On the Torelli Problem and Jacobian Nullwerte in Genus Three

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1. Introduction

It is known that the set of bitangent lines of a non-hyperelliptic genus-three curve determines completely the curve, since it admits a unique symplectic structure [CaS; L]. Given this structure, one can recover an equation for the curve following a method of Riemann ([Rie]; see also [R1]): one takes an Aronhold system of bitangent lines, determines some parameters by means of some linear systems, and then writes down a *Riemann model* of the curve. Unfortunately, the parameters involved in this construction are not defined in general over the field of definition of the curve but rather over the field of definition of the bitangent lines. This is inconvenient for arithmetical applications concerned with rationality questions (cf. [O], for instance).

We propose an alternative construction, giving a model of the curve directly from a certain set of bitangent lines; this model is already defined over the field of definition of the curve. In the particular case of complex curves, our construction provides a closed solution for the non-hyperelliptic Torelli problem on genus three as follows.

THEOREM 1.1. *Let C be a non-hyperelliptic genus-three curve defined over a field $K \subset \mathbb{C}$, and let $\omega_1, \omega_2, \omega_3$ be a K-basis of $H^0(C, \Omega^1_{/K})$ and $\gamma_1, \ldots, \gamma_6$ a symplectic basis of $H_1(C, \mathbb{Z})$. We denote by $\Omega = (\Omega_1 | \Omega_2) = \left(\int_{\gamma_j} \omega_k \right)_{j,k}$ the period matrix of C with respect to these bases and by $Z = \Omega_1^{-1}.\Omega_2$ the normalized period matrix. A model of C defined (up to normalization) over K is*

$$\left(\frac{[w_7 w_2 w_3][w_7 w'_2 w'_3]}{[w_1 w_2 w_3][w'_1 w'_2 w'_3]} X_1 Y_1 \right.$$

$$+ \frac{[w_1 w_7 w_3][w'_1 w_7 w'_3]}{[w_1 w_2 w_3][w'_1 w'_2 w'_3]} X_2 Y_2 - \left. \frac{[w_1 w_2 w_7][w'_1 w'_2 w_7]}{[w_1 w_2 w_3][w'_1 w'_2 w'_3]} X_3 Y_3 \right)^2$$

$$- 4 \frac{[w_7 w_2 w_3][w_7 w'_2 w'_3]}{[w_1 w_2 w_3][w'_1 w'_2 w'_3]} \frac{[w_1 w_7 w_3][w'_1 w_7 w'_3]}{[w_1 w_2 w_3][w'_1 w'_2 w'_3]} X_1 Y_1 X_2 Y_2 = 0,$$

where

$$w_1 = {}^t\left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right) + {}^t\left(0, 0, \tfrac{1}{2}\right).Z, \qquad w_1' = {}^t\left(0, 0, \tfrac{1}{2}\right) + {}^t\left(0, 0, \tfrac{1}{2}\right).Z,$$

$$w_2 = {}^t\left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right) + {}^t\left(0, \tfrac{1}{2}, \tfrac{1}{2}\right).Z, \qquad w_2' = {}^t\left(0, \tfrac{1}{2}, 0\right) + {}^t\left(0, \tfrac{1}{2}, \tfrac{1}{2}\right).Z,$$

$$w_3 = {}^t\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) + {}^t\left(0, \tfrac{1}{2}, 0\right).Z, \qquad w_3' = {}^t\left(0, \tfrac{1}{2}, \tfrac{1}{2}\right) + {}^t\left(0, \tfrac{1}{2}, 0\right).Z,$$

$$w_7 = {}^t\left(0, 0, \tfrac{1}{2}\right) + {}^t\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right).Z,$$

$$X_j = \mathrm{grad}_{z=0}\, \theta[w_j](z; Z).\Omega_1^{-1}.(X, Y, Z)^t,$$

$$Y_j = \mathrm{grad}_{z=0}\, \theta[w_j'](z; Z).\Omega_1^{-1}.(X, Y, Z)^t,$$

and $[u, v, w]$ *denotes the Jacobian Nullwert given by* u, v, *and* w *(as defined at the beginning of Part II).*

Of course, one could simplify the denominators in the main equation to obtain a simpler formula. The advantage of writing the fractions is that their values are algebraic over the field of definition of the curve.

The formula in the theorem can be interpreted as a universal curve over the moduli space of complex non-hyperelliptic curves of genus three (with certain level structure). Moreover, using the Frobenius formula, we can exchange the quotients of Jacobian Nullwerte appearing in the theorem's formula by quotients of Thetanullwerte (see Section 7).

As will become clear in the paper, many different choices for the w_k are possible. They are only restricted to some geometric conditions expressed in terms of the associated bitangent lines (see Section 4). Every election of the w_k leads to a model for the curve, though all these models agree up to a proportionality constant. The comparison of the models given for different elections provides a number of identities between Jacobian Nullwerte in dimension three. For instance, we prove the following theorem.

THEOREM 1.2.   *Take* $w_1, w_2, w_3, w_1', w_2', w_3'$ *as before, and write*

$$w_4 = {}^t\left(\tfrac{1}{2}, 0, 0\right) + {}^t\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right).Z, \qquad w_4' = {}^t\left(0, 0, 0\right) + {}^t\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right),$$

$$w_7' = {}^t\left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right) + {}^t\left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right).Z.$$

Then these equalities hold on the space $\mathbb{H}_3$:

$$[w_2 w_3 w_7](Z)[w_1 w_3' w_7'](Z) = [w_1 w_3 w_7](Z)[w_2 w_3' w_7'](Z),$$

$$[w_2' w_3 w_7'](Z)[w_1' w_3' w_7](Z) = [w_1' w_3 w_7'](Z)[w_2' w_3' w_7](Z);$$

$$[w_3 w_1 w_2](Z)[w_3' w_1 w_2](Z)[w_4 w_1 w_2'](Z)[w_4' w_1 w_2'](Z)$$
$$= [w_4 w_1 w_2](Z)[w_4' w_1 w_2](Z)[w_3 w_1 w_2'](Z)[w_3' w_1 w_2'](Z),$$

$$[w_3 w_1 w_2](Z)[w_3' w_1 w_2](Z)[w_4 w_2 w_1'](Z)[w_4' w_2 w_1'](Z)$$
$$= [w_4 w_1 w_2](Z)[w_4' w_1 w_2](Z)[w_3 w_2 w_1'](Z)[w_3' w_2 w_1'](Z),$$

$$[w_3 w_1 w_2](Z)[w_3' w_1 w_2](Z)[w_4 w_1' w_2'](Z)[w_4' w_1' w_2'](Z)$$
$$= [w_4 w_1 w_2](Z)[w_4' w_1 w_2](Z)[w_3 w_1' w_2'](Z)[w_3' w_1' w_2'](Z).$$

Again, there are many possible elections of the w_k, each one leading to a set of identities between Jacobian Nullwerte.

