mathbb{Z}_{>0}$ and $E_i \subset \operatorname{Supp}\Delta$ by Corollary 2.10. If some $a_i \geq 2$, then $|K_Y + \tilde{D} + \Delta| = \emptyset$, and hence by Lemma 2.1 $\tilde{D} + \Delta$ is a rational tree, in particular, $(\tilde{D}.\Delta) = 1$, and if, in addition, Δ is the fundamental cycle of (X, p) (i.e. $-\Delta$ is π-nef; this is the case if (X, p) has cyclic singularity or if the central curve of Δ has self-intersection at most -3), then by [KM99, Lemma 4.12], D is a smooth rational curve on X, but by Corollary 2.3 this contradicts our assumption since $|K_X + D| \neq \emptyset$. We already know that the singularity of X at p is cyclic, $\langle b; 2, 1; 3, 1; 3, 2\rangle$ or $\langle b; 2; 3; 5\rangle$, and hence, in the first case, all $a_i = 1$, and we claim that, in the latter two cases, at least one $a_i \geq 2$, and it follows that $b = 2$. Suppose that all $a_i = 1$. Then $K_Y + \tilde{D} + \Delta \sim 0$, but the LHS has a positive intersection with the central curve of Δ, a contradiction. □

We need a more careful analysis in the cyclic case, so assume for the moment that X has cyclic singularity at p. As before, D is an effective divisor in $|-K_X|$, and $\tilde{D}$ its birational transform on Y, whereas $\Delta = \pi^{-1}(p)_{\mathrm{red}}$.

LEMMA 2.12. *$\tilde{D}$ is a (-1)-curve on Y.*

Proof. Since $|K_Y + \tilde{D}| = |-\Delta| = \emptyset$, $\tilde{D}$ is a smooth rational curve by Lemma 2.1. We first show that $(\tilde{D}^2) < 0$. Suppose $(\tilde{D}^2) \geq 0$. Then the exact sequence $0 \to \mathcal{O}_Y \to \mathcal{O}_Y(\tilde{D}) \to \mathcal{O}_{\tilde{D}}(\tilde{D}) \to 0$ and $H^1(Y, \mathcal{O}_Y) = 0$ imply that $\mathcal{O}_Y(\tilde{D})$ is base point free, and hence we can choose D to pass through any point on X. By Corollary 2.10 this implies that p is the unique singular point of X. If C is a (-1)-curve on Y, then C is not contained in the support of $\tilde{D} + \Delta$, and since $K_Y + \tilde{D} + \Delta \sim 0$, we get $(K_Y + \tilde{D} + \Delta.C) = -1 + (\tilde{D} + \Delta.C) = 0$; hence, $(C.\Delta) \leq 1$, and $\pi(C)$ is a smooth rational curve on X (using [KM99, Lemma 4.12] as in the proof of

Proposition 2.11), a contradiction. It follows that Y contains no (-1)-curves, and hence $Y \cong \mathbb{F}_e (e \geq 2)$ and $X \cong \mathbb{P}(1,1,e)$, but then X contains many smooth rational curves, so these cases cannot occur. Hence $(\tilde{D}^2) < 0$. If $(\tilde{D}^2) \leq -2$, then by [Zha88, Lemma 1.3] $\tilde{D}$ is contained in E (the exceptional locus of π), so the lemma follows. □

LEMMA 2.13. *There exists a birational morphism $f : Y \to \bar{Y}$ such that*

(1) *$\bar{Y}$ is an $S(E_8)$;*
(2) $\mathrm{Ex}(f)$ *consists of all but one component of $\tilde{D} + E$;*
(3) *$f(\tilde{D})$ is a smooth point on $\bar{Y}$.*