Apart from its intrinsic theoretical interest, these results have different applications. From the computational viewpoint, for instance, Theorem 1.1 (or Corollary 7.2) can be used to determine equations for modular curves or to present three-dimensional factors of modular Jacobians as Jacobians of curves (thus improving the results in [O]). Ritzenthaler [R2] has already used them in the computation of optimal curves of genus three. In a more theoretical frame, the identities in Theorem 1.2 may lead to simplified expressions for the discriminant of genus-three curves (cf. [R1]).

The results just described are the analytic version of the corresponding results in an algebraic context. These algebraic results are proved in Part I. We start by recalling Riemann construction for curves of genus three and the basic concepts regarding the symplectic structure of the set of bitangent lines of these curves. The relation between different Steiner complexes is described explicitly in Section 3. In Sections 4 and 5 we combine the ideas in previous sections to provide the algebraic versions of our theorems. Part II contains the results in the analytic context. The proof of Theorems 4.1 and 1.2 is given in Section 6. We give the Thetanullwerte version of Theorem 4.1 in Section 7, which also contains a geometric description of the fundamental systems appearing in the Frobenius formula.

The author is much indebted to C. Ritzenthaler for directing his attention to the Torelli problem for non-hyperelliptic genus-three curves and to E. Nart for many fruitful discussions.

NOTATION AND CONVENTIONS. We will work with a nonsingular, genus-three curve $\mathcal{C}$ defined over a field K with $\mathrm{char}(K) \neq 2$ and given by a quartic equation $Q = 0$. We assume that the curve is embedded in the projective plane $\mathbb{P}^2(K)$ by its canonical map. Given two polynomials Q_1, Q_2, we will write $Q_1 \sim Q_2$ to express that they agree up to a constant proportionality factor. A number of geometric concepts and facts about genus-three curves will be used throughout the paper. We will take [D] as a basic reference and follow the notation introduced there.

PART I. ALGEBRAIC IDENTITIES

2. Riemann Model and Steiner Complexes

A triplet or tetrad of bitangent lines is called *syzygetic* whenever its contact points with the curve lie on a conic. Any pair of bitangent lines to $\mathcal{C}$ can be completed in five different ways with another pair to a syzygetic tetrad. Moreover, any two of the six pairs form a syzygetic tetrad. Such a set of six pairs of bitangent lines is called a *Steiner complex*.

Riemann [Rie] showed that, given three pairs of bitangent lines in a Steiner complex, one can determine proper equations $\{X_1 = 0, Y_1 = 0\}$, $\{X_2 = 0, Y_2 = 0\}$, and $\{X_3 = 0, Y_3 = 0\}$ of these lines such that the quartic equation $Q = 0$ defining the curve $\mathcal{C}$ can be written as

$$Q \sim (X_1Y_1 + X_2Y_2 - X_3Y_3)^2 - 4X_1Y_1X_2Y_2 = 0;$$

moreover, the proper equations satisfy the relation

$$X_1 + X_2 + X_3 = Y_1 + Y_2 + Y_3.$$

To find the proper equations of the bitangent lines, one takes arbitrary equations for them and solves certain linear systems to compute scaling factors leading to the well-adjusted equations.

Reciprocally, if we take three pairs $\{X_1, Y_1\}, \{X_2, Y_2\}, \{X_3, Y_3\}$ of linear polynomials over K such that no triplet formed by a polynomial in each pair is linearly dependent and $X_1 + X_2 + X_3 = Y_1 + Y_2 + Y_3$, then the quartic equation above gives a nonsingular, genus-three curve over K for which $\{X_1 = 0, Y_1 = 0\}$, $\{X_2 = 0, Y_2 = 0\}$, and $\{X_3 = 0, Y_3 = 0\}$ are three pairs of bitangent lines in a common Steiner complex.

Let us define $W = X_1 + X_2 + X_3 = 0$, $Z_1 = X_1 + Y_2 + Y_3$, $Z_2 = Y_1 + X_2 + Y_3$, and $Z_3 = Y_1 + Y_2 + X_3$. Then the trivial equalities

$$\begin{aligned}
Q \sim {} & (X_1Y_2 + X_2Y_1 - WZ_3)^2 - 4X_1X_2Y_1Y_2 \\
= {} & (X_1Y_3 + X_3Y_1 - WZ_2)^2 - 4X_1X_3Y_1Y_3 \\
= {} & (X_2Y_3 + X_3Y_2 - WZ_1)^2 - 4X_2X_3Y_2Y_3 \\
= {} & (X_1X_2 + Y_1Y_2 - Z_1Z_2)^2 - 4X_1X_2Y_1Y_2 \\
= {} & (X_1X_3 + Y_1Y_3 - Z_1Z_3)^2 - 4X_1X_3Y_1Y_3 \\
= {} & (X_2X_3 + Y_2Y_3 - Z_2Z_3)^2 - 4X_1X_3Y_1Y_3 \\
= {} & (X_1W + Y_2Z_3 - Y_3Z_2)^2 - 4X_1WY_2Z_3 \\
= {} & (X_2W + Y_1Z_3 - Y_3Z_1)^2 - 4X_2WY_1Z_3 \\
= {} & (X_3W + Y_1Z_2 - Y_2Z_1)^2 - 4X_3WY_1Z_2 \\
= {} & (X_1Z_3 + Y_2W - X_3Z_1)^2 - 4X_1Z_3Y_2W \\
= {} & (X_1Z_2 + Y_3W - X_2Z_1)^2 - 4X_1Z_2Y_3W \\
= {} & (X_1Z_1 + X_2Z_2 - X_3Z_3)^2 - 4X_1Z_1X_2Z_2 \\
= {} & (Y_1Z_1 + Y_2Z_2 - Y_3Z_3)^2 - 4X_1Y_1Y_2Z_2 & (1)
\end{aligned}$$

show that $W = 0$, $Z_1 = 0$, $Z_2 = 0$, and $Z_3 = 0$ are also bitangent lines to $\mathcal{C}$ and they make apparent a number of different Steiner complexes on $\mathcal{C}$.

On the other hand, one can check easily that in this situation any triplet formed by picking a line from each pair $\{X_i = 0, Y_i = 0\}$ is asyzygetic. Analogous conclusions can be derived from each of the preceding equations. For instance, we mention for our later convenience that $\{X_1 = 0, X_2 = 0, W = 0\}$, $\{X_1 = 0, X_3 = 0, W = 0\}$, $\{X_1 = 0, Y_2 = 0, Z_3 = 0\}$, and $\{X_1 = 0, Y_3 = 0, Z_3 = 0\}$ are asyzygetic triplets.

3. Relations between Steiner Complexes

Every Steiner complex S has an associated two-torsion element $O(D_S) \in \mathrm{Pic}^0(\mathcal{C})$; a divisor D_S defining it is given by the difference of the contact points of the two bitangent lines in any pair in S. We will call such a divisor an *associated divisor* of S. Indeed, the map $S \mapsto O(D_S)$ establishes a bijection between the set of Steiner complexes on $\mathcal{C}$ and $\mathrm{Pic}^0(\mathcal{C})[2] \setminus \{0\}$ (cf. [D]).

Given a pair $\{X = 0, Y = 0\}$ of bitangent lines, we shall denote by S_{XY} the Steiner complex determined by them. For any set $S = \{\{X_1 = 0, Y_1 = 0\}, \ldots, \{X_r = 0, Y_r = 0\}\}$ of pairs of bitangent lines, the subjacent set of lines will be denoted by $\bar{S} = \{X_1 = 0, Y_1 = 0, \ldots, X_r = 0, Y_r = 0\}$.

Two different Steiner complexes share four or six bitangent lines; they are called (respectively) syzygetic or asyzygetic. The following simple criterion can be used to check whether two Steiner complexes are syzygetic.

LEMMA 3.1 [D, p. 96]. *Let S_1, S_2 be two Steiner complexes, and let D_1, D_2 be associated divisors to them. Then $\#(\bar{S}_1 \cap \bar{S}_2) = 4$ if $e_2(D_1, D_2) = 0$ and $\#(\bar{S}_1 \cap \bar{S}_2) = 6$ if $e_2(D_1, D_2) = 1$, where e_2 denotes the Weil pairing on $\mathrm{Pic}^0(\mathcal{C})[2]$.*

We note that the Weil pairing can be computed by means of the Riemann–Mumford relation ([M]; see [ACGH, p. 290]): For any semicanonical divisor D, we have

$$e_2(D_1, D_2)$$
$$= h^0(D) + h^0(D + D_1 + D_2) + h^0(D + D_1) + h^0(D + D_2) \pmod{2}. \quad (2)$$

A priori, the bitangent lines that form a pair in a Steiner complex are completely indistinguishable. But if we consider the Steiner complex in relation to others, there appears to be an individualization of every line. This idea is made explicit in Corollary 3.4 and Proposition 3.7.

It is evident that, whenever two pairs of bitangent lines $\{X_1 = 0, Y_1 = 0\}$ and $\{X_2 = 0, Y_2 = 0\}$ form a syzygetic tetrad, the pairs $\{X_1 = 0, Y_2 = 0\}$ and $\{X_2 = 0, Y_1 = 0\}$ also form a syzygetic tetrad, as do the pairs $\{X_1 = 0, X_2 = 0\}$ and $\{Y_1 = 0, Y_2 = 0\}$. The corresponding Steiner complexes are tightly related as follows.

PROPOSITION 3.2. *Let $S_{X_1 Y_1} = \{\{X_1 = 0, Y_1 = 0\}, \ldots, \{X_6 = 0, Y_6 = 0\}\}$ be a Steiner complex.*

(a) *The Steiner complexes $S_{X_1 Y_1}$ and $S_{X_1 Y_j}$ are syzygetic and share the four lines $X_1 = 0$, $Y_1 = 0$, $X_j = 0$, and $Y_j = 0$; that is, $\bar{S}_{X_1 Y_1} \cap \bar{S}_{X_1 Y_j} = \{X_1 = 0, Y_1 = 0, X_j = 0, Y_j = 0\}$.*

(b) *The Steiner complexes $S_{X_1 Y_1}$ and $S_{X_1 X_2}$ are syzygetic and $\bar{S}_{X_1 Y_1} \cap \bar{S}_{X_1 X_2} = \{X_1 = 0, Y_1 = 0, X_2 = 0, Y_2 = 0\}$.*

(c) *The Steiner complexes $S_{X_1 X_2}$ and $S_{X_1 Y_2}$ are syzygetic and $\bar{S}_{X_1 Y_1} \cap \bar{S}_{X_1 X_2} = \{X_1 = 0, Y_1 = 0, X_2 = 0, Y_2 = 0\}$.*

(d) *For $j \neq k$ with $j, k \neq 1$, the Steiner complexes $S_{X_1 Y_j}$ and $S_{X_1 Y_k}$ are asyzygetic.*

Proof. Let us write $\operatorname{div}(X_i) := 2P_i + 2Q_i$ and $\operatorname{div}(Y_i) := 2R_i + 2S_i$. We apply the criterion of Lemma 3.1, using formula (2) to compute the Weil pairing of divisors D_{11} and D_{1j} associated to $S_{X_1 Y_1}$ and $S_{X_1 Y_j}$, respectively. We take $D = P_1 + Q_1$ and find

$$
\begin{aligned}
e_2&(D_{11}, D_{1j}) \\
&= h^0(3P_1 + 3Q_1 - R_1 - S_1 - R_j - S_j) - h^0(2P_1 + 2Q_1 - R_1 - S_1) \\
&\quad - h^0(2P_1 + 2Q_1 - R_j - S_j) + h^0(P_1 + Q_1) \\
&= h^0(K_{\mathcal{C}} + P_1 + Q_1 - R_1 - S_1 - R_j - S_j) - h^0(K_{\mathcal{C}} - R_1 - S_1) \\
&\quad - h^0(K_{\mathcal{C}} - R_j - S_j) + 1 \\
&= h^0(P_1 + Q_1 + R_1 + S_1 - R_j - S_j) - h^0(R_1 + S_1) - h^0(R_j + S_j) + 1 \\
&= h^0(P_1 + Q_1 + R_1 + S_1 - R_j - S_j) - 1 = h^0(P_j + Q_j) - 1 = 0,
\end{aligned}
$$

since $\{X_1 = 0, Y_1 = 0, X_j = 0, Y_j = 0\}$ forms a syzygetic tetrad. This proves part (a). Parts (b)–(d) are proved analogously. □

This result allows us to derive a more explicit version of [D, Thm. 6.1.8].