Proof. Let $Y \to Y_0$ be the contraction of all curves in $E \setminus \Delta$. Then every closed point of Y_0 is either smooth or an E_8-singularity (every Du Val singularity of X is an E_8-singularity). We run the K-negative MMP starting with Y_0:

$$Y_0 \xrightarrow{\phi_1} Y_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_m} Y_m \xrightarrow{g} Z,$$

where each step is the contraction of an extremal ray, ϕ_i are birational, and $\dim Z < 2$ (Y_0 is a Gorenstein rational surface, so the MMP stops at a Mori fiber space). By [KM99, Lemma 3.3] each ϕ_i is the contraction of a (-1)-curve contained in the smooth locus of Y_{i-1}. If this (-1)-curve is not a component of the image of $\tilde{D} + E$, then let C be its strict transform in Y. Then C is a smooth rational curve with negative self-intersection, and hence by [Zha88, Lemma 1.3] it is a (-1)-curve. Now the same argument as in Lemma 2.12 shows that $(C.\Delta) \leq 1$ and $\pi(C)$ is a smooth rational curve in X, a contradiction. So the exceptional locus of $Y_0 \to Y_m$ is contained in $\tilde{D} + E$. In particular, since $\tilde{D}$ is the only component of $\tilde{D} + E$ that is a (-1)-curve, ϕ_1 is the contraction of $\tilde{D}$.

We claim that Z is a point. Suppose it is not; then g is a $\mathbb{P}^1$-fibration. By [KM99, Lemma 3.4], since Y_m has only singularities of E_8 type, it is actually smooth and isomorphic to $\mathbb{F}_e$ for some $e \geq 0$. If $e = 1$, then we can choose to contract the (-1)-curve from Y_m, and then $Y_{m+1} = \mathbb{P}^2$, whereas Z is a point. So we may assume that $e = 0$ or $e \geq 2$. Since $\mathrm{Cl}(X)$ is generated by $-K_X$, we see that $\mathrm{Cl}(Y)$ is freely generated by $-K_Y$ and the components of E, or equivalently, by the components of $\tilde{D} + E$. Letting Γ be the image of $\tilde{D} + E$ on Y_m, we have $K_{Y_m} + \Gamma \sim 0$, and the irreducible components of Γ freely generate $\mathrm{Cl}(Y_m)$. Since $\rho(Y_m) = 2$ in this case, Γ has exactly two irreducible components. However, this contradicts the next lemma.

Hence Z is a point, and Y_m is a Gorenstein rank one del Pezzo. By construction $\mathrm{Cl}(Y_m)$ is generated by the effective divisor $\Gamma \sim -K_{Y_m}$; in other words, Y_m contains no smooth rational curves, and hence by the discussion on the Du Val case, Y_m is an $S(E_8)$, and the lemma follows by taking $\bar{Y} = Y_m$. □

The following lemma is used in the proof.

LEMMA 2.14. *Let $S = \mathbb{F}_e$ where $e = 0$ or $e \geq 2$. Then $-K_S$ cannot be written as the sum of two irreducible effective divisors that generate* $\mathrm{Pic}(S)$.

Proof. It is quite easy to see that when $e = 0$, such a decomposition of $-K_S$ is not possible, so we assume that $e \geq 2$. Let C_0 be the unique section of negative self-intersection, and F be a fiber. Then $\mathrm{Pic}(S)$ is freely generated by C_0 and F. If $M = aC_0 + bF$ represents an irreducible curve, then $M = aC_0$, or $b \geq ae \geq 0$. Suppose $-K_S \sim 2C_0 + (e+2)F \sim M_1 + M_2$, where M_1 and M_2 are irreducible and generate $\mathrm{Pic}(S)$. Then we must have $M_i = C_0 + m_i F$ with $m_i \geq e$ and $m_1 + m_2 = e + 2$, which is only possible when $m_1 = m_2 = e = 2$, but then $M_1 = M_2$ cannot generate $\mathrm{Pic}(S)$. □