COROLLARY 3.3.   *Let $\{\{X_1 = 0, Y_1 = 0\}, \{X_2 = 0, Y_2 = 0\}\}$ be a syzygetic tetrad of bitangent lines. Then the Steiner complexes $S_{X_1 Y_1}$, $S_{X_1 Y_2}$, $S_{X_1 X_2}$ satisfy*

$$
\bar{S}_{X_1 Y_1} \cup \bar{S}_{X_1 Y_2} \cup \bar{S}_{X_1 X_2} = \operatorname{Bit}(\mathcal{C}).
$$

Some immediate consequences of the preceding corollary are as follows.

COROLLARY 3.4.   *Let $S = \{\{X_1 = 0, Y_1 = 0\}, \ldots, \{X_6 = 0, Y_6 = 0\}\}$ be a Steiner complex.*
(a) *Any triplet $\{X_i = 0, Y_i = 0, X_j = 0\}$ is syzygetic.*
(b) *Any triplet $\{X_i = 0, X_j = 0, X_k = 0\}$ formed by picking a line from three different pairs of S is asyzygetic.*
(c) *Let $\{U = 0, V = 0\}$ be an arbitrary pair in the Steiner complex $S_{X_1 Y_2}$. Then $U = 0$ belongs to $S_{X_1 X_3}$ but not to $S_{X_1 Y_3}$ and $V = 0$ belongs to $S_{X_1 Y_3}$ but not to $S_{X_1 X_3}$ (or vice versa).*

We now derive a geometrical property of asyzygetic triplets of bitangent lines that we will need later.

COROLLARY 3.5.   *Every three asyzygetic bitangent lines $X_1 = 0$, $X_2 = 0$, and $X_3 = 0$ can be paired with other three bitangent lines $Y_1 = 0, Y_2 = 0$, and $Y_3 = 0$ such that the three pairs $\{X_i = 0, Y_i = 0\}$ belong to a common Steiner complex.*

PROPOSITION 3.6.   *Three asyzygetic bitangent lines do not cross in a point.*

Proof. Complete the lines to three pairs $\{\{X_1 = 0, Y_1 = 0\}, \{X_2 = 0, Y_2 = 0\}, \{X_3 = 0, Y_3 = 0\}\}$ in a common Steiner complex and then rescale the equations

to obtain a Riemann model for $\mathcal{C}$: $(X_1Y_1 + X_2Y_2 - X_3Y_3)^2 - 4X_1Y_1X_2Y_2 = 0$. If the lines $X_1 = 0$, $X_2 = 0$, and $X_3 = 0$ crossed in a point then this should be a singular point of $\mathcal{C}$, which is nonsingular by hypothesis. $\qquad\square$

We end this section with an explicit description of triplets of mutually asyzygetic Steiner complexes.

PROPOSITION 3.7. *Let $X_1 = 0$, $X_2 = 0$, and $X_3 = 0$ be three asyzygetic bitangent lines. After a proper labeling of the bitangent lines, the Steiner complexes $S_{X_1X_2}$, S_{X_2,X_3}, and $S_{X_3X_1}$ have the following respective shapes*:

$$S_{X_1X_2} = \begin{pmatrix} X_1 & X_2 \\ X_4 & X_9 \\ X_5 & X_{10} \\ X_6 & X_{11} \\ X_7 & X_{12} \\ X_8 & X_{13} \end{pmatrix}, \quad S_{X_2X_3} = \begin{pmatrix} X_2 & X_3 \\ X_9 & X_{14} \\ X_{10} & X_{15} \\ X_{11} & X_{16} \\ X_{12} & X_{17} \\ X_{13} & X_{18} \end{pmatrix}, \quad S_{X_3X_1} = \begin{pmatrix} X_3 & X_1 \\ X_{14} & X_4 \\ X_{15} & X_5 \\ X_{16} & X_6 \\ X_{17} & X_7 \\ X_{18} & X_8 \end{pmatrix}.$$

In particular, the three complexes are asyzygetic.

Proof. Take a second pair of bitangent lines $\{A = 0, B = 0\}$ in $S_{X_1X_2}$. By Corollary 3.3, any other bitangent line must lie in exactly one of the Steiner complexes

$$S_{X_1X_2} = \{\{X_1 = 0, X_2 = 0\}, \{A = 0, B = 0\}, \dots\},$$
$$S_{X_1A} = \{\{X_1 = 0, A = 0\}, \{X_2 = 0, B = 0\}, \dots\},$$
$$S_{X_1B} = \{\{X_1 = 0, B = 0\}, \{X_2 = 0, A = 0\}, \dots\}.$$

Since $X_3 = 0$ cannot be in $S_{X_1X_2}$ by hypothesis, we may suppose that we have a pair $\{X_3 = 0, C = 0\}$ in S_{X_1A} such that $X_1 = 0$, $A = 0$, $X_3 = 0$, and $C = 0$ constitute a syzygetic tetrad and thus $\{A = 0, C = 0\}$ belongs to $S_{X_1X_3}$. However, the combination of $X_2 = 0$, $B = 0$, $X_3 = 0$, and $C = 0$ is a syzygetic tetrad and hence $\{B = 0, C = 0\}$ belongs to $S_{X_2X_3}$. $\qquad\square$

4. From Bitangent Lines to Equations

The basic tool for the proof of Theorem 1.1 is the following algebraic version of it, which gives a general equation for a curve of genus three in terms of some of its bitangent lines.

THEOREM 4.1. *Let $\mathcal{C} : Q = 0$ be a nonsingular, genus three-plane curve defined over a field K with $\mathrm{char}(K) \neq 2$. Let $\{X_1 = 0, Y_1 = 0\}$, $\{X_2 = 0, Y_2 = 0\}$, and $\{X_3 = 0, Y_3 = 0\}$ be three pairs of bitangent lines from a given Steiner complex. Let $\{X_7 = 0, Y_7 = 0\}$ be a fourth pair of bitangent lines from the Steiner complex given by the pairs $\{\{X_1 = 0, Y_2 = 0\}, \{X_2 = 0, Y_1 = 0\}\}$ and ordered such that $\{X_1 = 0, X_3 = 0, X_7 = 0\}$ and $\{X_1 = 0, Y_3 = 0, Y_7 = 0\}$ are asyzygetic.*

The equation for $\mathcal{C}$ can now be written as

$$Q \sim \left(\frac{(X_7 X_2 X_3)(X_7 Y_2 Y_3)}{(X_1 X_2 X_3)(Y_1 Y_2 Y_3)} X_1 Y_1 \right.$$

$$\left. + \frac{(X_1 X_7 X_3)(Y_1 X_7 Y_3)}{(X_1 X_2 X_3)(Y_1 Y_2 Y_3)} X_2 Y_2 - \frac{(X_1 X_2 X_7)(Y_1 Y_2 X_7)}{(X_1 X_2 X_3)(Y_1 Y_2 Y_3)} X_3 Y_3 \right)^2$$

$$- 4 \frac{(X_7 X_2 X_3)(X_7 Y_2 Y_3)}{(X_1 X_2 X_3)(Y_1 Y_2 Y_3)} \frac{(X_1 X_7 X_3)(Y_1 X_7 Y_3)}{(X_1 X_2 X_3)(Y_1 Y_2 Y_3)} X_1 Y_1 X_2 Y_2 = 0, \quad (3)$$

where (ABC) denotes the determinant of the matrix formed by the coefficients of the homogeneous linear polynomials A, B, and C.

REMARK. All the determinants appearing in the theorem are nonzero because they are formed by asyzygetic triplets of bitangent lines, which are nonconcurrent by Proposition 3.6.

Proof of Theorem 4.1. Following Riemann's construction [Rie], we know that it is possible to find a proper rescaling $\bar{X}_i = \alpha_i X_i$ and $\bar{Y}_i = \beta_i Y_i$ that yields a Riemann model for $\mathcal{C}$:

$$Q = (\bar{X}_1 \bar{Y}_1 + \bar{X}_2 \bar{Y}_2 - \bar{X}_3 \bar{Y}_3)^2 - 4 \bar{X}_1 \bar{Y}_1 \bar{X}_2 \bar{Y}_2 = 0,$$

where $\bar{X}_7 = \bar{X}_1 + \bar{X}_2 + \bar{X}_3 = -\bar{Y}_1 - \bar{Y}_2 - \bar{Y}_3 = 0$. We look at these two equalities as equations in the scaling factors α_i, β_i to determine them, and we find that

$$\alpha_1 = \alpha_7 \frac{(X_7 X_2 X_3)}{(X_1 X_2 X_3)}, \quad \alpha_2 = \alpha_7 \frac{(X_1 X_7 X_3)}{(X_1 X_2 X_3)}, \quad \alpha_3 = \alpha_7 \frac{(X_1 X_2 X_7)}{(X_1 X_2 X_3)};$$

$$\beta_1 = -\alpha_7 \frac{(X_7 Y_2 Y_3)}{(Y_1 Y_2 Y_3)}, \quad \beta_2 = -\alpha_7 \frac{(Y_1 X_7 Y_3)}{(Y_1 Y_2 Y_3)}, \quad \beta_3 = -\alpha_7 \frac{(Y_1 Y_2 X_7)}{(Y_1 Y_2 Y_3)}.$$

We replace the $\bar{X}_i, \bar{Y}_i$ on the Riemann model with these values and then simplify the constant α_7 to get the desired equation. □

5. Determinants of Bitangent Lines

We have seen in Theorem 4.1 how to construct a presentation of the curve $\mathcal{C}$ from a certain set of bitangent lines, which we can choose in many different ways. If we make different elections then the comparison of the corresponding presentations, which all agree up to a constant, leads us to a number of equalities between the involved determinants.

5.1. Relations between Pairs in the Same Steiner Complex

THEOREM 5.1. *Let $\{X_1 = 0, Y_1 = 0\}$, ..., $\{X_4 = 0, Y_4 = 0\}$ be four different pairs in the Steiner complex $S = S_{X_i Y_i}$. Then*

$$\frac{(X_3X_1X_2)(Y_3X_1X_2)}{(X_4X_1X_2)(Y_4X_1X_2)} = \frac{(X_3X_1Y_2)(Y_3X_1Y_2)}{(X_4X_1Y_2)(Y_4X_1Y_2)}$$

$$= \frac{(X_3X_2Y_1)(Y_3X_2Y_1)}{(X_4X_2Y_1)(Y_4X_2Y_1)} = \frac{(X_3Y_1Y_2)(Y_3Y_1Y_2)}{(X_4Y_1Y_2)(Y_4Y_1Y_2)}.$$

Proof. It is known [D, p. 94] that all the pairs in a Steiner complex can be seen as one of the degenerate conics on a fixed pencil of conics. In particular, we must have a relation

$$\lambda^2 X_1Y_1 + X_2Y_2 + \lambda X_3Y_3 = X_4Y_4$$

for certain $\lambda \in \bar{K}^*$. Let P_1, P_2, P_3, and P_4 be the points of intersection of the pairs of lines $\{X_1 = X_2 = 0\}$, $\{X_1 = Y_2 = 0\}$, $\{Y_1 = X_2 = 0\}$, and $\{Y_1 = Y_2 = 0\}$. Substituting these points into the preceding equality yields

$$\lambda X_3(P_j)Y_3(P_j) = X_4(P_j)Y_4(P_j), \quad j = 1, \dots, 4,$$

so that for $j \neq k$ we have

$$\frac{X_3(P_j)Y_3(P_j)}{X_4(P_j)Y_4(P_j)} = \frac{X_3(P_k)Y_3(P_k)}{X_4(P_k)Y_4(P_k)}. \tag{4}$$

An elementary exercise in linear algebra shows that, for any two lines $U = 0$ and $V = 0$, we have the equalities

$$\frac{U(P_1)}{V(P_1)} = \frac{(UX_1X_2)}{(VX_1X_2)}, \qquad \frac{U(P_2)}{V(P_2)} = \frac{(UX_1Y_2)}{(VX_1Y_2)},$$

$$\frac{U(P_3)}{V(P_3)} = \frac{(UY_1X_2)}{(VY_1X_2)}, \qquad \frac{U(P_4)}{V(P_4)} = \frac{(UY_1Y_2)}{(VY_1Y_2)}.$$

The theorem follows if we apply these relations to the two pairs of lines $\{X_3 = 0, X_4 = 0\}$ and $\{Y_3 = 0, Y_4 = 0\}$ and then substitute the resulting relations into (4). $\square$

5.2. Relations between Pairs in Syzygetic Steiner Complexes

THEOREM 5.2. *Let $\{X_1 = 0, Y_1 = 0\}$, $\{X_2 = 0, Y_2 = 0\}$, and $\{X_3 = 0, Y_3 = 0\}$ be three pairs of bitangent lines from a given Steiner complex. Let $\{X_7 = 0, Y_7 = 0\}$ be a fourth pair of bitangent lines from the Steiner complex given by the pairs $\{\{X_1 = 0, Y_2 = 0\}, \{X_2 = 0, Y_1 = 0\}\}$ ordered such that $\{X_1 = 0, X_3 = 0, X_7 = 0\}$ and $\{X_1 = 0, Y_3 = 0, Y_7 = 0\}$ are asyzygetic. Then the following relations hold:*

$$\frac{(X_2X_3X_7)}{(X_1X_3X_7)} = \frac{(X_2Y_3Y_7)}{(X_1Y_3Y_7)}, \qquad \frac{(Y_2X_3X_7)}{(Y_1X_3X_7)} = \frac{(Y_2Y_3X_7)}{(Y_1Y_3X_7)}.$$

Proof. The validity of the equalities is not affected by a rescaling of the involved lines, so we can assume that $X_7 = X_1 + X_2 + X_3$ and $Y_7 = Y_1 + Y_2 + X_3 = -X_1 - X_2 - Y_3$. The result follows from the most elementary properties of the determinants. $\square$

Part II. Analytic Identities

We now prove the theorems stated in the Introduction, which turn out to be the translation of the results from Part I to the context of complex geometry.