Back to the general case. To finish the classification, let us now construct some surfaces that satisfy the conditions in Proposition 2.5. Let $\bar{Y}$ be an $S(E_8)$ with $\Gamma \in |-K_{\bar{Y}}|$ a rational curve. We have $\Gamma \subset \bar{Y}^0$ and $(\Gamma^2) = 1$. Let q be the unique double point of Γ. Let $Y \to \bar{Y}$ be the blowup at $q_1 = q, q_2, \cdots, q_m$ where each q_i is infinitely near q_{i-1} $(i > 1)$. If q is a node of Γ, then we also require that q_i always lies on the strict transform of either Γ or exceptional curves of previous blowup (there are two different choices of q_i for each $i > 1$). If q is a cusp, then we require that $m = 1, 2$, or 4 and that q_i lies on the strict transform of Γ for $i = 2, 3$ whereas q_4 is away from Γ and previous exceptional curves. Let E_i be the strict transform of the exceptional curve coming from the blowup of q_i. We define $X(\bar{Y}, \Gamma; q_1, \ldots, q_m)$ to be the contraction from Y of Γ and E_i $(i = 1, \ldots, m-1)$. It has two singular points, one of which is an E_8 singularity, and the other is a cyclic singularity except when Γ has a cusp at q and $m = 4$, in which case the second singularity has type $\langle 2; 2, 1; 3, 1; 5, 1 \rangle$. Arguing inductively, we get $-K_Y \sim \Gamma + \sum_{i=1}^{m} E_i$ unless Γ has a cusp at q and $m = 4$, in which case we instead have $-K_Y \sim \Gamma + E_1 + E_2 + 2E_3 + E_4$. Since $\mathrm{Cl}(Y)$ is generated by Γ and all E_i, it is not hard to verify that $X(\bar{Y}, \Gamma; q_1, \ldots, q_m)$ satisfies condition (2) in Proposition 2.5.

THEOREM 2.15. *If X is a surface with only quotient singularities that satisfies the conditions in Proposition 2.5, then it is either an $S(E_8)$ or one of the $X(\bar{Y}, \Gamma; q_1, \ldots, q_m)$ constructed before.*

Proof. If X is Gorenstein, then it is an $S(E_8)$, so we may assume that X is not Gorenstein. Let p be its unique non-Du Val singular point. By Proposition 2.11, there are three possibilities for the singularity of (X, p), and we analyze them one by one:

(1) (X, p) is a cyclic singularity. Let Y_0 be as in Lemma 2.13, by which there exists a birational morphism $f : Y_0 \to \bar{Y}$ where $\bar{Y}$ is an $S(E_8)$ such that f contracts all but one component of $\tilde{D} + \Delta$ to a smooth point (we use the same letters for strict transforms of $\tilde{D}$ and Δ on Y_0). By Proposition 2.11, $K_{Y_0} + \tilde{D} + \Delta \sim 0$, and thus the dual graph of $\tilde{D} + \Delta$ is a loop. It follows that $\Gamma = f(\tilde{D} + \Delta) \sim -K_{\bar{Y}}$ is a rational curve with a double point q. In addition, q is a cusp if and only if $\tilde{D} + \Delta$ consists of two rational curves that are tangent to each other or three rational curves that intersect at the same

point. In particular, $\tilde{D}+\Delta$ has at most three components when q is a cusp. Since f is a composition of blowing down of (-1)-curves, we recover Y_0 as a successive blowup from $\bar{Y}$ of nodes on the images of $\tilde{D}+\Delta$. Let $q_1,\dots,q_m$ be the centers of these blowups. Clearly, $q_1=q$, and if q is a cusp, then $m\le 2$. Since $\tilde{D}$ is the only (-1)-curve among the components of $\tilde{D}+\Delta$, each q_i is infinitely near q_{i-1}. It is now easy to see that X is a $X(\bar{Y},\Gamma;q_1,\dots,q_m)$ with Γ nodal or Γ cuspidal and $m\le 2$.