From now on, we suppose that $\mathcal{C}$ is a non-hyperelliptic plane curve of genus three defined over a field $K \subset \mathbb{C}$. We choose the basis $\omega_1, \omega_2, \omega_3 \in H^0(\mathcal{C}, \Omega^1_{/K})$ of holomorphic differential forms such that $\mathcal{C}$ can then be identified with the image of the associated canonical map $\iota \colon \mathcal{C} \to \mathbb{P}^2(K)$. We also fix a symplectic basis $\gamma_1, \ldots, \gamma_6$ of the singular homology $H_1(\mathcal{C}, \mathbb{Z})$. We denote by $\Omega = (\Omega_1 | \Omega_2) = \left(\int_{\gamma_j} \omega_k \right)_{j,k}$ the period matrix of $\mathcal{C}$ with respect to these bases and by $Z = \Omega_1^{-1}.\Omega_2$ the normalized period matrix. We consider the Jacobian variety $J_\mathcal{C}$, represented by the complex torus $\mathbb{C}^3/(1|Z)$, with the Abel–Jacobi map:

$$
\begin{aligned}
\mathcal{C}^2 &\xrightarrow{\Pi} J_\mathcal{C}, \\
D &\longrightarrow \int_\kappa^D (\omega_1, \omega_2, \omega_3);
\end{aligned}
\tag{5}
$$

here κ is the Riemann divisor, which guarantees that $\Theta = \Pi(\mathcal{C}^2)$ is the divisor cut by the Riemann theta function $\theta(z; Z)$ and that $\Pi(K_\mathcal{C} - D) = -\Pi(D)$.

We shall describe, as customary, the elements of $J_\mathcal{C}[2]$ by means of *characteristics*: every $w \in J_\mathcal{C}[2]$ is determined uniquely by a six-dimensional vector $\varepsilon = (\varepsilon', \varepsilon'') \in \{0, 1/2\}^6$ and the relation $w = \varepsilon + Z\varepsilon''$. With this notation, the Weil pairing on $J_\mathcal{C}[2]$ is given by

$$
\tilde{e}_2(w_1, w_2) := \tilde{e}_2(\varepsilon_1, \varepsilon_2) := \varepsilon_1' \cdot \varepsilon_2'' + \varepsilon_1'' \cdot \varepsilon_2' \pmod 1.
\tag{6}
$$

Every $w \in J_\mathcal{C}[2]$ defines a translate of the Riemann theta function:

$$
\begin{aligned}
\theta[w](z; Z) &:= e^{\pi i {}^t\varepsilon'.Z.\varepsilon' + 2\pi i {}^t\varepsilon'.(z+\varepsilon'')} \theta(z + Z\varepsilon' + \varepsilon'') \\
&= \sum_{n \in \mathbb{Z}^g} e^{\pi i {}^t(n+\varepsilon').Z.(n+\varepsilon') + 2\pi i {}^t(n+\varepsilon').(z+\varepsilon'')}.
\end{aligned}
$$

The values $\theta[w](0, Z)$ are usually called *Thetanullwerte* and denoted more compactly by $\theta[w](Z)$. For a sequence of three points $w_1, w_2, w_3 \in J_\mathcal{C}[2]$, the modified determinant $[w_1, w_2, w_3](Z) := \pi^3 \det J[w_1, w_2, w_3](Z)$ of the matrix

$$
J[w_1, w_2, w_3](Z) := \begin{pmatrix} \dfrac{\partial \theta[w_1]}{\partial z_1}(0; Z) & \dfrac{\partial \theta[w_1]}{\partial z_2}(0; Z) & \dfrac{\partial \theta[w_1]}{\partial z_3}(0; Z) \\[2.2ex] \dfrac{\partial \theta[w_2]}{\partial z_1}(0; Z) & \dfrac{\partial \theta[w_2]}{\partial z_2}(0; Z) & \dfrac{\partial \theta[w_2]}{\partial z_3}(0; Z) \\[2.2ex] \dfrac{\partial \theta[w_3]}{\partial z_1}(0; Z) & \dfrac{\partial \theta[w_3]}{\partial z_2}(0; Z) & \dfrac{\partial \theta[w_3]}{\partial z_3}(0; Z) \end{pmatrix}
$$

is called *Jacobian Nullwert*. (These definitions can be given for g-dimensional complex abelian varieties, but for brevity we restrict them to our case of dimension three.)

6. Proof of Main Results

It is well known that the Abel–Jacobi map establishes a bijection between semi-canonical divisors on $\mathcal{C}$ and the set $J_{\mathcal{C}}[2]$. In our particular case of a non-hyperelliptic genus-three curve, the effective semicanonical divisors are given by the bitangent lines: given a bitangent $X = 0$, we have $\mathrm{div}(X) = 2D$ with D a semicanonical divisor that goes to a $w := \Pi(D) \in J_{\mathcal{C}}[2]^{\mathrm{odd}}$. Reciprocally, given $w \in J_{\mathcal{C}}[2]^{\mathrm{odd}}$, the line

$$H_w := \left(\frac{\partial \theta[w]}{\partial z_1}(0; Z), \frac{\partial \theta[w]}{\partial z_2}(0; Z), \frac{\partial \theta[w]}{\partial z_3}(0; Z) \right).\Omega_1^{-1}. \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 0$$

is a bitangent line to $\mathcal{C}$ whose corresponding semicanonical divisor D satisfies $\Pi(D) = w$ (cf. [Gu1, Prop. 3.1]). Moreover, we have

$$e_2(\Pi(w_1), \Pi(w_2)) = \tilde{e}_2(w_1, w_2).$$

Note also that, given $w_i \in J_{\mathcal{C}}[2]^{\mathrm{odd}}$, we have

$$\frac{[w_1, w_2, w_3](Z)}{[w_4, w_5, w_6](Z)} = \frac{(H_{w_1} H_{w_2} H_{w_3})}{(H_{w_4} H_{w_5} H_{w_6})}. \tag{7}$$

The theorems stated in the Introduction are the translation to the analytic context of Theorems 4.1, 5.1, and 5.2. In order to prove them, we need only check that the bitangent lines associated with the odd two-torsion points $w_k \in J_{\mathcal{C}}[2]^{\mathrm{odd}}$ appearing in the theorems satisfy the respective hypotheses of those theorems. But this is immediate if we use the analytic description (6) of the Weil pairing.

7. Some Geometry around the Frobenius Formula

Although the closed solution to the Torelli problem given by Theorem 1.1 is satisfactory from the geometrical viewpoint, it would be desirable to have also a formula involving only Thetanullwerte instead of Jacobian Nullwerte. We cannot avoid the gradients giving the bitangent lines, but at least we can replace the quotients of Jacobian Nullwerte by quotients of Thetanullwerte—taking into account the Frobenius formula, which we recall briefly.

Both the Thetanullwerte and the Jacobian Nullwerte can be interpreted as functions defined on the Siegel upper half-space $\mathbb{H}_3$: the idea consists of fixing the necessary characteristics and letting Z run through $\mathbb{H}_3$. The Frobenius formula relates certain Jacobian Nullwerte to products of Thetanullwerte.

Three characteristics $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are called *asyzygetic* if $\tilde{e}_2(\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3) = -1$. A sequence of characteristics is called asyzygetic if every triplet contained in it is asyzygetic. A fundamental system of characteristics is an asyzygetic sequence $S = \{\varepsilon_1, \ldots, \varepsilon_8\}$ of characteristics with $\varepsilon_1, \varepsilon_2, \varepsilon_3$ odd and $\varepsilon_4, \ldots, \varepsilon_8$ even.

THEOREM 7.1 [F; I]. *Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be three odd characteristics, and take $w_i :=$*
$w_i(Z) := \varepsilon_i' + Z.\varepsilon_i''$. Then there is an equality on $\mathbb{H}_3$ of the form

$$[w_1, w_2, w_3](Z) = \pm \pi^3 \theta[w_4](Z) \cdots \theta[w_8](Z)$$

if and only if $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are asyzygetic, and in this case the characteristics $\varepsilon_4, \ldots, \varepsilon_8$ corresponding to $w_4, \ldots, w_8$ are the unique completion of $\varepsilon_1, \varepsilon_2, \varepsilon_3$ to a fundamental system.

The sign in the preceding equality can be determined for every particular fundamental system.

As mentioned after Theorem 4.1, all the triplets of bitangent lines involved in the expression displayed in Theorem 7.1 are asyzygetic. This implies that we can express the corresponding Jacobian Nullwerte as products of Thetanullwerte. We have the following identities:

$$[w_1, w_2, w_3] = \pi^3 \prod_{k=1}^{5} \theta[a_k](Z), \qquad [w_1', w_2', w_3'] = -\pi^3 \prod_{k=1}^{5} \theta[b_k](Z),$$

$$[w_7, w_2, w_3] = \pi^3 \prod_{k=1}^{5} \theta[c_k](Z), \qquad [w_7, w_2', w_3'] = \pi^3 \prod_{k=1}^{5} \theta[d_k](Z),$$

$$[w_1, w_7, w_3] = -\pi^3 \prod_{k=1}^{5} \theta[e_k](Z), \qquad [w_1', w_7, w_3'] = -\pi^3 \prod_{k=1}^{5} \theta[f_k](Z),$$

$$[w_1, w_2, w_7] = \pi^3 \prod_{k=1}^{5} \theta[g_k](Z), \qquad [w_1', w_2', w_7] = \pi^3 \prod_{k=1}^{5} \theta[h_k](Z),$$

where

$$
\begin{aligned}
{}^t a_1 &= {}^t(0,0,0) + {}^t(0,0,0).Z, & {}^t a_2 &= {}^t(0,0,0) + {}^t\left(\tfrac{1}{2},0,0\right).Z, \\
{}^t a_3 &= {}^t\left(\tfrac{1}{2},0,\tfrac{1}{2}\right) + {}^t\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right).Z, & {}^t a_4 &= {}^t\left(\tfrac{1}{2},\tfrac{1}{2},0\right) + {}^t\left(\tfrac{1}{2},\tfrac{1}{2},0\right).Z, \\
{}^t a_5 &= {}^t\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right) + {}^t\left(\tfrac{1}{2},0,\tfrac{1}{2}\right).Z;
\end{aligned}
$$

$$
\begin{aligned}
{}^t b_1 &= {}^t(0,0,0) + {}^t\left(\tfrac{1}{2},0,0\right).Z, & {}^t b_2 &= {}^t\left(\tfrac{1}{2},0,0\right) + {}^t(0,0,0).Z, \\
{}^t b_3 &= {}^t\left(\tfrac{1}{2},0,\tfrac{1}{2}\right) + {}^t\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right).Z, & {}^t b_4 &= {}^t\left(\tfrac{1}{2},\tfrac{1}{2},0\right) + {}^t\left(\tfrac{1}{2},\tfrac{1}{2},0\right).Z, \\
{}^t b_5 &= {}^t\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right) + {}^t\left(\tfrac{1}{2},0,\tfrac{1}{2}\right).Z;
\end{aligned}
$$

$$
\begin{aligned}
{}^t c_1 &= {}^t(0,0,0) + {}^t(0,0,0).Z, & {}^t c_2 &= {}^t\left(0,\tfrac{1}{2},0\right) + {}^t(0,0,0).Z, \\
{}^t c_3 &= {}^t\left(0,\tfrac{1}{2},\tfrac{1}{2}\right) + {}^t\left(0,\tfrac{1}{2},\tfrac{1}{2}\right).Z, & {}^t c_4 &= {}^t\left(\tfrac{1}{2},0,0\right) + {}^t\left(0,\tfrac{1}{2},0\right).Z, \\
{}^t c_5 &= {}^t\left(\tfrac{1}{2},0,\tfrac{1}{2}\right) + {}^t\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right).Z;
\end{aligned}
$$

$$
\begin{aligned}
{}^t d_1 &= {}^t(0,0,0) + {}^t\left(0,\tfrac{1}{2},0\right).Z, & {}^t d_2 &= {}^t\left(\tfrac{1}{2},0,0\right) + {}^t(0,0,0).Z, \\
{}^t d_3 &= {}^t\left(\tfrac{1}{2},0,\tfrac{1}{2}\right) + {}^t\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right).Z, & {}^t d_4 &= {}^t\left(\tfrac{1}{2},\tfrac{1}{2},0\right) + {}^t(0,0,0).Z, \\
{}^t d_5 &= {}^t\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right) + {}^t\left(0,\tfrac{1}{2},\tfrac{1}{2}\right).Z;
\end{aligned}
$$