(2) (X,p) has type $\langle 2;2,1;3,1;3,2\rangle$. By assumption $H^2(Y,\mathbb{Z})=\mathrm{Pic}(Y)$ is freely generated by K_Y and the components in E. Since the intersection paring on $H^2(Y,\mathbb{Z})$ is unimodular, the intersection matrix of K_Y and E has determinant ± 1. Write $K_Y+G=\pi^*K_X$ where G is supported on E (and can be easily computed from the given singularity type). Since π^*K_X is the orthogonal projection of K_Y to the span of the components of E, we must then have $(K_X^2)=((K_Y+G)^2)=(K_Y^2)+(K_Y.G)=10-\rho+(K_Y.G)=\frac{1}{r}$, where $r=|\det((E_i.E_j))|$, and ρ is the Picard number of Y. It is straightforward to compute that $G=\frac{5}{9}E_1+\cdots$, where E_1 is the only component of E with self-intersection (-3) (and this is the only component whose coefficient is relevant to us), $(K_Y.G)=\frac{5}{9}$, and $r=9$. But ρ is an integer, so this case cannot occur.

(3) (X,p) has type $\langle 2;2;3;5\rangle$. A similar computation as in case (2) shows that to have $10-\rho+(K_Y.G)=\frac{1}{r}$, (X,p) must has type $\langle 2;2,1;3,1;5,1\rangle$ and $\rho=13$. Since the other singularities of X are of E_8-type, X has exactly one E_8-singularity. By the same proof as that of Lemma 2.12, $\tilde{D}$ is a (-1)-curve. Let E_1 be the central curve of Δ, and E_2, E_3, E_5 be the other three components of Δ with self-intersections -2, -3, and -5, respectively. Write $K_Y+\tilde{D}+\sum a_iE_i=\pi^*(K_X+D)\sim 0$ as before. We have $a_1\ge 2$ since otherwise the LHS has positive intersection with E_1. By Lemma 2.1, $\tilde{D}+\Delta$ is a rational tree, and thus D intersects transversally with exactly one component of Δ. It is straightforward to find the discrepancies a_i once we know which component $\tilde{D}$ intersects. But as a_i are integers, we find that $\tilde{D}$ intersects E_1 by enumerating all the possibilities and that $a_2=a_3=a_5=1$. Now, as in Lemma 2.13, we may contract $\tilde{D}$, E_1, E_2, E_3 and all components of $E\setminus\Delta$ from Y to obtain $\bar{Y}$, which is an $S(E_8)$, such that the image of E_5 is a cuspidal rational curve $\Gamma\sim -K_{\bar{Y}}$. Reversing this blowing down procedure, we see that X is isomorphic to some $X(\bar{Y},\Gamma;q_1,\dots,q_4)$ where $\bar{Y}$ is an $S(E_8)$ and Γ is cuspidal. □

It is well known that E_8 is the only surface quotient singularity that does not admit a smooth curve germ [GSLJ94]. Hence the following corollary immediately follows from the theorem.

COROLLARY 2.16. *Let X be a surface with only quotient singularities. Assume that*

(1) *K_X is not pseudo-effective*;

(2) *For every closed point $p \in X$, there is a smooth curve (locally analytically) passing through p.*

Then X contains at least one smooth rational curve.

3. Examples and Questions

If X is a log del Pezzo surface, then a curve of minimal degree on X seems to be a natural candidate for the smooth rational curve (such a curve is used extensively in the study of log del Pezzo surfaces). However, the following example shows that this is not always the case, even if X is known to contain some smooth rational curve.

EXAMPLE 3.1. Let $Y \neq S(E_8)$ be a Gorenstein log del Pezzo surface of degree 1 such that the linear system $|-K_Y|$ contains a nodal curve D. Let $\bar{Y} \to Y$ be the blow up of the node of D. Let E be the exceptional curve, and $\bar{D}$ the strict transform of D. Contract the (-3)-curve $\bar{D}$ to get our surface X. It is straightforward to verify that the image of E under the contraction is the only curve of minimal degree on X. But since E intersects $\bar{D}$ at two points, its image on X is not smooth. In fact, the smooth rational curves on X are usually given by the strict transform of (-1)-curves on the minimal resolution of Y. Observe that since $K_X + E \sim 0$, we have that $K_X + C$ is ample for any smooth rational curve C on X.