$$^te_1 = {}^t(0,0,0) + {}^t(0,0,0).Z, \qquad {}^te_2 = {}^t(0,0,0) + {}^t(0,0,\tfrac{1}{2}).Z,$$

$$^te_3 = {}^t(0,0,\tfrac{1}{2}) + {}^t(0,\tfrac{1}{2},0).Z, \qquad {}^te_4 = {}^t(\tfrac{1}{2},0,\tfrac{1}{2}) + {}^t(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}).Z,$$

$$^te_5 = {}^t(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}) + {}^t(0,0,0).Z;$$

$$^tf_1 = {}^t(0,\tfrac{1}{2},\tfrac{1}{2}) + {}^t(0,0,0).Z, \qquad {}^tf_2 = {}^t(\tfrac{1}{2},0,0) + {}^t(0,0,0).Z,$$

$$^tf_3 = {}^t(\tfrac{1}{2},0,0) + {}^t(0,0,\tfrac{1}{2}).Z, \qquad {}^tf_4 = {}^t(\tfrac{1}{2},0,\tfrac{1}{2}) + {}^t(0,\tfrac{1}{2},0).Z,$$

$$^tf_5 = {}^t(\tfrac{1}{2},0,\tfrac{1}{2}) + {}^t(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}).Z;$$

$$^tg_1 = {}^t(0,0,0) + {}^t(0,0,0).Z, \qquad {}^tg_2 = {}^t(0,0,0) + {}^t(0,\tfrac{1}{2},\tfrac{1}{2}).Z,$$

$$^tg_3 = {}^t(0,0,\tfrac{1}{2}) + {}^t(0,0,0).Z, \qquad {}^tg_4 = {}^t(\tfrac{1}{2},0,\tfrac{1}{2}) + {}^t(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}).Z,$$

$$^tg_5 = {}^t(\tfrac{1}{2},\tfrac{1}{2},0) + {}^t(0,0,\tfrac{1}{2}).Z;$$

$$^th_1 = {}^t(0,\tfrac{1}{2},0) + {}^t(0,0,\tfrac{1}{2}).Z, \qquad {}^th_2 = {}^t(\tfrac{1}{2},0,0) + {}^t(0,0,0).Z,$$

$$^th_3 = {}^t(\tfrac{1}{2},0,0) + {}^t(0,\tfrac{1}{2},\tfrac{1}{2}).Z, \qquad {}^th_4 = {}^t(\tfrac{1}{2},0,\tfrac{1}{2}) + {}^t(0,0,0).Z,$$

$$^th_5 = {}^t(\tfrac{1}{2},0,\tfrac{1}{2}) + {}^t(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}).Z.$$

Observe that there are some coincidences between the fundamental systems giving these identities; for instance, $c_1 = a_1$ and $c_5 = a_3$. In fact, every two share two of their even characteristics. Thus, when we plug the preceding equalities into Theorem 1.1, we can make some simplifications as follows.

COROLLARY 7.2. *Let*

$$A_1 = \theta[c_2](Z)\theta[c_3](Z)\theta[c_4](Z)\theta[d_1](Z)\theta[d_4](Z)\theta[d_5](Z),$$

$$A_2 = \theta[e_2](Z)\theta[e_3](Z)\theta[e_5](Z)\theta[f_1](Z)\theta[f_3](Z)\theta[f_4](Z),$$

$$A_3 = \theta[g_2](Z)\theta[g_3](Z)\theta[g_5](Z)\theta[h_1](Z)\theta[h_3](Z)\theta[h_4](Z).$$

An equation for C, *defined over* K *up to a normalization, is*

$$C : \left(X_1Y_1 + \frac{A_2}{A_1}X_2Y_2 - \frac{A_3}{A_1}X_3Y_3\right)^2 - 4\frac{A_2}{A_1}X_1X_2Y_1Y_2 = 0.$$

The quotients of Thetanullwerte appearing in this presentation of C can be interpreted as modular functions for certain level congruence subgroups, as is the case for the coefficients of the formula in Theorem 1.1.

We finish with a geometrical description of the fundamental systems appearing in the Frobenius formula. We have considered the general hyperelliptic case in [Gu2], and next we explain the situation for non-hyperelliptic genus-three curves.

THEOREM 7.3. *Let* $X_1 = 0$, $X_2 = 0$, *and* $X_3 = 0$ *be three asyzygetic bitangent lines. Write*

$$S_{X_1X_2} \cap S_{X_1,X_3} = \{X_1 = 0,\ X_4 = 0,\ \ldots,\ X_8 = 0\}$$

according to Proposition 3.7. Let

$$W_j = \begin{cases} \frac{1}{2}\operatorname{div}(X_j) & \text{if } j = 1, 2, 3; \\ \frac{1}{2}\operatorname{div}\left(\frac{X_2 X_3}{X_j}\right) = W_2 + W_3 - \frac{1}{2}\operatorname{div}(X_j) & \text{if } j = 4, \dots, 8. \end{cases}$$

The odd two-torsion points $w_1, \dots, w_8 \in J_{\mathcal{C}}$ corresponding to these semicanonical divisors through the Abel–Jacobi map (5) *form a fundamental system.*

Proof. We must show that $\tilde{e}_2(w_1 - w_i, w_1 - w_j) = 1$ for all pairs $i, j \in \{1, \dots, 8\}$. By construction of the W_i, it is enough to see that $\tilde{e}_2(w_1 - w_2, w_1 - w_3) = 1$, $\tilde{e}_2(w_1 - w_2, w_1 - w_4) = 1$, and $\tilde{e}_2(w_1 - w_4, w_1 - w_5) = 1$. The first equality is immediate because $X_1 = 0$, $X_2 = 0$, and $X_3 = 0$ are asyzygetic. The remaining two are derived easily: we compute the Weil pairing by applying the Riemann–Mumford relation (2) with $D = W_1$. We have, for instance:

$$\begin{aligned} \tilde{e}_2(w_1 &- w_2, w_1 - w_4) \\ &= e_2(W_1 - W_2, W_1 - W_4) \\ &= h^0(W_1) + h^0(3W_1 - W_2 - W_4) + h^0(2W_1 - W_2) + h^0(2W_1 - W_4) \\ &= 1 + h^0(W_1 - W_4) + h^0(W_2) + h^0(W_4) \\ &= 3 + h^0(W_1 - W_4) = 1 \pmod{2}, \end{aligned}$$

since $W_1 - W_4$ is not effective. $\qquad\square$

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