We are also interested in whether the smooth rational curve C we find supports a tiger of the log del Pezzo surface X (i.e. there exists $D \sim_{\mathbb{Q}} -K_X$ such that $\mathrm{Supp}(D) = C$ and (X, D) is not klt. See [KM99, Definition 1.13]). At least, when C passes through at most one singular point, we have a positive answer:

LEMMA 3.2. *Let C be a smooth rational curve on a rank 1 log del Pezzo surface X. Assume that C passes through at most one singular point of X. If $\alpha \in \mathbb{Q}$ is chosen such that $K_X + \alpha C \equiv 0$, then the pair $(X, \alpha C)$ is not klt.*

Proof. If C lies in the smooth locus of X, then by adjunction $(K_X + C.C) = -2 < 0$, hence $\alpha > 1$, and the result is clear. Otherwise, we may assume that $C \cap \mathrm{Sing}(X) = \{p\}$. Let β be the log canonical threshold of the pair (X, C), and $\pi : \tilde{X} \to X$ the minimal resolution. It suffices to show that $(K_X + \beta C.C) \le 0$. Since C is a smooth rational curve, π is also a log resolution of (X, C). Write $\pi^*(K_X + \beta C) = K_{\tilde{X}} + \beta\tilde{C} + \sum a_i E_i$, where the E_i are the exceptional curves of π. We have $a_i \le 1$ by the choice of β, and $\tilde{C}$ only intersects one E_i. Now since X is of rank 1, we have $(\tilde{C}^2) \ge -1$ by [Zha88, Lemma 1.3] and $(K_{\tilde{X}} + \tilde{C}.\tilde{C}) = -2$ by adjunction; thus

$$\begin{aligned}(K_X + \beta C.C) &= \left(K_{\tilde{X}} + \beta\tilde{C} + \sum a_i E_i.C\right) \le -1 - \beta + \sum (E_i.\tilde{C})\\ &\le -\beta < 0.\end{aligned}$$

□

On the other hand, once C passes through more singular points of X, the situation becomes more complicated. The following example suggests that even if X has a tiger, then in general there is no guarantee that the tiger can be supported on C.

EXAMPLE 3.3. Similarly to the previous example, let Y be a Gorenstein log del Pezzo surface of degree 1 with an A_8-singularity and $D \in |-K_Y|$ a nodal curve. Blow up the node and one of its infinitely near points to get a new surface $\bar{Y}$ and let X be the contraction of the strict transform of D and the first exceptional curve. The second singularity of X has a dual graph of type $\langle 4, 2\rangle$. Every smooth rational curve on X is a (-1)-curve on the minimal resolution and intersects both singular points of X. By direct computation we have $\beta = \frac{1}{2}$ (where $\beta = \mathrm{lct}(X, C)$ as in the proof of Lemma 3.2) and $(K_X + \beta C.C) > 0$, and hence by the same reasoning as in Lemma 3.2 we know that C does not support a tiger. However, $-K_X$ is effective, so X does have a tiger.

In view of Proposition 2.5, we may ask for a similar classification of surfaces with rational singularities that contain no smooth rational curves. The next example shows that we do get additional cases.

EXAMPLE 3.4. The construction is similar to that of $X(\bar{Y}, \Gamma; q_1, \dots, q_m)$. Let $\bar{Y}$ be an $S(E_8)$ with $\Gamma \in |-K_{\bar{Y}}|$ a cuspidal rational curve, and let q be the cusp of Γ. Let $Y \to \bar{Y}$ be the blowup at $q_1 = q, q_2, \dots, q_m$ $(m \geq 5)$ where each q_i is infinitely near q_{i-1} $(i > 1)$ such that q_i lies on the strict transform of Γ for $i < m$ whereas q_m is away from Γ and the previous exceptional curves. Let E_i be the strict transform of the exceptional curve coming from the blowup of q_i. The dual graph of Γ and E_i $(i = 1, \dots, m-1)$ is given as follows:

$$\begin{array}{l} (-2) \\ \qquad \diagdown \\ \qquad\quad (-2) - (-2) - \cdots - (-2) - (-m-1) \\ \qquad \diagup \\ (-3) \end{array}$$

We define X to be the contraction from Y of these curves. It has two singular points, one of which is an E_8 singularity, and the other is not a quotient singularity since we assume that $m \geq 5$. Nevertheless, it is a rational singularity (one way to see this is to attach m auxiliary (-1)-curves to Γ and notice that the corresponding configuration of curves contracts to a smooth point, and hence any subset of these curves also contracts to a rational singularity by [Art66, Proposition 1]). We also have $-K_Y \sim \Gamma + E_1 + E_2 + 2\sum_{i=3}^{m-1} E_i + E_m$ by induction on m, and it follows as before that $\mathrm{Cl}(X)$ is generated by the image of E_m, which is linearly equivalent to $-K_X$. By Proposition 2.5, X is a surface with rational singularities that contains no smooth rational curves.

We observe that the surfaces in this example still contain E_8 singularities and thus violate the second assumption of Corollary 2.16. In addition, the construction does not seem to have many variants. It is therefore natural to ask the following question.

QUESTION 3.5. Let X be a surface with rational singularities. Assume that K_X is not pseudo-effective and every closed point of X admits a smooth curve germ. Is it true that X contains a smooth rational curve? More aggressively, classify all surfaces with rational singularities that contain no smooth rational curves.

Finally, we investigate what happens if we remove the assumption on K_X in our main theorem. Clearly, there are many smooth surfaces (e.g. Abelian surfaces, ball quotients, etc.) with nef canonical divisors that even contain no rational curves. Since we are mostly interested in the existence of *smooth* rational curves, we restrict ourselves to rational surfaces. We construct some examples of rational surfaces with cyclic quotient singularities that contain no smooth rational curves. These rational surfaces X are the quotient of certain singular K3 surfaces and satisfy $K_X \sim_{\mathbb{Q}} 0$, and hence assumption (1) in our main theorem is necessary.

EXAMPLE 3.6. Let T be a smooth del Pezzo surface of degree 1. For general choice of T, the linear system $|-K_T|$ contains at least two nodal rational curves C_i $(i = 1, 2)$. Let Q_i be the node of C_i, and $P = C_1 \cap C_2$. Let $\pi : Y \to T$ be the blowup of both Q_i with exceptional divisors E_i, and let $\tilde{C}_i$ be the strict transform of C_i on Y. Then $K_Y = \pi^* K_T + E_1 + E_2$ and $\tilde{C}_i = \pi^* C_i - 2E_i = \pi^*(-K_T) - 2E_i$, and thus $-2K_Y \sim \tilde{C}_1 + \tilde{C}_2$. We also have $(\tilde{C}_i^2) = -3$ and $(\tilde{C}_1.\tilde{C}_2) = 1$, and hence we can contract both $\tilde{C}_i$ simultaneously to get a rational surface X with a cyclic singularity p of type $\frac{1}{8}(1, 3)$. The next three lemmas tell us that for very general choice of T and C_i, such X contains no smooth rational curves.

LEMMA 3.7. *Every smooth curve on X is away from p.*

Proof. Let $p \in C$ be a smooth curve on X, and $\tilde{C}$ its strict transform on Y. Then $(\tilde{C}.\tilde{C}_1 + \tilde{C}_2) = 1$. But we have $\tilde{C}_1 + \tilde{C}_2 = -2K_Y$, so the intersection must be even, a contradiction. □

Let $Y' \to Y$ be the blowup of P, and C_i' the strict transform of $\tilde{C}_i$. Then $C_1' + C_2' = -2K_{Y'}$, and hence we can take the double cover $f : S \to Y'$ ramified along $C_1' + C_2'$. The surface S is smooth since C_1' and C_2' are smooth and disjoint, and S is indeed a K3 surface since $K_S = f^* K_{Y'}(\frac{1}{2}(C_1' + C_2')) \sim 0$ and $H^1(S, \mathcal{O}_S) = H^1(Y', \mathcal{O}_{Y'}) \oplus H^1(Y', \mathcal{O}_{Y'}(K_{Y'})) = 0$.

LEMMA 3.8. *For very general choice of T and C_i, the K3 surface S has Picard number* 12.

Proof. We have $\rho(S) \geq 12$ since it is a double cover of Y' and $\rho(Y') = 12$. Since the moduli space of K3 surfaces is 20-dimensional, the locus of those with Picard number at least 13 is a countable union of subvarieties of dimension at most 7. On the other hand, the above construction gives us an eight-dimensional family of K3 surfaces: we have an eight-dimensional family of del Pezzo surfaces of degree 1. Hence, for very general choice of T, we get a K3 surface S with $\rho(S) = 12$. □

We now get the following:

LEMMA 3.9. *For very general choice of C_i, the rational surface X constructed contains no smooth rational curve.*

Proof. Suppose $C \subseteq X$ is a smooth rational curve. By Lemma 3.7, $p \notin C$, and hence its strict transform C' in Y' is disjoint from $C_1' + C_2'$. Since $f : S \to Y'$ is étale outside $C_1' + C_2'$, $f^{-1}(C')$ splits into a disjoint union of two smooth rational curves D_1, D_2. This implies $\rho(S) \geq 13$ (D_1, D_2, and the pullback of the orthogonal complement of C' in $\mathrm{Pic}(Y')$ generate a sublattice of rank 13), which cannot happen for very general choice of T and C_i by Lemma 3.8. □

By allowing more singular points we can give a similar construction with a simpler proof of nonexistence of smooth rational curves.

EXAMPLE 3.10. Instead of taking a smooth del Pezzo surface of degree 1, let T be a Gorenstein rank one log del Pezzo surface of degree 1. Assume that either T has a unique singular point or it has exactly two A_n-type singular points. Then a similar argument as the proof of [KM99, Lemma 3.6] implies that, for general choice of T, $|-K_T|$ contains two nodal rational curves C_i $(i = 1, 2)$ lying inside the smooth locus of T. Let X be the surface obtained by the same construction in Example 3.6 (i.e. blow up the nodes Q_i of C_i and contract both $\tilde{C}_i$). Then it has the same singularities as T and a cyclic singularity p of type $\frac{1}{8}(1, 3)$. Suppose X contains a smooth rational curve C. As before, we know that $C \subset U = X \setminus p$, and since $2K_X \sim 0$, we have a double cover $g : Y \to X$ that is unramified over U (since K_X is Cartier over U). Since $C \cong \mathbb{P}^1$ is simply connected, we see that $g^{-1}(C)$ consists of two disjoint copies of $\mathbb{P}^1$. By construction, X has Picard number one; hence, C is ample, and thus g^*C is also ample on Y, but this contradicts [Har77, III.7.9]. In some cases, we can also derive a contradiction without using the double cover. For example, suppose T has a unique A_8-type singularity q. Then modulo torsion $\mathrm{Cl}(X)$ is generated by E, the strict transform of the exceptional curve over either one of the Q_i, and $(E^2) = \frac{1}{2}$. It follows that

$$\deg(K_C + \mathrm{Diff}_C(0)) = (K_X + C.C) = (C^2) \geq \frac{1}{2}, \tag{3.1}$$

but $\deg K_C = -2$, and since C is smooth at q, the dual graph of (X, C) at q is a fork with C being one of the branches. It is then straightforward to compute that $\deg \mathrm{Diff}_C(0) = (\frac{1}{m} + \frac{1}{n})^{-1} \leq \frac{20}{9}$, where m, n are the indices of the other two branches of the dual graph (i.e. one larger than the number of vertices in the branch), which contradicts (3.1).

Inspired by these examples, we may expect to take certain quotients of Calabi–Yau varieties and construct higher-dimensional rationally connected varieties with klt singularities that contain no smooth rational curves. Unfortunately, we are unable to identify such an example and therefore leave it as the following question.

QUESTION 3.11. Let X be a rationally connected variety of dimension ≥ 3 with klt singularities. Does X always contain a smooth rational curve?

We remark that if X is indeed log Fano, then a folklore conjecture predicts that the smooth locus of X is rationally connected and thus contains a smooth rational curve since $\dim X \geq 3$.

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Department of Mathematics
Princeton University
Princeton, NJ 08544-1000
USA

zzhuang@math.princeton.edu